PÓLYA SEQUENCES, TOEPLITZ KERNELS AND GAP THEOREMS

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Abstract. A separated sequence Λ on the real line is called a Pólya sequence if any entire function of zero exponential type bounded on Λ is constant. In this paper we solve the problem by Pólya and Levinson that asks for a description of Pólya sets. We also show that the Pólya-Levinson problem is equivalent to a version of the so-called Beurling gap problem on Fourier transforms of measures. The solution is obtained via a recently developed approach based on the use of Toeplitz kernels and de Branges spaces of entire functions.

1. Introduction and background

1.1. Introduction. An entire function F is said to have exponential type zero if

$$\limsup_{|z|\to\infty} \frac{\log|F(z)|}{|z|} = 0.$$

We call a separated real sequence $\Lambda = \{\lambda_n\}_{n=-\infty}^{\infty}$ (a sequence is separated if it satisfies $|\lambda_n - \lambda_m| \geq \delta > 0, \ (n \neq m)$) a *Pólya sequence* if any entire function of exponential type zero that is bounded on Λ is constant. In this paper we consider the problem of description of Pólya sequences.

Historically, first results on Pólya sequences were obtained in the work of Valiron [22], where it was proved that the set of integers $\mathbb{Z}$ is a Pólya sequence. Later this result was popularized by Pólya, who posted it as a problem in [21]. Subsequently many different proofs and generalizations were given (see for example section 21.2 of [14] or chapter 10 of [4] and references therein).

In his 1940' book [15] Levinson showed that if $|\lambda_n - n| \leq p(n)$, where $p(t)$ satisfies $\int \frac{p(t)}{1+t^2} \log|\frac{t}{p(t)}|dt < \infty$ and some smoothness conditions, then $\Lambda = \{\lambda_n\}$ is a Pólya sequence. In the same time for each such $p(t)$ satisfying $\int p(t)dt/(1+t^2) = \infty$ he was able to construct a sequence $\Lambda = \{\lambda_n\}$ that is not Pólya sequence. As it often happens in problems from this area, the construction took a considerable effort (see [15], pp. 153-185). Closing the gap between Levinson's sufficient condition and the counterexample remained an open problem for almost 25 years until de Branges [6] essentially solved it by showing that Λ is a Pólya sequence if $\int p(t)dt/(1+t^2) < \infty$ (but assuming extra regularity conditions on the sequence).

The results of [15] and [6] remain strongest to date. However, none of them gives a complete answer, since there are Pólya sequences for which $\int p(t)dt/(1 + t^2) = \infty$. For example, as will be clear from our results below, the sequence

$$\lambda_n := n + n/\log(|n| + 2), \ n \in \mathbb{Z}$$

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is a Pólya sequence.

In the opposite direction, [6] contains the following necessary condition. A sequence of disjoint intervals I_n on the real line is called long (in the sense of Beurling and Malliavin) if

$$\sum_n \frac{|I_n|^2}{1+\operatorname{dist}^2(I_n,0)} = \infty,$$

and it is called short otherwise. Here $|I_n|$ denotes the length of the interval I_n. De Branges [6] proved that if the complement of a closed set $X \subset \mathbb{R}$ is long then there exists a non constant zero type entire function that is bounded on X. In particular, if the complement of a sequence Λ is long then Λ is not a Pólya sequence. The sequence $\lambda_n := n^2$ shows that this condition is not sufficient. Indeed, $\lambda_n = n^2$ is the zero set of the zero type function $F(z) := \cos\sqrt{2\pi z}\cos\sqrt{-2\pi z}$ and thus is not a Pólya sequence. On the other hand, the real complement of this sequence is short.

In this paper we give the following answer to the Pólya-Levinson question, see the corollary of Theorem A below. We show that a separated sequence of real numbers Λ is *not* a Pólya sequence if and only if there exists a long sequence of intervals $\{I_n\}$ such that

$$\frac{\#(\Lambda\cap I_n)}{|I_n|} \to 0.$$

Our approach is similar to the one developed by the second author and Makarov in [16] and [17], where it was used to obtain extensions and applications of the Beurling-Malliavin theory. One of our main tools is the connection between the Pólya-Levinson problem, the gap problem and the problem of injectivity of Toeplitz operators, see Theorems A and C. To promote the Toeplitz approach and to make the paper more self-contained we often include full proofs rather than referring to existing results.

