ENTIRE FUNCTIONS AND COMPLETENESS PROBLEMS

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Lecture 8

Out of the three classical completeness problems formulated in the first lecture it remains to discuss the Type Problem. A solution to the Type Problem was recently found in [21] and we plan to present it in our lectures. As it turns out, to approach the Type Problem one needs first to treat another well-known problem of Fourier Analysis, the so-called Gap Problem, which we will consider in this lecture.

First, let us recall the statement of the Type Problem. We consider the family $\mathcal{E}_\Lambda$ of exponential functions $\exp(2\pi i\lambda t)$ on $\mathbb{R}$ whose frequencies λ belong to a certain set $\Lambda \subset \mathbb{C}$:

$$\mathcal{E}_\Lambda = \{\exp(2\pi i\lambda t)|\ \lambda \in \Lambda\}.$$

In particular, we denote by $\mathcal{E}_a = \mathcal{E}_{[0,a]}$ the family of exponential functions whose frequencies belong to the interval from 0 to a. If μ is a finite positive measure on $\mathbb{R}$ we denote by T_μ its exponential type that is defined as

$$T_\mu = \inf\{\ a > 0 \mid \mathcal{E}_a \text{ is complete in } L^2(\mu)\ \} \tag{0.1}$$

if the set of such a is non-empty and as infinity otherwise. The type problem asks to calculate T_μ in terms of μ. Various reformulations of this problem appear in many fields of analysis. We discussed some of such connections in the first lecture. For more information see [3, 21, 1].

General case $p \neq 2$.

The family $\mathcal{E}_a$ is incomplete in $L^2(\mu)$ if and only if there exists a function $f \in L^2(\mu)$ orthogonal to all elements of $\mathcal{E}_a$. Expanding to other $1 \leqslant p \leqslant \infty$ we define

$$\mathbf{G}^p_\mu = \sup\{\ a \mid \exists\ f \in L^p(\mu), \int f(x)e^{2\pi i\lambda x}d\mu(x) = 0, \forall\ \lambda \in [0,a]\ \}. \tag{0.2}$$

We put $\mathbf{G}^p_\mu = 0$ if the set in (0.2) is empty. By duality, for $1 < p \leqslant \infty$, $\mathbf{G}^p_\mu$ can still be defined as the infimum of a such that $\mathcal{E}_a$ is complete in $L^q(\mu)$, $\frac{1}{p} + \frac{1}{q} = 1$. In particular, $\mathbf{G}^2_\mu = T_\mu$. The cases $p \neq 2$ were considered in several papers, see for instance articles by Koosis [9] or Levin [14] for the case $p = \infty$ or [20] for $p = 1$.

Since μ is a finite measure we have

$$\mathbf{G}^p_\mu \leqslant \mathbf{G}^q_\mu \text{ for } p \geqslant q. \tag{0.3}$$

Apart from this obvious observation, the problems of finding $\mathbf{G}^p_\mu$ for different p were generally considered non-equivalent until recently! One of the consequences of the main result of [21] is that, in some sense, there are only two significantly different cases, $p = 1$ (the gap problem) and $1 < p \leqslant \infty$ (the general type problem).

The Gap Problem.

Not only is the case $p = 1$ important and interesting by itself, but, as was mentioned before, it seems to be a necessary step towards a solution for the Type Problem, $p = 2$. Let us start with the following reformulation of the Gap Problem.

Let X be a closed subset of the real line. Denote

$$\mathbf{G}_X = \sup\{\ a \mid \exists\ \mu \neq 0,\ \operatorname{supp}\mu \subset X, \hat{\mu} = 0 \text{ on } [0, a]\ \}.$$

Here and in the rest of the paper $\hat{\mu}$ denotes the (inverse) Fourier transform of a finite measure μ on $\mathbb{R}$:

$$\hat{\mu}(z) = \int_{\mathbb{R}} e^{2\pi i z t} d\mu(t).$$

As was shown in [20], for any finite measure μ on $\mathbb{R}$, $\mathbf{G}^1_\mu$, as defined in the previous section, depends only on its support:

$$\mathbf{G}^1_\mu = \mathbf{G}_X,\ X = \operatorname{supp}\mu.$$

This property separates the gap problem from all the cases $p > 1$. (See exercises.)

