# A Continuous Version of Duality of $H^1$ with $BMO$ on the Bidisc


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# Dual Spaces

The question of duality is a natural one: what are the linear functionals acting on a space. For a topological vector space, one can ask a related question that takes advantage of the further structure given by a topology: This is, what are the continuous linear functionals?

**Problem**

Characterize the continuous dual space of $H^1(\mathbb{D}^2)$

The answer was given by Chang and Fefferman in 1980. We present their results below.

# Motivation: the 1-dimensional case

## Theorem

The space BMO is the dual space of the Hardy space $H^1(\mathbb{R})$. The pairing is what you expect: $\langle f, g \rangle = \int\limits_{-\infty}^{\infty} f(x)g(x)dx$, defined for $g \in BMO$ and $f \in C^\infty$, a dense subset of $H^1$.

Obvious questions:

1: What do we mean by $H^1$, and what is the proper analogy on the bidisc?

2: What is BMO, and what its higher-dimensional analog?

3: Does this theorem lift to the bidisc?

**Definition**

The Hardy space $H^1(\mathbb{R})$ is defined to be the boundary values of $H^1(\mathbb{R}^2_+)$. This latter space is defined as functions $f \in Hol(\mathbb{R}^2_+)$ that satisfy $\sup_{y>0} \int f(x+iy)dx < \infty$.

One useful property of $H^1(\mathbb{R})$ is that it has an atomic decomposition, as follows:

# Atomic Decomposition of $H^1(\mathbb{R})$

## Definition

An $H^1$ atom is a function $a(x)$ that satisfies:

- The support of $a$ lies in a bounded interval $I$
- $\int_I a(x)dx = 0$
- $||a||_\infty \leq \frac{1}{|I|}$

## Characterization of $H^1$

Given this definition, the useful fact is that for $f \in H^1$ we have $f = \sum \lambda_k a_k(x)$. Moreover $\sum |\lambda_k| \leq C||f||_{H^1}$.

# Definition of $H^1(\mathbb{R}^2)$

## Analogy in Higher dimensions

The space $H^1(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ will be defined to be biharmonic functions $u \in H^1$ having their non-tangential maximal functions $N(u)$ in $L^1(\mathbb{R}^2)$. We say a function is biharmonic if it satisfies Laplace's equation

We note that this is one of several equivalent definitions of this space.

With the right definition of an atom, we shall also have an atomic decomposition.

# An Auxiliary Function

To work in higher dimensions we will first need a few preliminaries. Let $\psi \in C^1(\mathbb{R})$ be an even function supported on $[-1, 1]$ with mean 0. Then for $x = (x_1, x_2)$ in $\mathbb{R}^2$ and $y = (y_1, y_2)$ in $(\mathbb{R}_+)^2$, define

$$\psi_y(x) = \frac{1}{y_1 y_2} \psi\left(\frac{x_1}{y_1}\right) \psi\left(\frac{x_2}{y_2}\right)$$

The function $\psi$ will be fixed, and normalized so that $\int_0^\infty \left|\hat{\psi}(\xi)\right|^2 \frac{d\xi}{\xi} = 1.$

# Extensions from Boundary Values

We define the extension of a function $f$ defined on the boundary to be $f(x, y) = f * \psi_y$. We hence have for $f \in H^1$

$$f(x, y) = \iint_{(t,y) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+} f(t, y)\psi_y(x - t)\frac{dtdy}{y_1 y_2}$$

# The double-S function

In one dimension the sweep (or balayage) of a measure is useful: the sweep of a the absolute value of a measure is in BMO if the measure itself in Carleson. In higher dimensions we will have the following important analogue, which Chang and Fefferman call the "double-S" function.

$$S^2(f)(x_1, x_2) = \iint_{(t,y)\in\Gamma(x_1)\times\Gamma(x_2)} |f(t,y)|^2 \frac{dtdy}{y_1^2 y_2^2}$$

Here $\Gamma(x_i)$ is the usual nontangential approach region. This function will play a vital role in our computations.