The Beurling gap problem that we consider here may be formulated as follows. Under what conditions on a separated real sequence $\Lambda = \{\lambda_n\}_{n=-\infty}^{\infty}$ does there exist a nonzero finite measure μ supported on Λ such that the Fourier transform of μ vanishes on an interval of positive length? Of course, one can ask the same question for an arbitrary closed set $X \subset \mathbb{R}$. It was de Branges who first observed the connection between this problem and the Pólya-Levinson problem. The sufficient condition that he gave for the Pólya-Levinson problem was the same as Beurling's sufficient condition for the gap problem, see [1] or [13]. Namely, Beurling proved that if the complement of a closed set $X \subset \mathbb{R}$ is long then there exists no nonzero finite measure μ supported on X such that the Fourier transform of μ vanishes on an interval of positive length. The natural question is whether these two problems are equivalent. As the reader will see below, for sequences the answer is positive.

Besides giving a solution to the gap problem for separated sequences we also improve Beurling's gap theorem for general closed sets, see Theorem B and its corollaries.

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1.2. **Background.** We use the standard notation $N^+(\mathbb{C}_+)$ to denote the Smirnov-Nevanlinna class in the upper half-plane $\mathbb{C}_+ = \{z | \Im z > 0\}$ consisting of analytic functions $f(z)$ that can be represented as a ratio $g(z)/h(z)$ of two bounded analytic functions with $h(z)$ being outer. Each function in $N^+(\mathbb{C}_+)$ has non-tangential boundary values almost everywhere on $\mathbb{R}$ that completely determine the function. A mean type of a function $f(z)$ in $N^+(\mathbb{C}_+)$ is defined as $\tau := \limsup_{y\to\infty} \log|F(iy)|/y$. It is easy to see that every function in $N^+(\mathbb{C}_+)$ has a non-positive mean type which is exactly the exponent of $S(z) := e^{iz}$ in the inner-outer factorization of $f(z)$ taken with a negative sign. Here and throughout the paper $S(z)$ denotes the singular inner function e^{iz}.

A Hardy space $H^2(\mathbb{C}_+)$ consists exactly of those functions in $N^+(\mathbb{C}_+)$ which are square-integrable on $\mathbb{R}$ (for more on Smirnov-Nevanlinna and Hardy spaces see, e.g., [9]).

A classical theorem of Krein gives a connection between the Smirnov-Nevanlinna class $N^+(\mathbb{C}_+)$ and the Cartwright class C_a consisting of all entire functions $F(z)$ of exponential type $\le a$ that satisfy $\log|F(t)| \in L^1(dt/(1+t^2))$. An entire function $F(z)$ belongs to the Cartwright class C_a if and only if

$$\frac{F(z)}{S^{-a}(z)} \in N^+(\mathbb{C}_+), \qquad \frac{F^\#(z)}{S^{-a}(z)} \in N^+(\mathbb{C}_+),$$

where $F^\#(z) = \overline{F(\bar{z})}$.

As an immediate consequence one obtains a connection between the Hardy space $H^2(\mathbb{C}_+)$ and the Paley-Wiener space PW_a. Namely, an entire function $F(z)$ belongs to the Paley-Wiener class PW_a if and only if

$$\frac{F(z)}{S^{-a}(z)} \in H^2(\mathbb{C}_+), \qquad \frac{F^\#(z)}{S^{-a}(z)} \in H^2(\mathbb{C}_+).$$

The definition of the de Branges spaces of entire functions may be viewed as a generalization of the above definition of the Payley-Wiener spaces with $S^{-a}(z)$ replaced by a more general entire function. Consider an entire function $E(z)$ satisfying the inequality

$$|E(z)| > |E(\bar{z})|, \qquad z \in \mathbb{C}_+.$$

Such functions are usually called de Branges functions. The de Branges space B_E associated with $E(z)$ is defined to be the space of entire functions $F(z)$ satisfying

$$\frac{F(z)}{E(z)} \in H^2(\mathbb{C}_+), \qquad \frac{F^\#(z)}{E(z)} \in H^2(\mathbb{C}_+).$$

It is a Hilbert space equipped with the norm $\|F\|_E := \|F/E\|_{L^2(\mathbb{R})}$. If $E(z)$ is of exponential type then all the functions in the de Branges space B_E will be of exponential type not greater then the type of $E(z)$ (see, for example, the last part in the proof of Lemma 3.5 in [10]). A de Branges space is called short (or regular) if together with every function $F(z)$ it contains $(F(z) - F(a))/(z - a)$ for any $a \in \mathbb{C}$.

We will utilize the following well-known result from the theory of de Branges spaces of entire functions.

Theorem I. [7] *Let μ be a positive measure on $\mathbb{R}$ satisfying $\int d\mu(t)/(1+t^2) < \infty$. Then there exists a short de Branges space B_E contained isometrically in $L^2(\mu)$, with de Branges function $E(z)$ being of Cartwright class and having no real zeros. Moreover, if there exists such a space B_E with $E(z)$ of positive exponential type, then there also exists such a space B_E that is contained properly in $L^2(\mu)$.*

Remark. The existence part follows from Theorem XII of [7]. The second part follows from Theorems IV and X in the same paper. Finally, the shortness can be derived from the proof of Theorem XII [7] or from problem 71 in [5] by taking $S(z) = 1$.

