ENTIRE FUNCTIONS AND COMPLETENESS PROBLEMS

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Lecture 7

In this lecture we return to the classical problems of Harmonic Analysis outlined in the first lecture and discuss Bernstein's problem on weighted polynomial approximation.

Let us recall the statement of the problem.

In this lecture we allow the weight function W to be semi-continuous from below instead of continuous as in most classical papers and in our first lecture. Throughout the rest of the lecture we use the following definition.

We say that a function $W \geqslant 1$ on $\mathbb{R}$ is a *weight* if W is lower semi-continuous and $x^n = o(W)$ as $|x| \to \infty$ for any $n \in \mathbb{N}$.

Our weights are also allowed to take infinite values at finite points on $\mathbb{R}$, which makes it possible to study approximation on subsets of the line within the same general formulation of the problem. For instance, the classical Weierstrass theorem answers the question of density of polynomials in C_W with W equal to 1 on an interval and infinity elsewhere. Another important case of the problem is approximation on discrete sequences (see, for instance, [4]), which corresponds to the weights that are infinite outside of a discrete sequence.

With a semi-continuous and $\hat{\mathbb{R}}$-valued W ($\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$), the quantity $||f||_W$, defined on the set of all continuous f such that $f/W \to 0$ at $\pm\infty$ as

$$||f||_W = \sup_{\mathbb{R}} \frac{|f|}{W} \tag{0.1}$$

(see the first lecture), ceases being a norm and becomes a semi-norm: the set is no-longer complete. Those functions supported on $\{W = \infty\}$ will satisfy $||f||_W = 0$.

The semi-norm defined by (0.1) can be made a norm following a standard procedure. First the space of continuous functions g, such that $g/W \to 0$ at $\pm\infty$, needs to be factorized to obtain a space of equivalence classes: $f \sim g$ if and only if $||f - g||_W = 0$. After that the factor-space needs to be completed. We denote by C_W the resulting space.

Note that if W is continuous and takes only finite values, C_W coincides with the space of continuous functions defined in the first lecture. In the general case, we still have the following property.

If W is a weight we say that a measure μ on $\mathbb{R}$ is W-finite if

$$\int W d|\mu| < \infty.$$

Proposition 1. *The dual space of C_W consists of W-finite measures.*

Proof. Consider a sequence of continuous weights W_n such that

$$W_{n+1}(x) \geqslant W_n(x)$$

and $W_n(x) \to W(x)$ for any $x \in \mathbb{R}$. Note that any bounded linear functional μ on C_W induces a linear bounded functional on C_{W_n} for any n. Because of monotonicity, $C_{W_n} \subset C_{W_{n+1}}$. Since any linear bounded functional on C_{W_n} can be identified with a W_n-finite measure, again using monotonicity of W_n, we conclude that μ can be identified with a W-finite measure on the set $\cup C_{W_n}$. Since the last set is dense in C_W (or, more precisely, the set of equivalence classes, containing the elements from $\cup C_{W_n}$, is dense in C_W), μ can be identified with a W-finite measure on the whole C_W.

□

Note that in the general case of semi-continuous $\hat{\mathbb{R}}$-valued weights, when we say that polynomials are not dense in C_W that statement still means that there exists a continuous g and $\varepsilon > 0$ such that $g/W \to 0$ at $\pm\infty$ and $||g - p||_W > \varepsilon$ for every polynomial. The crucial dual statement, that characterizes non-completeness in the case of continuous weights, still holds for general W: Polynomials are not dense in C_W if and only if there exists a non-zero W-finite measure that annihilates polynomials.

Here is a well-known fact in the theory. The notation $K\mu$ stands for the Cauchy integral of μ in $\mathbb{C}_+$:

$$K\mu(z) = \int \frac{d\mu(x)}{x - z}.$$

Recall that a measure μ has finite moments if $x^n \in L^1(|\mu|)$ for all $n = 0, 1, 2, \ldots$.

Lemma 1. *A measure μ with finite moments annihilates polynomials if and only if*

$$K\mu(iy) = o(y^{-n})$$

for any $n > 0$ as $y \to \infty$.

Proof. Suppose that μ annihilates polynomials. Since $(t^n - z^n)/(t - z)$ is a polynomial of t for every fixed z,

$$0 = \int \frac{t^n - z^n}{t - z} d\mu(t) = [Kt^n\mu](z) - z^n K\mu(z).$$

Since any Cauchy integral of a finite measure tends to zero along $i\mathbb{R}_+$, so does $Kt^n\mu$. Hence $K\mu(z) = o(z^{-n})$ as $z \to \infty,\ z \in i\mathbb{R}_+$.