For a long time both the gap problem and the type problem were considered by experts to be "transcendental," i.e. not having a closed form solution. Following an approach developed in [16] and [17], a solution to the gap problem was recently suggested in [20], see below.

Classic examples.

As before, we say that a function f on $\mathbb{R}$ is Poisson-summable if it is summable with respect to the Poisson measure Π,

$$d\Pi = dx/(1+x^2).$$

We say that a sequence of real numbers $A = \{a_n\}$ is discrete if it does not have finite accumulation points. We always assume that a discrete sequence is enumerated in the natural increasing order: $a_n \geqslant a_{n-1}$. Since the sequences considered here have $\pm\infty$ as their density points, the indices run over $\mathbb{Z}$. In most of our statements and definitions, the sequences do not have multiple points. We call a discrete sequence $\{a_n\} \subset \mathbb{R}$ separated if $|a_n - a_k| > c$ for some $c > 0$ and any $n \neq k$.

The following statement combines results by Krein (part I in the statement below, case $p = 2$) and by Levinson and McKean (part II, $p = 2$).

Theorem 1 (Krein [10], Levinson-McKean [4])**.** *Let μ be a finite measure on $\mathbb{R}$, $\mu = w(x)dx$, where $w(x) \geqslant 0$. Then*

I) If $\log w$ is Poisson-summable then for any $1 \leqslant p \leqslant \infty$, $\mathbf{G}^p_\mu = \infty$.

II) If $\log w$ is monotone and Poisson-unsummable on a half-axis $(-\infty, x)$ or (x, ∞) for some $x \in \mathbb{R}$ then for any $1 < p \leqslant \infty$, $\mathbf{G}^p_\mu = 0$.

(See Exercises.)

A theorem by Duffin and Schaeffer [5] implies that if μ is a measure such that for any $x \in \mathbb{R}$

$$\mu([x-L, x+L]) > d$$

for some $L, d > 0$ then $\mathbf{G}^2_\mu \geqslant 1/L$.

For discrete measures, in the case $\operatorname{supp}\mu = \mathbb{Z}$, a deep result by Koosis shows an analogue of Krein's result: if $\mu = \sum w(n)\delta_n$, where

$$\sum \frac{\log w(n)}{1+n^2} > -\infty,$$

then $\mathbf{G}^p_\mu = 1$ for all p, $1 \leqslant p \leqslant \infty$ [9]. Not much was known about supports other than $\mathbb{Z}$ besides a recent result from [18], which implies that if

$$\mu = \sum \frac{\delta_{a_n}}{1+a_n^2}$$

for a separated sequence $A = \{a_n\} \subset \mathbb{R}$ then $\mathbf{G}^p_\mu = D_*(A)$, where D_* is the interior Beurling-Malliavin density of A, see lecture 2 for the definition.

In addition to these few examples, classical theorems by Levinson-McKean, Beurling and de Branges show that if a measure has long gaps in its support or decays too fast, then $\mathbf{G}^p_\mu = 0$. We will discuss these theorems in our next lecture. Examples of measures of positive type can be constructed using the results by Benedicks [2]. The most significant recent development, that allows one to modify existing examples, is the result by Borichev and Sodin [1], which says that "exponentially small" changes in weight or support do not change the type of a measure.

The gap problem and d-uniform sequences.