General treatment of de Branges' theory is given in [5].

Every de Branges function gives rise to a meromorphic inner function $\Theta(z) = E^{\#}(z)/E(z)$. We say that an inner function $\Theta(z)$ in $\mathbb{C}_+$ is a meromorphic inner function if it allows a meromorphic extension to the whole complex plane. The meromorphic extension to the lower half-plane $\mathbb{C}_-$ is given by:

$$\Theta(z) = \frac{1}{\Theta^{\#}(z)}.$$

Conversely, by a classical Theorem of Krein, every meromorphic inner function $\Theta(z)$ can be represented in the form $\Theta(z) = E^{\#}(z)/E(z)$, for some de Branges function $E(z)$ (see, for instance, Section 27.2 in [14] or Lemma 2.1 in [12]). Such a function is unique up to a factor of an entire function that is real on $\mathbb{R}$ and has only real zeros.

Each inner function $\Theta(z)$ determines a model subspace

$$K_\Theta = H^2 \ominus \Theta H^2$$

of the Hardy space $H^2(\mathbb{C}_+)$. These subspaces play a prominent role in complex and harmonic analysis, as well as in operator theory, see [18, 19]. There is an important relationship between the model subspaces K_Θ and the de Branges spaces B_E of entire functions. If $E(z)$ is a de Branges function and $\Theta(z) = E^{\#}(z)/E(z)$ is the corresponding meromorphic inner function, then the multiplication operator $f \mapsto Ef$ is an isometric isomorphism $K_\Theta \to B_E$.

Each inner function $\Theta(z)$ determines a positive harmonic function $\Re\frac{1+\Theta(z)}{1-\Theta(z)}$ and by Herglotz representation a positive measure σ such that

$$\Re\frac{1+\Theta(z)}{1-\Theta(z)} = py + \frac{1}{\pi}\int \frac{yd\sigma(t)}{(x-t)^2+y^2}, \qquad z = x+iy, \tag{1.1}$$

for some $p \geq 0$. The number p can be viewed as a point mass at infinity. The measure σ is singular, supported on $\{\Theta = 1\} \subset \mathbb{R}$, and satisfies $\int d\sigma(t)/(1+t^2) < \infty$. It is usually called a Clark measure for $\Theta(z)$. Conversely, for every positive singular measure σ with $\int d\sigma(t)/(1+t^2) < \infty$ and a number $p \geq 0$, there exists an inner function $\Theta(z)$ determined by the formula (1.1). Below, when we say that an inner function $\Theta(z)$ corresponds to σ we always assume $p = 0$.

Every function $f \in K_\Theta$ can be represented by the formula

$$f(z) = \frac{p}{2\pi i}(1-\Theta(z))\int f(t)\overline{(1-\Theta(t))}dt + \frac{1-\Theta(z)}{2\pi i}\int \frac{f(t)}{t-z}d\sigma(t). \tag{1.2}$$

If $1-\Theta(t) \notin L^2(\mathbb{R})$ then $p = 0$ and hence we have a nicer looking formula

$$f(z) = \frac{1-\Theta(z)}{2\pi i}\int \frac{f(t)}{t-z}d\sigma(t).$$

This gives an isometry of $L^2(\sigma)$ onto K_Θ. In the case of meromorphic $\Theta(z)$, every function $f \in K_\Theta$ also has a meromorphic extension in $\mathbb{C}$, and it is given by the formula (1.2). The corresponding Clark measure is discrete with atoms at the points of $\{\Theta = 1\}$ given by $\sigma(\{x\}) = \frac{2\pi}{|\Theta'(x)|}$.

Each meromorphic inner function $\Theta(z)$ can be written as $\Theta(t) = e^{i\phi(t)}$ on $\mathbb{R}$, where $\phi(t)$ is a real analytic and strictly increasing function. The function $\phi(t) = \arg\Theta(t)$ is the continuous argument of $\Theta(z)$. The phase function of $E(z)$ is defined as $-\frac{1}{2}\arg\Theta(t)$, where $\Theta(z)$ is the corresponding meromorphic inner function.

A subset of $\mathbb{R}$ is called discrete if it has no finite density points. For every discrete set $\Lambda \subset \mathbb{R}$, there exists a (far from unique) meromorphic inner function $\Theta(z)$ such that $\{\Theta = 1\} = \Lambda$. In the case of a separated sequence Λ, there is a meromorphic inner function $\Theta(z)$ with $\{\Theta = 1\} = \Lambda$ whose continuous argument $\arg\Theta(t)$ has a bounded derivative (see for instance Lemma 16 in [7]).