# The "Rectangle" Function

The last preliminary we shall need is the 'rectangle' function. For $f \in H^1$ and $R$ a rectangle , define

$$f_R(x,y) = \iint_{(t,y)\in R_+} f(t,y)\psi_y(x-t)\frac{dtdy}{y_1 y_2}$$

Here $R_+ = \left\{(t,y) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+ : t \in R, \frac{|I|}{2} \leq y_1 \leq |I|, \frac{|J|}{2} \leq y_2 \leq |J|\right\}$.
The region $R_+$ is part of a Carleson box. This function will be key in writing down an atomic decomposition.

# Some remarks on $\psi$

## An explanatory formula

For $f \in H^1$

$$f(x,y) = \iint_{(t,y)\in\mathbb{R}^2_+\times\mathbb{R}^2_+} f(t,y)\psi_y(x-t)\frac{dtdy}{y_1 y_2}$$

# Atomic Decomposition of $H^1(\mathbb{R}^2)$

Of course, finding the "right" atom takes some work:

## Definition

An atom on $\mathbb{R}^2$ is a function $a(x_1, x_2)$ satisfying:

- The support of $a$ is contained in an open set $\Omega$
- $\int_I a(x_1, x_2) dx_1 = 0$ where $I$ is any component interval of any $x_1$ cross-section of $\Omega$.
- $\int_J a(x_1, x_2) dx_1 = 0$ where $J$ is any component interval of any $x_2$ cross-section of $\Omega$.
- Further geometric requirements:

# More on atoms

## Further requirements

- a$=\sum_R a_R$, where each $a_R$ is supported on a rectangle $R \subset \Omega$. Say $R = I \times J$.
  - The rectangles R form a collection of maximal dyadic rectangles.
  - $\int_I a(x_1, x_2) dx_1 = 0$ for each $x_2 \in J$
  - $\int_J a(x_1, x_2) dx_1 = 0$ for each $x_1 \in I$
  - $a_R$ is $C^1$ with $||a_R||_\infty \leq |R|^{1/2}$
  - $\left|\left|\frac{\partial a_R}{\partial x_1}\right|\right| \leq \frac{C_R}{|I||R|^{1/2}}$ and $\left|\left|\frac{\partial a_R}{\partial x_2}\right|\right| \leq \frac{C_R}{|J||R|^{1/2}}$
  - $\left|\left|\frac{\partial^2 a_R}{\partial x_1 \partial x_2}\right|\right| \leq \frac{C_R}{|R|^{3/2}}$
  - $\sum_R C_R^2 \leq \frac{A}{|\Omega|}$

# Characterization of $H^1(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$

With all that in place, we get the following:

**Theorem (Chang-Fefferman)**

Let $f \in H^1$. Then $f$ can be written as $f = \sum \lambda_k a_k$ where the $a_k$ are atoms and $\lambda_k \geq 0$ satisfy $\sum \lambda_k \leq C||f||_{H^1}$

Ingredients of Proof: The proof of this theorem contains much of the geometric content used in the larger duality theorem. The key ingredient is finding a collection of maximal dyadic rectangles, and using the 'rectangle' function as defined above to be the atom.

# Bounded Mean Oscillation

## Definition

For a function $\phi \in L^1_{loc}(\mathbb{R})$, we say $\phi$ is of bounded mean oscillation (BMO) if

$$\sup_I \frac{1}{|I|} \int_I |\phi - \phi_I|^2 dx = ||\phi||_*^2 < \infty.$$

Here the supremum is taken over all finite intervals $I$, and $\phi_I$ is the average value of $\phi$ over such an interval.

By replacing intervals with rectangles, one would hope to get an analogy for BMO. Unfortunately such functions may not act continuously on $H^1$ of the bidisc. (This follows from work by Carleson).