Conversely, suppose that $K\mu(iy) = o(y^{-n})$ for any $n > 0$ as $y \to \infty$. Without loss of generality, μ is real (otherwise consider $\mu - \bar{\mu}$ or $i(\mu + \bar{\mu})$). Then

$$K\mu(-iy) = \overline{K\mu(iy)} = o(y^{-n})$$

as well. Since μ has finite moments we may consider the function

$$H(z) = \int \frac{t^n - z^n}{t - z} d\mu(t).$$

It is easy to show that H is entire of exponential type zero. Noticing again that

$$H(z) = [Kt^n\mu](z) - z^n K\mu(z),$$

we see that H is bounded on $i\mathbb{R}$. Hence H is a constant by the Phragmen-Lindellöf principle. Since $H(iy)$ tends to zero, H is zero. Therefore

$$z^n K\mu(z) = [Kt^n\mu](z) = \int \frac{t^n}{t - z} d\mu(t).$$

Putting $z = 0$ in the last equation we get that μ annihilates t^{n-1} for any $n > 0$. □

Equivalence between weighted uniform and L^p-approximation

In 1924 when Bernstein published his problem the L^p-spaces did not play the same dominating role in analysis as they do now. In later years many approximation problems in weighted situations were replaced with problems on L^p-approximation. Nonetheless, the original form of Bernstein's problem has survived all the major changes in analysis over the last 90 years and is still used today. One of the reasons for such longevity is that it implies its more modern L^p-reformulations!

Close connections between L^p- and weighted uniform approximation have been known to the experts for a long time. Nevertheless, the formal result that reduces the problem of polynomial approximation in L^p-spaces to Bernstein's problem was found by A. Bakan only recently. This result allows us to concentrate on the latter problem for the rest of the lecture.

Theorem 1. [2] *Let $0 < p < \infty$ be a constant and let μ be a positive finite measure on $\mathbb{R}$ such that $L^p(\mu)$ contains all polynomials. Polynomials are dense in $L^p(\mu)$ if and only if μ can be represented as $\mu = W^{-p}\nu$ for some finite positive measure ν and a weight W such that polynomials are dense in C_W.*

Let us point out that the weights appearing in the theorem are lower semicontinuous. Hence, to study the L^p- and uniform versions as one problem one needs the general definition of C_W discussed in this lecture, as opposed to its more traditional version with a continuous W. Here is a short proof of Bakan's result.

Proof. If polynomials are dense in C_W for some weight W such that $\mu = W^{-p}\nu$ then for any bounded continuous function f there exists a sequence of polynomials $\{s_n\}$ such that s_n/W converges to f/W uniformly. Then

$$\int |f - s_n|^p d\mu = \int \frac{|f - s_n|^p}{W^p} W^p d\mu = \int \left| \frac{f}{W} - \frac{s_n}{W} \right|^p d\nu \to 0.$$

Hence polynomials are dense in $L^p(\mu)$.

Suppose that polynomials are dense in $L^p(\mu)$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a set of bounded continuous functions on $\mathbb{R}$, that is dense in any C_W (see exercises). Let $\{s_{n,k}\}_{n,k\in\mathbb{N}}$ be a family of polynomials such that

$$||f_n - s_{n,k}||_{L^p(\mu)} < 4^{-(n+k)}.$$

Define

$$W = 1 + \sum_{n,k\in\mathbb{N}} 2^{n+k}|f_n - s_{n,k}|.$$

Notice that then $W \in L^p(\mu)$, W is lower semi-continuous and $s_{n,k}/W \to f_n/W$ uniformly as $k \to \infty$. Without loss of generality, $L^p(\mu)$ is not finite dimensional. Then $\{s_{n,k}\}$ contains polynomials of arbitrarily large degrees and $x^n = o(W)$ for any n. Thus W is a weight. Since $\{f_n\}$ is dense in C_W, polynomials are dense in C_W. The measure ν can be chosen as $W^p\mu$.

□

Most of the results on Bernstein's problem belong to one of the two following groups. The first group, containing classical theorems by Akhiezer, Mergelyan and Pollard as well as more recent results by Koosis, provides conditions on W in terms of the norms of point evaluation functionals. The second group uses the approach pioneered by de Branges (see [7] or theorem 66 in [6]) and further developed by Borichev, Sodin and Yuditski. These results are formulated in terms of existence of entire functions belonging to certain classes (see references given in the first lecture).