Recall that the Toeplitz operator T_U with a symbol $U \in L^\infty(\mathbb{R})$ is the map

$$T_U : H^2 \to H^2, \qquad F \mapsto P_+(UF),$$

where P_+ is the orthogonal projection in $L^2(\mathbb{R})$ onto the Hardy space $H^2 = H^2(\mathbb{C}_+)$.

We will use the following notation for kernels of Toeplitz operators (or *Toeplitz kernels* in H^2):

$$N[U] = \ker T_U.$$

For example, $N[\bar{\Theta}] = K_\Theta$ if Θ is an inner function. Along with H^2-kernels, one defines Toeplitz kernels in the Smirnov class $N^+(\mathbb{C}_+)$,

$$N^+[U] = \{f \in N^+ \cap L^1_{loc}(\mathbb{R}) : \bar{U}\bar{f} \in N^+\}.$$

1.3. **Beurling-Malliavin densities.** Before we formulate our results let us discuss the following notion of density of a discrete sequence and related theorems.

Following [3] we say that a discrete sequence $\Lambda \subset \mathbb{R}$ is *a-regular* if for every $\epsilon > 0$ any sequence of disjoint intervals $\{I_n\}$ that satisfies

$$\left| \frac{\#(\Lambda \cap I_n)}{|I_n|} - a \right| \geq \epsilon$$

for all n, is short.

A slightly different a-regularity can be defined in the following way, that is more convenient in some settings. For a discrete sequence $\Lambda \subset \mathbb{R}$ we denote by $n_\Lambda(x)$ its continuous counting function, i.e. the function that is continuous on $\mathbb{R}$, grows linearly by 1 between each pair of neighboring points of Λ and is equal to 0 at 0. We say that Λ is *strongly a-regular* if

$$\int \frac{|n_\Lambda(x) - ax|}{1 + x^2} < \infty.$$

Conditions like this can be found in many related results, see for instance [5] or [13]. Even though a-regularity is not equivalent to strong a-regularity, in the following definitions of densities changing "a-regular" to "strongly a-regular" will lead to equivalent definitions.

The interior BM (Beurling-Malliavin) density is defined as

$$D_*(\Lambda) := \sup\{a \mid \exists\ a\text{-regular subsequence } \ \Lambda' \subset \Lambda\}. \tag{1.3}$$

Similarly, the exterior BM density is defined as

$$D^*(\Lambda) := \inf\{a \mid \exists\ a\text{-regular supsequence } \ \Lambda' \supset \Lambda\}. \tag{1.4}$$

If no such sequence exists $D^*(\Lambda) := \infty$, see [3]. Exterior density was used in the Beurling-Malliavin solution of the completeness problem for families of exponential functions in L^2 on an interval, see [3], [11] or [13].

The following simple observation will be useful in the next section: $D_*(\Lambda) = 0$ if and only if there exists a long sequence of intervals $\{I_n\}$ such that

$$\#(\Lambda \cap I_n) = o(|I_n|) \quad as \quad |n| \to \infty.$$

A description of $D^*(\Lambda)$ in terms of Toeplitz kernels is given by the following formula, see [16, Section 4.6]:

$$D^*(\Lambda) = \frac{1}{2\pi} \sup\{a : N[\bar{S}^a \Theta] = 0\},$$

where $\Theta(z)$ denotes some/any meromorphic inner function with $\{\Theta = 1\} = \Lambda$. Below (see Theorems B and C) we give a similar description of the interior BM density for separated sequences Λ. Namely,

$$D_*(\Lambda) = \frac{1}{2\pi} \sup\{a : N[\bar{\Theta} S^a] = 0\},$$

where $\Theta(z)$ denotes some/any meromorphic inner function with $\{\Theta = 1\} = \Lambda$.

An equivalent way to define the interior BM density is as follows. Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\gamma(\mp\infty) = \pm\infty$. i.e.

$$\lim_{x \to -\infty} \gamma(x) = +\infty, \qquad \lim_{x \to +\infty} \gamma(x) = -\infty.$$

The family $\mathcal{BM}(\gamma)$ is defined as the collection of the connected components of the open set

$$\left\{ x \in \mathbb{R} : \ \gamma(x) \neq \max_{t \in [x, +\infty)} \gamma(t) \right\}.$$

We say that γ is almost decreasing if $\gamma(\mp\infty) = \pm\infty$ and the family of the intervals $\mathcal{BM}(\gamma)$ is short.

Now we can state an equivalent definition for interior BM density:

$$D_*(\Lambda) := \sup\{a \mid ax - n_\Lambda(x) \text{ is almost decreasing}\}. \tag{1.5}$$

Equivalence of this definition and (1.3) can be easily verified.