It is not difficult to calculate the gap characteristic of an arithmetic progression $\Lambda = a + dn,\ a \in \mathbb{R},\ d > 0$: $\mathbf{G}_\Lambda = 1/d$, see exercises. It follows that if X contains an arithmetic progression Λ then $\mathbf{G}_X \geq 1/d$. It would be nice if $\mathbf{G}_X$ for a general X could be calculated as a supremum of such numbers $1/d$ taken over all arithmetic progressions contained in X. Unfortunately, this is not the case. However, as it turns out, this simple idea is the right step towards a solution. We just need to replace arithmetic progressions with a slightly larger class of sequences, the d-uniform sequences defined in this section.

Let $\Lambda = \{\lambda_1, ..., \lambda_n\}$ be a finite set of distinct points on $\mathbb{R}$. Define

$$E(\Lambda) = \sum_{\lambda_k, \lambda_j \in \Lambda,\ k \neq j} \log |\lambda_k - \lambda_j|. \tag{0.4}$$

According to the 2D Coulomb law, the quantity $E(\Lambda)$ can be interpreted as potential energy of the system of "flat electrons" placed at Λ, see [20]. That observation motivates the term we use for the condition (0.7) below.

The following example is included to illustrate our next definition.

Key example:

Let $I \subset \mathbb{R}$ be an interval and let $\Lambda = d^{-1}\mathbb{Z} \cap I$ for some $d > 0$. Then

$$\Delta = \#\Lambda = d|I| + O(1)$$

and

$$E = E(\Lambda) = \sum_{1 \leqslant m \leqslant \Delta} \log \left[d^{-\Delta+1}(m-1)!(\Delta - m)! \right] = \Delta^2 \log |I| + O(|I|^2) \tag{0.5}$$

as follows from Stirling's formula. Here the notation $O(\cdot)$ corresponds to the direction $|I| \to \infty$.

Remark 1. *The uniform distribution of points on the interval does not maximize the energy* $E(\Lambda)$ *but comes within* $O(|I|^2)$ *from the maximum, which is negligible for our purposes, see the main definition and its discussion below. It is interesting to observe that the maximal energy for* k *points is achieved when the points are placed at the endpoints of* I *and the zeros of the Jacobi* $(1,1)$*-polynomial of degree* $k-2$*, see for example* [12].

Let

$$... < a_{-2} < a_{-1} < a_0 = 0 < a_1 < a_2 < ...$$

be a discrete sequence of real points. We say that the intervals $I_n = (a_n, a_{n+1}]$ form a short partition of $\mathbb{R}$ if $|I_n| \to \infty$ as $n \to \pm\infty$ and the sequence $\{I_n\}$ is short (in the sense of Beurling and Malliavin, as was defined in lecture 2).

Main Definition:

Let $\Lambda = \{\lambda_n\}$ be a discrete sequence of real points. We say that Λ is d-uniform if there exists a short partition $\{I_n\}$ such that

$$\Delta_n = d|I_n| + o(|I_n|) \quad \text{for all} \quad n \ \ \text{(density condition)} \tag{0.6}$$

as $n \to \pm\infty$ and

$$\sum_n \frac{\Delta_n^2 \log|I_n| - E_n}{1 + \operatorname{dist}^2(0, I_n)} < \infty \ \ \text{(energy condition)} \tag{0.7}$$

where Δ_n and E_n are defined as

$$\Delta_n = \#(\Lambda \cap I_n) \quad \text{and} \quad E_n = E(\Lambda \cap I_n) = \sum_{\lambda_k, \lambda_l \in I_n,\ \lambda_k \neq \lambda_l} \log|\lambda_k - \lambda_l|.$$

Remark 2. *Note that the series in the energy condition is positive: every term in the sum defining* E_n *is at most* $\log|I_n|$ *and there are less than* Δ_n^2 *terms.*

As follows from the example above, the first term in the numerator of (0.7) *is approximately equal to the energy of* Δ_n *electrons spread uniformly over* I_n*. The second term is the energy of electrons placed at* $\Lambda \cap I_n$*. Thus the energy condition is a requirement that the placement of the points of* Λ *is close to uniform, in the sense that the work needed to spread the points of* Λ *uniformly on each interval is summable with respect to the Poisson weight. For a more detailed discussion of this definition see* [20]

