ENTIRE FUNCTIONS AND COMPLETENESS PROBLEMS

A. POLTORATSKI

Lecture 9

Short proofs for classical gap theorems.

The proof of the gap theorem presented in the last lecture is too long and technical to include in this course. To give the reader some taste of the proofs, in this lecture we discuss classical theorems by Krein, Levinson and McKean, Beurling and de Branges on the same subject.

We first formulate an auxiliary statement, theorem 1 below, and give it a short elementary proof. We then show how to deduce the classical results from theorem 1, thus giving simple proofs to those theorems as well. Instead of deducing the classical theorems from each other we prefer to give each a direct closed proof through theorem 1, which itself could be considered an equivalent reformulation of de Branges' theorem 5 below.

In our estimates we write $a(n) \lesssim b(n)$ if $a(n) < Cb(n)$ for some positive constant C, not depending on n, and large enough $|n|$. Similarly, we write $a(n) \asymp b(n)$ if $ca(n) < b(n) < Ca(n)$ for some $C \geq c > 0$. Some formulas will have other parameters in place of n or no parameters at all.

Recall that a sequence of disjoint intervals $\{I_n\}$ on the real line is long (in the sense of Beurling and Malliavin) if

$$\sum_n \frac{|I_n|^2}{1+\operatorname{dist}^2(0, I_n)} = \infty \tag{0.1}$$

where $|I_n|$ stands for the length of I_n. If the sum is finite we call $\{I_n\}$ short.

If I is an interval on $\mathbb{R}$ and $C > 0$ we denote by CI the interval with the same center as I of length $C|I|$.

Theorem 1. *Let μ be a finite measure on $\mathbb{R}$ whose Fourier transform vanishes on an interval. Suppose that there exists a sequence of disjoint intervals $\{I_n\}$ such that*

$$\sum \frac{|I_n|}{1+\operatorname{dist}^2(I_n,0)} \min\left(|I_n|, \log\frac{1}{|\mu|(I_n)}\right) = \infty. \tag{0.2}$$

Then $\mu \equiv 0$.

Roughly speaking, the formula in 0.2 and the conclusion of the theorem say that if μ decays fast along a large sequence of intervals, then it cannot have a spectral gap (unless it is identically zero). In a sense, this is a hybrid of a theorem by Beruling, which says that a measure with a spectral gap may not vanish on a large sequence of intervals, and a theorem by Levinson, which says that such a measure may not decay fast along the whole line (see below).

The proof borrows an idea from the proof of Beurling's gap theorem by Benedicks in [1].

Proof. Without loss of generality $|I_n| > 1$ for all n, because the sum in (0.2) taken over all intervals of length less than 1 is finite. Suppose that $\hat{\mu}$ vanishes on $[-a, a]$. Then, once again, its Cauchy integral $K\mu$ is divisible by e^{iaz} in $\mathbb{C}_+$, in the sense that

$$K\mu = e^{iaz} K\nu,$$

where ν is a finite measure, $\nu = e^{-iaz}\mu$, see for instance lemma 2 in [8] that was already discussed in previous lectures.

Denote by J_n the interval on $\mathbb{R} + i$:

$$J_n = \left\{ z \mid \Im z = 1,\ \Re z \in \frac{1}{2} I_n \right\}.$$

Denote by μ_n the restriction of μ on I_n and put $\eta_n = \mu - \mu_n$. Notice that $K\eta_n(z)$ is holomorphic in $(\mathbb{C} \setminus \mathbb{R}) \cup I_n$. Hence $-\log|K\eta_n(z)|$ is superharmonic in $\{|z-\xi| \leqslant |I_n|/4\}$ for any $\xi \in J_n$. Since

$$-\log|K\mu(z)| = -\log|K\nu(z)| - \log|e^{iaz}| \gtrsim a|I_n|$$

in the half-plane

$$\{\Im z > |I_n|/8\},$$

we obtain

$$-\log|K\eta_n(\xi)| \geqslant -\frac{1}{2\pi}\int_0^{2\pi} \log\left|K\eta_n\left(\xi + \frac{|I_n|}{4} e^{i\phi}\right)\right| d\phi =$$

$$-\frac{1}{2\pi}\int_0^{2\pi}\log\left|K\mu\left(\xi+\frac{|I_n|}{4}e^{i\phi}\right)-K\mu_n\left(\xi+\frac{|I_n|}{4}e^{i\phi}\right)\right|d\phi\gtrsim$$
$$\min\left(a|I_n|,-\log\frac{|\mu|(I_n)}{|I_n|}\right)$$

for any $\xi\in J_n$. On the other hand,

$$|K\mu(\xi)|=|K\eta_n(\xi)+K\mu_n(\xi)|\leqslant|K\eta_n(\xi)|+|\mu|(I_n)|$$

and

$$-\log|K\mu(\xi)|\gtrsim\min(|I_n|,-\log\frac{|\mu|(I_n)}{|I_n|},-\log|\mu_n|(I_n))\gtrsim$$
$$\min(|I_n|,-\log|\mu|(I_n))$$

(recall that $|I_n|>1|$).