We will use the following formulations of the Beurling-Malliavin theorems [2, 3]:

Theorem II ([16, Section 4.2])**.** *Suppose that* $\Theta(z)$ *is a meromorphic inner function with the derivative of* $\arg\Theta(t)$ *bounded on* $\mathbb{R}$*. Then for any meromorphic inner function* $J(z)$*, we have*

$$N^+[\bar{\Theta} J] \neq 0 \qquad \Rightarrow \qquad \forall \epsilon > 0, \qquad N[\bar{S}^\epsilon \bar{\Theta} J] \neq 0.$$

Theorem III ([16, Section 4.3])**.** *Suppose* $\gamma'(t) > -\text{const}$.

(i) If γ *is not almost decreasing, then for every* $\epsilon > 0$, $N^+[S^\epsilon e^{i\gamma}] = 0$.

(ii) If γ *is almost decreasing, then for every* $\epsilon > 0$, $N^+[\bar{S}^\epsilon e^{i\gamma}] \neq 0$.

Remark. As noted in [16, Section 4.3], the part (i) of Theorem III holds without the assumption $\gamma'(t) > -\text{const}$.

2. Results and proofs

2.1. **Main results.** As was mentioned in the introduction, a sequence of real numbers is called separated if $|\lambda_n - \lambda_m| \geq \delta > 0, \ \ (n \neq m)$. It is natural to introduce a separation condition in the Pólya-Levinson problem because of the following obvious reasons. If one takes a zero set of a zero-type entire function and adds a large number of points close enough to each zero, the entire function will still be bounded on the new sequence. At the same time, this way one can obtain non-Pólya sequences of arbitrarily large density, in any reasonable definition of density. Hence, if one hopes to obtain a description of Pólya sequences based on densities or similar terms, it is necessary to include a separation condition, as it was done in the classical results cited above.

Recall that a separated sequence $\{\lambda_n\}_{n=-\infty}^{\infty}$ is called a *Pólya sequence* if every zero-type entire function bounded on $\{\lambda_n\}$ is constant.

Theorem A. *Let $\Lambda = \{\lambda_n\}_{n=-\infty}^{\infty} \subset \mathbb{R}$ be a separated sequence of real numbers. The following are equivalent:*

- *(i)* $\Lambda = \{\lambda_n\}$ *is a Pólya sequence.*
- *(ii) There exists a non-zero measure μ of finite total variation, supported on Λ, such that the Fourier transform of μ vanishes on an interval of positive length.*
- *(iii) The interior Beurling-Malliavin density of Λ, $D_*(\Lambda)$, is positive.*
- *(iv) There exists a meromorphic inner function $\Theta(z)$ with $\{\Theta = 1\} = \{\lambda_n\}$ such that $N[\bar{\Theta} S^{2c}] \neq 0$, for some $c > 0$.*

As an immediate consequence we obtain that the sequence of integers $\mathbb{Z}$ is a Pólya sequence, as known from Valiron's original statement. This follows from (iii) and also from (iv) by taking $\Theta(z) = S^{2\pi}(z)$. Another consequence is that a separated real sequence with density zero cannot be a Pólya sequence. However, there are sequences with positive density (and hence positive exterior Beurling-Malliavin density) which are not Pólya. As was mentioned in the introduction, the first example of such a sequence was given by Levinson [15]. New examples in both directions can now be constructed using the following description.

Corollary. *Let $\Lambda = \{\lambda_n\}_{n=-\infty}^{\infty}$ be a separated sequence of real numbers. Then Λ is a Pólya sequence if and only if for every long sequence of intervals $\{I_n\}$ the sequence $\frac{\#(\Lambda \cap I_n)}{|I_n|}$ is not a null sequence, i.e., $\frac{\#(\Lambda \cap I_n)}{|I_n|} \not\to 0$.*

In regard to the gap problem we will prove the following result. Let M denote the set of all complex measures of finite total variation on $\mathbb{R}$. For $\mu \in M$ its Fourier transform $\hat{\mu}(x)$ is defined as

$$\hat{\mu}(x) = \int e^{ixt} d\mu(t).$$

If X is a closed subset of the real line denote by $G(X)$ the gap characteristic of X:

$$G(X) := \sup\{a \mid \exists\ \mu \in M,\ \mu \not\equiv 0,\ \operatorname{supp} \mu \subset X, \ \text{ such that } \hat{\mu} = 0 \ \text{ on } \ [0, a]\}.$$

Theorem B. *The following are true:*

(i) For any separated sequence $\Lambda \subset \mathbb{R}$, $G(\Lambda) \geq 2\pi D_*(\Lambda)$.

(ii) For any closed set $X \subset \mathbb{R}$, $G(X) \leq 2\pi D_*(X)$.

Corollary. *For separated sequences* $\Lambda \subset \mathbb{R}$, $G(\Lambda) = 2\pi D_*(\Lambda)$.