In [20], d-uniform sequences were used to solve the gap problem mentioned in the introduction. Recall that with any closed $X \subset \mathbb{R}$ one can associate its (spectral) gap characteristic $\mathbf{G}_X$ defined as the supremum of the size of the spectral gap taken over all finite non-zero measures supported on X. The main result of [20] is the following statement:

Theorem 2. [20] *Let X be a closed set on $\mathbb{R}$. Then*

$$\mathbf{G}_X = \sup\{\ d \mid X \textit{ contains a } d - \textit{uniform sequence}\ \}.$$

Recall that, as was proved in [20], $\mathbf{G}_X = \mathbf{G}^1_\mu$ for any μ such that $\operatorname{supp}\mu = X$.

Remark 3.

- *If Λ is a d-uniform sequence then $D_*(\Lambda) = d$, as follows easily from the density condition* (0.6).
- *Among other things, the energy condition ensures that the points of Λ are not too close to each other. In particular, if Λ is d-uniform for some $d > 0$ and $\Lambda' = \{\lambda_{n_k}\}$ is a subsequence such that for all k,*
$$\lambda_{n_k+1} - \lambda_{n_k} \leqslant e^{-c|\lambda_{n_k}|}$$
for some $c > 0$, then $D_*(\Lambda') = 0$.
- *An exponentially small perturbation of a d-uniform sequence contains a d-uniform subsequence. More precisely, if $c > 0$ and Λ is a d-uniform sequence then any sequence $A = \{\alpha_n\}$ such that $|\lambda_n - \alpha_n| \leqslant e^{-c|\lambda_n|}$ contains a d-uniform subsequence A' consisting of all α_{n_k} such that*
$$\lambda_{n_k+1} - \lambda_{n_k} \geqslant e^{-(c-\varepsilon)|\lambda_{n_k}|}.$$
- *As discussed in* [20]*, the energy condition always holds for separated sequences. If Λ is separated then it is d-uniform if and only if $D_*(\Lambda) = d$.*

Exercises.

1) Show that
$$\mathbf{G}^1_\mu = \mathbf{G}_X,\ \ X = \operatorname{supp}\mu.$$
(This is proposition 1 in [20].)

2) Prove the statements in the last remark.

3) The following statement connects the size of the spectral gap of a measure with the asymptotic behavior of its Cauchy integral. It is

similar to the lemma in the last lecture on the measures that annihilate polynomials:

Lemma 1. *Let μ be a measure with finite total variation. Then the Fourier transform of μ vanishes on $[-a, a]$ if and only if*

$$\lim_{y\to\pm\infty} e^{xy} \int \frac{d\mu(t)}{t - iy} = 0,$$

for every $x \in [-a, a]$.

Try to prove this statement. (This is lemma 2 in [18].)

4) Show that $\mathbf{G}_{\mathbb{Z}} = 1$. (Hint: show that $\csc(\pi z)$ is a Schwarz integral of a Poisson-finite measure. Make an adjustment to obtain a Cauchy integral and use the last exercise.) Obtain the formula for $\mathbf{G}_\Lambda$, where Λ is an arithmetic progression.

5) Prove the classical result by Krein [10] which says that if $d\mu = w(x)dx$ and $\log w$ is Poisson-summable then $\mathbf{G}^p_\mu = \infty$ for all p, $1 \leqslant p \leqslant \infty$. (Hint: consider the outer function $W = e^{\mathcal{S}w}$, where $\mathcal{S}w$ denotes the Schwarz integral defined in lecture 3. Then for any $a > 0$ the measure $e^{2\pi aiz}W dx$ annihilates the family of exponentials $\mathcal{E}_a$.)

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