Both approaches have produced significant progress towards a full solution, although the conditions of density remained rather implicit. Besides specific examples, the only general explicit results in the literature are a classical theorem by Hall [8] and a theorem on log-convex weights published by Carleson [5], see below.

Our goal for the rest of this lecture is to discuss an example of an explicit result on Bernstein's problem. We discuss a theorem recently obtained in [12]. It is closely related to a theorem of de Branges [6] that gives an answer in terms of zero sets of entire functions. Before we state the result we need the following definitions.

Characteristic sequences

Recall that a real sequence is *discrete* if it does not have finite accumulation points. To simplify the definitions we will always assume that a discrete sequence is infinite and does not have multiple points. A discrete sequence is called one-sided if it is bounded from below or from above and two-sided otherwise.

If $\Lambda = \{\lambda_n\}$ is a discrete sequence we will always assume that it is *enumerated in the natural order*, i.e. $\lambda_n < \lambda_{n+1}$, non-negative elements are indexed with non-negative integers and negative elements with negative integers.

For instance, if $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ is a two sided sequence then

$$...\lambda_{-n-1} < \lambda_{-n} < ... < \lambda_{-1} < 0 \leqslant \lambda_0 < \lambda_1 < ...\lambda_n < \lambda_{n+1} < ...$$

Thus a one-sided sequence bounded from below (above) will be enumerated with $n \in \mathbb{Z}, n \geqslant -N$ ($n \in \mathbb{Z}, n < N$), where N is the number of negative (non-negative) elements in the sequence.

As before, we say that a sequence $\Lambda = \{\lambda_n\}$ has upper density d if

$$\limsup_{A\to\infty} \frac{\#[\Lambda \cap (-A, A)]}{2A} = d.$$

If $d = 0$ we say that the sequence has zero density.

A discrete sequence $\Lambda = \{\lambda_n\}$ is called *balanced* if the limit

$$\lim_{N\to\infty} \sum_{|n|<N} \frac{\lambda_n}{1+\lambda_n^2} \tag{0.2}$$

exists.

Observe that any even sequence (any sequence Λ satisfying $-\Lambda = \Lambda$) is balanced. So is any two-sided sequence sufficiently close to even. At the same time, a one-sided sequence has to tend to infinity fast enough to be balanced (the series $\sum \lambda_n^{-1}$ must converge).

Let $\Lambda = \{\lambda_n\}$ be a balanced sequence of finite upper density. For each $n,\ \lambda_n \in \Lambda$, put

$$p_n = \frac{1}{2}\left[\log(1+\lambda_n^2) + \sum_{n\neq k,\ \lambda_k\in\Lambda} \log\frac{1+\lambda_k^2}{(\lambda_k-\lambda_n)^2}\right],$$

where the sum is understood in the sense of principle value, i.e. as

$$\lim_{N\to\infty} \sum_{0<|n-k|<N} \log\frac{1+\lambda_k^2}{(\lambda_k-\lambda_n)^2}.$$

We will call the sequence of such numbers $P = \{p_n\}$ the *characteristic sequence* of Λ.

Note that for a sequence of finite upper density the last limit exists for every n if and only if it exists for some n *if and only if the sequence is balanced.*

The paper [12] contains the following result on Bernstein's problem. Recall that per our agreement all sequences are assumed to be infinite.

Theorem 2. *Polynomials are not dense in C_W if and only if there exists a balanced sequence $\Lambda = \{\lambda_n\}$ of zero density such that Λ and its characteristic sequence $P = \{p_n\}$ satisfy*

$$\sum W(\lambda_n)\exp(p_n) < \infty. \tag{0.3}$$

The proof is not difficult but is still too long to include in this short course. We send interested readers to [12]. In the rest of this lecture let us discuss some implications and relations of the above result.

Examples and corollaries

This section contains further discussion of theorem 2 including its relations with some of the known results.

A classical theorem by Hall [8] says that if

$$\int_{-\infty}^{\infty} \frac{\log W}{1+x^2} dx < \infty$$

for a weight W then polynomials are not dense in C_W. Indeed, if F is an outer function in $\mathbb{C}_+$ satisfying

$$|F| = \frac{1}{(1+x^2)W},$$

then the measure $e^{ix}F(x)dx$ is a W-finite measure that annihilates polynomials by lemma 1.