The formula for $G(X)$ for a general closed set X is more involved, see [20]. Another immediate consequence of Theorem B is the following extension of Beurling's gap theorem:

Corollary. *Let X be a closed subset of the real line. If there exists a long sequence of intervals $\{I_n\}$ such that*

$$\frac{\#(X \cap I_n)}{|I_n|} \to 0$$

then any measure μ of finite total variation supported on X, whose Fourier transform vanishes on an interval of positive length, is trivial.

Finally, our next result connects the gap problem to the problem on injectivity of Toeplitz operators. It provides one of the main tools for our proofs.

If $X \subset \mathbb{R}$ is closed, define

$$T(X) = \sup\{a \mid \exists \text{ meromorphic inner } \Theta(z) \text{ with } \{\Theta = 1\} \subset X \text{ and } N[\bar{\Theta}S^a] \neq 0\}.$$

Theorem C. *For any closed* $X \subset \mathbb{R}$,

$$T(X) = G(X).$$

Theorems A, B and C will be proved in the last section.

2.2. **Technical lemmas.** For the main proofs we will need the following lemmas.

Lemma 1. *Let $\Theta(z)$ be a meromorphic inner function with $1 - \Theta(t) \notin L^2(\mathbb{R})$ and let σ be the corresponding Clark measure. If $N[\bar{\Theta}S^{2a}] \neq 0$ for some $a > 0$, then for any $\epsilon > 0$ there exists $h \in L^2(\sigma)$ such that*

$$\lim_{y \to \pm\infty} e^{xy} \int \frac{h(t)}{t - iy} d\sigma(t) = 0$$

for every $x \in (-a + \epsilon, a - \epsilon)$ and the measure $hd\sigma$ has finite total variation.

Proof: The idea of the proof is truly simple: If the Toeplitz kernel from the statement is non-trivial then K_Θ contains a function divisible by S^{2a}. The desired measure $hd\sigma$ is then obtained from the Clark representation of that function. The details are as follows.

Let

$$b(z) := \frac{z - i}{z + i}.$$

Since $N[\bar{\Theta}S^{2a}] \neq 0$, $N[\bar{\Theta}S^{2a-2\epsilon}b] \neq 0$ for any $\epsilon > 0$. Hence there exists a non-zero

$$f \in N[\bar{\Theta}S^{2a-2\epsilon}b] \subset H^2(\mathbb{C}_+).$$

Then $S^{2a-2\epsilon}bf \in K_\Theta$. Define $h := S^{a-\epsilon}bf/(z-i)$. Clearly h belongs to K_Θ, and therefore

$$h(z) = \frac{1-\Theta(z)}{2\pi i}\int \frac{h(t)}{t-z}d\sigma(t)$$

where σ is the Clark measure of Θ. In particular, for $x < a-\epsilon$,

$$\lim_{y\to\infty} e^{xy}\int \frac{h(t)}{t-iy}d\sigma(t) = 0$$

because $f(iy) \to 0$, since $f \in H^2(\mathbb{C}_+)$, and because the outer function $1-\Theta(iy)$ cannot go to zero exponentially fast.

Denote $g = \bar{\Theta}h \in \bar{H}^2 = H^2(\mathbb{C}_-)$. Then $h = \Theta g$ in the lower half-plane. Note that $g = S^{-a+\epsilon}k$ where

$$k = \bar{\Theta}S^{2a-2\epsilon}bf/(z-i) \in \bar{H}^2 = H^2(\mathbb{C}_-).$$

Hence for $x > -a+\epsilon$,

$$\lim_{y\to-\infty} e^{xy}\int \frac{h(t)}{t-iy}d\sigma(t) = \lim_{y\to-\infty} 2\pi i\frac{k(iy)\Theta(iy)}{e^{(a-\epsilon+x)y}(1-\Theta(iy))} = 0.$$

The last equality follows from the facts that $k(z) \in H^2(\mathbb{C}_-)$ and that

$$\frac{1-\Theta(iy)}{\Theta(iy)} = \overline{\Theta}(-iy) - 1.$$

It is left to notice that $h(z) = l(z)/(z-i)$ where both $l(t) = S^{2a-2\epsilon}bf$ and $(z-i)^{-1}$ belong to $L^2(\sigma)$. Thus $h \in L^1(\sigma)$ and $hd\sigma$ has finite total variation. □

The following Lemma is a well known fact whose proof we include here for completeness.

Lemma 2. *Let μ be a measure with finite total variation. Then the Fourier transform of μ vanishes on $[-a,a]$ if and only if*

$$\lim_{y\to\pm\infty} e^{xy}\int \frac{d\mu(t)}{t-iy} = 0,$$

for every $x \in [-a,a]$.