# The 'right' BMO: preliminaries

We now define the space BMO that will be the correct continuous dual space. Note now though the name now only holds through the duality analogy.

We first recall the definition of BMO:

**Definition**

The space BMO is the space of locally integrable functions $\phi$ such that $\sup_I \frac{1}{|I|} \int\limits_I |\phi - \phi_I| dx$

# A candidate for the continuous dual space

Our first candidate is motivated by the 1-d definition:

## Definition

The space $BMO_{(a)}$ is the space of locally integrable functions $\phi$ such that

$$\sup_{\Omega} \frac{1}{|\Omega|} \left|\left| \sum_{R \subset \Omega} \phi_R \right|\right|_2^2 = ||\phi||_*^2 < \infty$$

Here the supremum ranges over all open sets of finite measure, and we sum over the rectangle as described above.

# A second candidate for the continuous dual space

Motivated by the atomic decomposition, we have:

## Definition

The space $BMO_{(b)}$ is the space of locally integrable functions $\phi$ such that given any open set $\Omega \subset \mathbb{R}^2$ there exists a function $\tilde{\phi}_\Omega$ satisfying the following:

$$\frac{1}{|\Omega|} \int_\Omega |\phi(t) - \tilde{\phi}_\Omega(t)|^2 dt \leq M$$

for some constant $M$ independent of $\Omega$

This definition is motivated by the atomic decomposition, so there are also conditions we ask about $\tilde{\phi}_R$

# Requirements on $\tilde{\phi}_R$

- $\tilde{\phi}_R = \sum \tilde{\phi}_i$, where each $\tilde{\phi}_i$ is supported on the triple $\tilde{R}_i$ of distinct dyadic rectangles $R_i$ with $|\tilde{R}_i \cap \Omega| < \frac{1}{2}|\tilde{R}_i|$
- Each $\tilde{\phi}_i$ has mean value zero over each horizontal and vertical slice of $\tilde{R}_i$
- $||\tilde{\phi}_i||_\infty \leq \dfrac{C_{R_i}}{|R_i|^{1/2}}$
- Similar smoothness conditions as for atoms, but different requirement on the constants $C_{R_i}$

# The Main Result

## Theorem (Chang, Fefferman '80)

Let $\phi \in L^2(\mathbb{R}^2)$ satisfy
$\int \phi(y_1, x_2) dy_1 = \int \phi(x_1, y_2) dx_2 = 0$ for all $(x_1, x_2) \in \mathbb{R}^2$.
Then the following are equivalent:

(i) $\phi \in BMO_{(a)}$

(ii) $\phi \in BMO_{(b)}$

(iii) $\displaystyle \frac{1}{|\Omega|} \sum_{R \subset \Omega} S_R^2(\phi) < \infty$, where the supremum ranges over all the finite open sets $\Omega$, and for each dyadic rectangle $R$

$$S_R^2(\phi) = \iint\limits_{R_+} |\phi(t, y)|^2 \frac{dtdy}{y_1 y_2}$$

(iv) $\phi$ is in the (continuous) dual of $H^1$

# Ideas in the proof

## Sketch of proof

- $(i) \Leftrightarrow (iii)$ is proven by a careful study decomposition of $\Omega$ into rectangles.
- $(ii) \Rightarrow (iv)$ can be established easily by checking on atoms.
- The other implications shown in the paper are $(iii) \Rightarrow (ii)$ and $(iv) \Rightarrow (iii)$.

# Product BMO

## A final comment

The theorem proves the equivalence of $BMO_{(a)}$ with $BMO_{(b)}$; in the literature this space is sometimes referred to as product BMO.

# References

This talk was based off of the paper:

Chang, S.-Y and Robert Fefferman. "A continuous version of duality of $H^1$ with $BMO$ on the bidisc." Annals of Mathematics, (1980), no. 1, pg. 179–201.

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