Now (0.2) implies that $\log|K\mu|$ is not Poisson-summable on the line $\{\Im z=1\}$. But any Cauchy integral of a non-zero measure must have a Poisson-summable logarithm on any horizontal line in $\mathbb{C}_+$, unless it is identically zero in $\mathbb{C}_+$, see for instance [5]. Similarly, it is zero in $\mathbb{C}_-$. If $K\mu$ is zero in both half-planes, $\mu\equiv 0$.

□

Now assume that the compliment of $\operatorname{supp}\mu$ is long. Then the compliment can be taken as $\{I_n\}$ in (0.2). We obtain

Theorem 2 (Beurling's Gap Theorem [2])**.** *If μ is a finite measure supported on a set with long gaps and the Fourier transform of μ vanishes on an interval, then $\mu\equiv 0$.*

If instead of having porous support μ decays too fast at infinity, one can arrive at the same conclusion:

Theorem 3 (Levinson, [7])**.** *Let μ be a finite measure on $\mathbb{R}$ whose Fourier transform vanishes on an interval. Denote*

$$M(x)=|\mu|((x,\infty)).$$

If $\log M$ is not Poisson-summable on $\mathbb{R}_+$ then $\mu\equiv 0$.

Proof. Suppose that $\log M$ is not Poisson-summable on $\mathbb{R}_+$. Without loss of generality, $M(0)=1$. Let $0=a_0<a_1<a_2<...$ be the points such that $M(a_n)=2^{-n}$ and denote by $I_n=(a_n,a_{n+1}]$ the corresponding partition of $\mathbb{R}_+$. If

$$\sum\frac{n|I_n|}{1+\operatorname{dist}^2(I_n,0)}<\infty$$

then $\log M$ is Poisson-summable and we have a contradiction.

If the last sum is infinite, but but the sum in 0.2 is finite, i.e. the partition I_n is short, then any long sup-partition of I_n will satisfy (0.2). If the last sum is infinite and I_n is long, then (0.2) is satisfied. $\square$

Levinson's result above was later improved by Beurling [2] who showed that an interval can be replaced with a set of positive Lebesgue measure.

Recall the following definition given in previous lectures. If μ is a finite positive measure on $\mathbb{R}$ we define

$$\mathbf{G}_\mu^p = \sup\{\ a \mid \exists\ f \in L^p(\mu), \int f(x)e^{2\pi i\lambda x}d\mu(x) = 0, \forall\ \lambda \in [0,a]\ \}.$$

For $p = 2$, $\mathbf{G}_\mu^p$ is equal to T_μ, the exponential type of μ, the infimum of a, such that the family of exponentials with frequencies from $[0,a]$ is complete in $L^2(\mu)$.

Our next corollary combines results by Krein (part I, $p = 2$) and by Levinson and McKean (part II, $p = 2$).

Theorem 4 (Krein [6], Levinson-McKean [4])**.** *Let μ be a finite measure on $\mathbb{R}$, $\mu = w(x)dx$ where $w(x) \geqslant 0$. Then*

I) If $\log w$ is Poisson-summable then for any $1 \leqslant p \leqslant \infty$, $\mathbf{G}_\mu^p = \infty$.

II) If $\log w$ is monotone and Poisson-unsummable on a half-axis $(-\infty, x)$ or (x,∞) for some $x \in \mathbb{R}$ then for any $1 < p \leqslant \infty$, $\mathbf{G}_\mu^p = 0$.

Proof. If $\log w$ is Poisson-summable, denote by $W(z)$ the outer function in $\mathbb{C}_+$ satisfying $|W| = w$ on $\mathbb{R}$. Then for any $a > 0$ the measure $e^{iax}W(x)dx$ annihilates all exponentials with frequencies from $[0,a)$.

It is left to show that if $\log w$ is Poisson-unsummable and monotone on a half-axis then $\mathbf{G}_\mu^p = 0$ for all $p > 1$. Without loss of generality, the half-axis is $\mathbb{R}_+$. Since for any $f \in L^p(\mu), p > 1$, $\log(fw)$ is unsummable as well, we will simply assume that the Fourier transform of μ itself vanishes on an interval and arrive at a contradiction.

Choose real points $a_0 = 0 < a_1 < ... < a_n < ...$ in the following way. Put $a_0 = 0$. After $a_n, n \geqslant 0$ is chosen, choose a_{n+1} to be the number such that $a_{n+1} - a_n = -\log\mu((a_n, a_{n+1}))$. Note that such a number always exists except in the trivial case when the support of μ is bounded, see exercises.