Proof: Suppose that $\int e^{ixt}d\mu(t) = 0$ for all $x \in [-a,a]$. Then

$$e^{-ixz}\int_{-\infty}^{+\infty}\frac{e^{ixt}-e^{ixz}}{i(t-z)} = \int_{-\infty}^{+\infty}\int_0^x e^{iu(t-z)}dud\mu(t) =$$

$$= \int_0^x\int_{-\infty}^{+\infty} e^{iut}d\mu(t)e^{-iuz}du = 0,$$

for every $x \in [-a,a]$ and $z \in \mathbb{C}$. Therefore,

$$\int \frac{e^{ixt}-e^{ixz}}{t-z}d\mu(t) = 0$$

for every $x \in [-a,a]$. Obviously,

$$\lim_{y\to\pm\infty}\int \frac{e^{ixt}}{t-iy}d\mu(t) = 0 \tag{2.1}$$

and therefore

$$\lim_{y\to\pm\infty} e^{xy}\int \frac{d\mu(t)}{t-iy} = 0$$

for every $x \in [-a, a]$.

Conversely, for $x \in [-a, a]$, define

$$H(z) := \int \frac{e^{ixt} - e^{ixz}}{t - z} d\mu(t).$$

Then $H(z)$ is an entire function of Cartwright class. To show that $H(z)$ is identically zero it suffices to check that $\lim_{y \to \pm\infty} H(iy) = 0$. Recall that for $x \in [-a, a]$,

$$\lim_{y \to \pm\infty} \int \frac{e^{ix(iy)}}{t - iy} d\mu(t) = 0.$$

Together with (2.1) this implies $H \equiv 0$.

Thus

$$\int e^{ixt} d\mu(t) = \lim_{y \to \infty} -iy \int \frac{e^{ixt} d\mu(t)}{t - iy} = \lim_{y \to \infty} -iye^{-xy} \int \frac{d\mu(t)}{t - iy} = 0$$

for all $-a \le x \le a$. □

2.3. **Main proofs.** Now we are ready to prove our main theorems. We will do it in the reverse order.

Proof of Theorem C: The inequality $T(X) \le G(X)$ follows from Lemma 1 and Lemma 2. To prove the opposite inequality, let $G(X) = a$. Then for any $\epsilon > 0$ there exists a non-zero complex measure of total variation no greater than 1 supported on X whose Fourier transform vanishes on $[0, a - \epsilon]$. Consider the set of all such measures. Since this set is closed, convex and contains non-zero elements, by the Krein-Milman theorem it has an extreme point, a non-zero measure ν. Similarly to the proof of Theorem 66 in [5], we can show that the extremality of ν implies that it is supported on a discrete subset of X. Let $\Theta(z)$ be the meromorphic inner function whose Clark measure is $|\nu|$. Then $\{\Theta = 1\} \subset X$. It is left to notice that the function

$$f(z) = \frac{1 - \Theta(z)}{2\pi i} \int \frac{d\nu(t)}{t - z}$$

belongs to K_Θ and is divisible by $S^{a-\epsilon}$ (as follows, for instance, from the proof of Lemma 2). Hence $f/S^{a-\epsilon} \in N[\bar{\Theta} S^{a-\epsilon}] \ne 0$. □

Proof of Theorem B: (i) By Theorem C it is enough to prove that $T(\Lambda) \ge 2\pi D_*(\Lambda)$. Suppose that $D_*(\Lambda) = a/2\pi$. By the second definition (1.5) of D_* the function

$$\phi(x) = -2\pi n_\Lambda(x) + (a - \epsilon)x$$

is almost decreasing for any $\epsilon > 0$. Consider a meromorphic inner function Θ with $\{\Theta = 1\} = \Lambda$ and bounded derivative on $\mathbb{R}$. Then $\arg \bar{\Theta} S^{a-\epsilon}$ differs from ϕ by a bounded function. Hence $\arg \bar{\Theta} S^{a-2\epsilon}$ is almost decreasing. By Theorem II and Theorem III

$$N[\bar{\Theta} S^{a-3\epsilon}] \ne 0.$$

(ii) Again we will prove that $T(X) \le 2\pi D_*(X)$. If $T(X) = a$ then for any $\epsilon > 0$ there exists a meromorphic inner $\Theta(z)$ such that $\Gamma := \{\Theta = 1\} \subset X$ and

$$N[\bar{\Theta} S^{a-\epsilon}] \ne 0.$$

By Theorem II (and remark after it) this means that $\arg \bar{\Theta} S^{a-2\epsilon}$ is almost decreasing. Hence

$$-2\pi n_\Gamma(x) + (a-3\epsilon)x$$

is almost decreasing. Since ϵ is arbitrary, $D_*(X) \geq a/2\pi$. □

Remark. It was pointed out by the referee that a different proof of Theorem B can be obtained combining Beurling and Malliavin Theorem [3, Theorem I], Theorems 66 and 67 in [5] and a Theorem of Krein [14, Theorem 3 in Section 16].