A direct inverse to this statement is false. Even if one requires that $\log W$ is poisson unsummable and W is monotone on $\mathbb{R}_\pm$, the polynomials may still not be dense in C_W, as follows from an example given in [10].

We say that $f : E \subset \mathbb{R}_+ \to \mathbb{R}$ is log-convex if it is convex as a function of $\log x$, i.e. if the function $g(t) = f(e^t)$ is convex on $S = \log E = \{\log x |\ x \in E\}$. In particular, a twice differentiable function f is log-convex on an interval $(a, b) \subset \mathbb{R}_+$ if $f'(x) + xf''(x) \geqslant 0$ for all $x \in (a, b)$.

The following classical result, published by L. Carleson in [5], but seemingly known earlier to several other mathematicians (see for instance [9]), is a partial inverse to Hall's theorem.

Theorem 3. *Let W be an even weight that is log-convex on $\mathbb{R}_+$. Then polynomials are not dense in C_W if and only if $\log W \in L^1(\Pi)$.*

Proof. If $S = \{s_n\}$ is an even discrete sequence of finite density denote by v_S the function

$$v_S(x) = \frac{1}{2}\sum \log\left|\frac{(s_n - x)^2}{1+s_n^2}\right|,$$

where the sum is understood in terms of normal convergence of partial sums $\sum_{|n|<N}$ in $\mathbb{C}\setminus\Lambda$. Simple computations show that $-v_S$ is log-convex on every interval $(s_n, s_{n+1}),\ \lambda_n \geqslant 0$.

To prove the theorem, notice that in one direction it follows from Hall's result. In the opposite direction, suppose that polynomials are not dense in C_W. Then there exists a sequence Λ like in the statement of theorem 2. It is not difficult to prove that Λ can be chosen to be even.

Fix $n > 0$ and denote $\Gamma_n = \Lambda \setminus \{\lambda_n, \lambda_{-n}, \lambda_{n+1}, \lambda_{-n-1}\}$. Then (0.3) implies

$$\log W(\lambda_k) \leqslant v_{\Gamma_n}(\lambda_k) + \frac{1}{2}\log_- \frac{(\lambda_n - \lambda_{n+1})^2}{1+\lambda_{n+1}^2} + \text{const}, \text{ for } k = n, n+1.$$

Since both W and $-v_{\Gamma_n}$ are log-convex on $(\lambda_n, \lambda_{n+1})$ the inequality can be extended to the whole interval $(\lambda_n, \lambda_{n+1})$ for every n. Since $v_\Lambda \in L^1(\Pi)$, the quantity

$$\sum_n \int_{\lambda_n}^{\lambda_{n+1}} |v_\Lambda - v_{\Gamma_n}| d\Pi$$

is finite and $\log W \geqslant 0$, this implies that $\log W \in L^1(\Pi)$.

□

A direct proof of the log-convex theorem can be found in [10].

Asymptotics of characteristic sequences and applications

Let u be a monotone increasing function on $\mathbb{R}$. Suppose that the harmonic conjugate function $\tilde{u}$ is Poisson-summable, i.e. $\tilde{u} \in L^1(\Pi)$. (Recall that $d\Pi = dx/(1+x^2)$.) Let $\Lambda = \{\lambda_n\}$ be a sequence such that $u(\lambda_n) = n\pi$.

It is not difficult to show that then Λ is a zero density balanced sequence. (This condition is actually equivalent to $\tilde{u} \in L^1(\Pi)$.) Let $P = \{p_n\}$ be the characteristic sequence of Λ.

Elementary estimates yield:

Proposition 2. *Suppose that $u'(x)$ exists and is bounded for large enough $|x|$. Then*

$$p_n = \tilde{u}(\lambda_n) + O(\log|\lambda_n|)$$

as $|n| \to \infty$.

Theorem 2 gives the following

Corollary 1.

I) If W is a regular weight such that $\log W(\lambda_n) \leqslant \tilde{u}(\lambda_n) + O(\log|\lambda_n|)$ then polynomials are not dense in C_W.

II) If $\mu = \sum \alpha_n \delta_{\lambda_n}$ is a finite positive measure such that

$$\sum \alpha_n^{1-q} \exp q p_n < \infty$$

for some $1 < q < \infty$ then polynomials are not dense in $L^p(\mu)$, $\frac{1}{p} + \frac{1}{q} = 1$.

III) If

$$\alpha_n = O(\exp p_n)$$

then polynomials are not dense in $L^1(\mu)$.