Notice that if $\{I_n\}$ is long we are done by theorem 1. It is left to show that since $\log w$ is Poisson-unsummable and w is monotone, I_n cannot be short. We leave this part to the reader as an elementary exercise.

$\square$

Several of the statements above are also implied by the following theorem of de Branges:

Theorem 5 (de Branges, theorem 63 [3]). *Let $K(x)$ be a continuous function on $\mathbb{R}$ such that $K(x) \geqslant 1$, $\log K$ is uniformly continuous and Poisson-unsummable. Then there is no nonzero finite measure μ on $\mathbb{R}$ such that*

$$\int_{-\infty}^{\infty} K d|\mu| < \infty \tag{0.3}$$

and $\hat{\mu}$ vanishes on an interval.

Proof. Without loss of generality $K \geqslant 2$ and K is Poisson-unsummable on $\mathbb{R}_+$. Choose points $a_0, a_1, ...$ on $\mathbb{R}_+$ in the following way. Put $a_0 = 0$. After a_{n-1} is chosen, choose a_n to be the smallest point greater than a_{n-1} such that

$$\log K(a_n) \notin \left(\frac{\log K(a_{n-1})}{2}, 2\log K(a_{n-1}) \right).$$

Note that such a_n always exists because K is unbounded on any ray $[x, \infty)$. Denote by L the step function, minorating $\log K$ defined as

$$L(x) = L_n = \min_{I_n} \log K$$

on each $I_n = (a_{n-1}, a_n]$. Notice that by the choice of $\{I_n\}$, $\log L \asymp \log K$. In particular, $\log L$ is Poisson-unsummable. By (0.3), $\mu(I_n) \lesssim 1/L_n$. Also, because of uniform continuity of $\log K$, $\log L_n \lesssim |I_n|$. Hence the sum in (0.2) is minorated by

$$\sum \frac{|I_n| \log L_n}{1 + \operatorname{dist}^2(I_n, 0)} \gtrsim \int \log L(x) \frac{dx}{1+x^2} = \infty.$$

□

Theorem 1 has the following partial inverse.

Proposition 1. *Let $\mu = w(x)dx$ be an absolutely continuous finite measure with $w > 0$ and $\log |w|$ absolutely continuous. Suppose that the sequence of intervals I_n satisfying (0.2) does not exist. Then $G_\mu^\infty = \infty$.*

Proof. Similarly to the last proof, it is not difficult to show that $\log |w|$ is Poisson-summable. After that for any $C > 0$ consider the measure $u\mu$ with

$$u = e^{iCx} F/w,$$

where F is the outer function in the upper half-plane satisfying $|F| = w$. □

Exercises.

1) Without using any of the theorems of the last two lectures, prove that if μ has bounded support (from below or from above) and its Fourier transform vanishes on an interval, then $\mu \equiv 0$. In a sense, Beurling's and Levinson's theorems above generalize this statement.

2) Show that if μ has a spectral gap (a gap in the support of its Fourirer transform) then it annihilates polynomials. Give an example of a measure that annihilates polynomials but does not have a spectral gap.

3) Show that the second statement of theorem 4 needs the restriction that $\log w$ is not monotone.

4) Show that the condition that $\log K$ is uniformly continuous cannot be dropped from the statement of theorem 5.

5) Finish the proof of theorem 4.

References

[1] Benedicks, M. *The support of functions and distributions with a spectral gap,* Math. Scand., 55 (1984), 285–309

[2] Beurling, A. *On quasianalyticity and general distributions,* Mimeographed lecture notes, Summer institute, Stanford University (1961)

[3] De Branges, L. *Hilbert spaces of entire functions.* Prentice-Hall, Englewood Cliffs, NJ, 1968

[4] Dym H, McKean H.P. *Gaussian processes, function theory and the inverse spectral problem* Academic Press, New York, 1976

[5] Koosis, P. *The logarithmic integral, Vol. I & II,* Cambridge Univ. Press, Cambridge, 1988

[6] Krein, M. G. *On an interpolation problem of A. N. Kolmogorov,* Dokl. Akad. Nauk SSSR 46 (1945), 306–309 (Russian).

[7] Levinson, N. *Gap and density theorems,* AMS Colloquium Publications, 26 (1940)

[8] Mitkovski, M., Poltoratski, A. *Polya sequences, Toeplitz kernels and gap theorems,* to appear in Advances of Math., arXiv:0903.4499

[9] Poltoratski, A. *Boundary behavior of pseudocontinuable functions*, Algebra i Analiz 5 (1993), no. 2, 189–210; translation in St. Petersburg Math. J. 5 (1994), no. 4, 389–408