Proof of Theorem A: *(ii)* ⇔ *(iii)* follows from Theorem B and *(ii)* ⇔ *(iv)* from Theorem C.

(i) ⇒ *(iii)* Assume *(iii)* is not true, i.e. for every meromorphic inner function $\Theta(z)$ with $\{\Theta = 1\} = \Lambda$, $N[\bar{\Theta} S^{2c}] = 0$ for every $c > 0$. In this case we will construct a non-constant zero type entire function bounded on Λ, which will mean that Λ is not a Pólya set. Define a measure μ to be the counting measure of Λ. Then clearly $\int d\mu(t)/(1+t^2) < \infty$. By Theorem I there exists a short de Branges space B_E contained isometrically in $L^2(\mu)$. First, let us show that B_E cannot contain a function of positive exponential type. If $E(z)$ has type zero then all functions in B_E have type zero (see the background part). If the type of $E(z)$ is positive, then by Theorem I we can assume that B_E is contained properly in $L^2(\mu)$.

Suppose that $F(z) \in B_E$ has positive type. We can assume that $F(iy)$ grows exponentially in y as $y \to \infty$. Since $B_E \neq L^2(\mu)$, there exists $g \in L^2(\mu)$ with $\bar{g} \perp B_E$. Then

$$0 = \int \frac{F(t)-F(w)}{t-w} g(t) d\mu(t) = \int \frac{F(t)}{t-w} g(t) d\mu(t) - F(w) \int \frac{1}{t-w} g(t) d\mu(t)$$

for any $w \in \mathbb{C}$ and therefore

$$F(w) = \frac{\int \frac{F(t)}{t-w} g(t) d\mu(t)}{\int \frac{1}{t-w} g(t) d\mu(t)}.$$

Since $F(w)$ grows exponentially along $i\mathbb{R}_+$, the integral in the denominator must decay exponentially in w along $i\mathbb{R}_+$. Thus the function

$$G(z) := \frac{1-\Theta(z)}{2\pi i} \int \frac{1}{t-z} g(t) d\mu(t)$$

can be represented as $G(z) = S^c(z) h(z)$ for some nonzero $h(z) \in H^2(\mathbb{C}_+)$ and $c > 0$, and belongs to K_Θ, where $\Theta(z)$ is the inner function corresponding to the measure μ. Hence $h \in N[\bar{\Theta} S^c]$ and we have a contradiction.

Therefore any $F(z) \in B_E$ has zero type. It is left to notice that

$$|F(\lambda_n)| \leq \sqrt{\sum_m |F(\lambda_m)|^2} = \|F\|_{L^2(\mu)} < \infty,$$

which means that $F(z)$ is bounded on Λ.

(ii) ⇒ *(i)* This is Theorem XI in [6]. For reader's convenience, we include de Branges' proof. Let $F(z)$ be a zero type entire function bounded on Λ by some constant $M > 0$. For any integer $n \in \mathbb{N}$, $F^n(z)$ is also a zero type function. Let μ be a nonzero measure with finite total variation whose Fourier transform

vanishes identically on $[-a,a]$ for some $a>0$. Then, by the proof of Lemma 2, $\int(e^{ixt}-e^{ixz})/(t-z)d\mu(t)=0$ for every $x\in(-a,a)$. Define

$$H(z):=\int\frac{F^n(t)-F^n(z)}{t-z}d\mu(t)$$

for all $z\in\mathbb{C}$. It is clear that $H(z)$ is an entire function of zero type. To show that $H(z)\equiv 0$ it is enough to check that $H(iy)\to 0$ as $y\to\pm\infty$. This follows from

$$\lim_{y\to\pm\infty}H(iy)=\lim_{y\to\pm\infty}\left[\int\frac{F^n(t)}{t-iy}d\mu(t)-F^n(iy)e^{xy}\int\frac{e^{ixt}}{t-iy}d\mu(t)\right]=0.$$

Therefore,

$$\int\frac{F^n(t)-F^n(z)}{t-z}d\mu(t)\equiv 0.$$

Now,

$$\left|F(z)\left(\int\frac{d\mu(t)}{t-z}\right)^{1/n}\right|\leq M\left(\frac{\|\mu\|}{|\Im z|}\right)^{1/n},$$

for every non real z. Since this is true for all $n\in\mathbb{N}$, we have that $|F(z)|\leq M$ for all non real $z\in\mathbb{C}$ for which $\int d\mu(t)/(t-z)\neq 0$. Since μ is a non-zero measure, by continuity, $F(z)$ is bounded in the whole plane. □

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