For many examples of discrete sequences Λ one can easily find a suitable function u and the values of its conjugate at Λ. If, for instance, $\Lambda = \{n^{1/\alpha}\}_{n\geqslant 0},\ 0 < \alpha < 1/2$ then one may consider u defined as

$$u(x) = \begin{cases} \pi x^\alpha \text{ if } x \in \mathbb{R}_+ \\ 0 \text{ if } x \in \mathbb{R}_- \end{cases}$$

and find that

$$\tilde{u}(n^{1/\alpha}) = -\pi n \tan\left(\alpha\pi - \frac{\pi}{2}\right).$$

In the two-sided case $\Lambda = \{\pm n^{1/\alpha}\}_{n\geqslant 0},\ 0 < \alpha < 1$, one may use u defined as

$$u(x) = \begin{cases} \pi x^\alpha \text{ if } x \in \mathbb{R}_+ \\ -\pi |x|^\alpha \text{ if } x \in \mathbb{R}_- \end{cases}.$$

Then

$$\tilde{u}(\pm n^{1/\alpha}) = -\pi n \tan\left(\alpha\frac{\pi}{2} - \frac{\pi}{2}\right).$$

Such simple calculations and estimates, together with statements from this section, yield majority of the examples of discrete measures, whose L^p spaces are not spanned by polynomials, existing in the literature. See [4] for more examples.

Exercises

1) Let μ be a finite positive measure concentrated on $\mathbb{Z}$, $\mu = \sum_{n\in\mathbb{Z}} e^{-\sqrt{|n|}}\delta_n$. Show that polynomials are not dense in $L^p(\mu)$ for any $0 < p \leq \infty$. How much smaller can we make the point masses for this statement to still hold? (Hint: switch to Bernstein's form and find a way to use Hall's or log-convex theorem. No complete answer to the last question is expected.)

2) In the definition of C_W one requires that all functions from that set satisfied $f/W \to 0$ at $\pm\infty$. Why? Similarly, why can't we drop the condition that W is lower semi-continuous?

3) Produce a countable set of bounded continuous functions on $\mathbb{R}$ that is dense in C_W for any weight W (we needed such a set in the proof of Bakan's theorem).

4) Consider $\Lambda = \{n^3\}_{n>0}$. Let the weight W be defined as n^γ at n^3 and as ∞ outside of Λ. Using the results from this lecture, discuss for what real γ the polynomials will be dense (not dense) in C_W. Using Bakan's theorem, reformulate your statements in terms of L^p approximation.

References

[1] N. I. Akhiezer, *On the weighted approximation of continuous functions by polynomials on the real axis,* Uspekhi Mat. Nauk 11(56), 3-43, AMS Transl. (ser 2), 22 (1962), 95-137.

[2] A. G. Bakan, *Representation of measures with polynomial denseness in* $L^p(\mathbb{R},\mu), 0 < p < \infty$*, and its application to determinate moment problems*, Proc. Amer. Math. Soc., 136 (2008), no. 10, 35793589.

[3] S. N. Bernstein, *Le probleme de lapproximation des fonctions continues sur tout laxe reel et lune de ses applications,* Bull. Math. Soc. France, 52(1924), 399410.

[4] A. Borichev, M. Sodin, *The Hamburger moment problem and weighted polynomial approxi- mation on discrete subsets of the real line,* J. Anal. Math., 76(1998), 219264.

[5] L. Carleson, *Bernsteins approximation problem,* Proc. Amer. Math. Soc., 2(1951), 953961.

[6] De Branges, L. *Hilbert spaces of entire functions.* Prentice-Hall, Englewood Cliffs, NJ, 1968

[7] De Branges, L. *The Bernstein problem*, Proc. Amer. Math. Soc., 10(1959), 825832.

[8] T. Hall, *Sur l'approximation polynômiale des fonctions continues d'une variable réelle,* Neuvième Congrès des Mathématiciens Scandinaves 1938, Helsingfors (1939), 367-369

[9] S. Izumi, T. Kawata, *Quasi-analytic class and closure of tn in the interval* $(-\infty,\infty)$, Tohoku Math. J., 43(1937), 267.273.

[10] Koosis, P. *The logarithmic integral, Vol. I & II.* Cambridge Univ. Press, Cambridge, 1988

[11] D. S. Lubinsky *A Survey of Weighted Polynomial Approximation with Exponential Weights,* Surveys in Approximation Theory, 3, 1105 (2007)

[12] A. Poltoratski *Bernstein's problem on weighted polynomial approximation,* preprint