ENTIRE FUNCTIONS AND COMPLETENESS PROBLEMS

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Lecture 6

In the second part of this lecture we arrive at one of the main points of the course. We show how spectral problems for differential operators connect to problems on completeness of complex exponentials and discuss recently discovered relations between these two important classical areas of mathematics.

Recall that if Φ is a meromorphic inner function, $\Phi = e^{i\phi}$ on $\mathbb{R}$ for a smooth real function ϕ, we say that Λ is a *defining set* for Φ if for any other meromorphic inner function $\tilde{\Phi}$, $\tilde{\Phi} = e^{i\tilde{\phi}}$,

$$\tilde{\phi} = \phi \quad \text{on} \quad \Lambda \qquad \Rightarrow \qquad \Phi \equiv \tilde{\Phi}.$$

As before, $K_\Phi = K^2_\Phi$ is the model space $H^2 \ominus \Phi H^2$ corresponding to Φ. In general, K^p_Φ is defined as the closure of finite linear combinations of reproducing kernels in $L^p(\mathbb{R}),\ 0 < p \leq \infty$.

Relation to uniqueness sets

Proposition. Λ *is not defining for* Φ *if there is a non-constant function* $G \in K^\infty_\Phi$ *such that*

$$G = \bar{G} \quad \text{on} \quad \Lambda. \tag{0.1}$$

Proof. We can assume $\|G\|_\infty < 1$. Let F be a bounded analytic function in $\mathbf{C}_+$ such that $\bar{\Phi} G = \bar{F}$ on $\mathbb{R}$, and consider

$$\tilde{\Phi} = \frac{\Phi + F}{1 + G}.$$

Then $\tilde{\Phi}$ is an inner function because it is in the Smirnov class $\mathcal{N}^+$ and

$$|\Phi + F| = |\Phi + \Phi \bar{G}| = |1 + G| \qquad \text{on} \quad \mathbb{R}.$$

Also, $\tilde\Phi \neq \Phi$ because otherwise we would have $F = G\Phi$, which together with $F = \Phi\bar G$ implies $G = \bar G$, so $G = \text{const}$. Finally, we have

$$\tilde\Phi = \Phi\frac{1+\bar\Phi F}{1+G} = \Phi\frac{1+\bar G}{1+G} = \Phi \qquad \text{on} \quad \Lambda,$$

and since

$$\|\arg\ \tilde\Phi - \arg\ \Phi\|_{L^\infty(\mathbb{R})} < 2\pi$$

by construction, we get $\arg\ \tilde\Phi = \arg\ \Phi$ on Λ. □

Remark. We say that a set is a uniqueness set for a set of functions if any function that is zero on that set is identically zero. The condition (0.1) is very close to the condition that Λ is not a uniqueness set for $K^\infty_{\Phi^2}$. The precise relation between the two statements is an interesting question, which we will not discuss here. We only mention that if $p \in (1,\infty)$, then

$$\exists G \in K^p_\Phi, \quad G \not\equiv \text{const}, \quad G = \bar G \quad \text{on} \quad \Lambda,$$

iff

$$\exists F \in K^p_{\Phi^2}, \qquad F \not\equiv 0, \quad F = 0 \quad \text{on} \quad \Lambda.$$

The above proposition gives a necessary condition for a set Λ to be defining for Φ. To get sufficient conditions one can use the following simple observation (prove it).

Lemma. *If* $\tilde\Phi = \Phi$ *on* Λ *and* $F = \tilde\Phi - \Phi$, *then*

$$F \in K^\infty_{\tilde\Phi\Phi}, \qquad F = 0 \quad \text{on} \quad \Lambda.$$

If we also have $\arg\ \tilde\Phi = \arg\ \Phi$ on Λ (as in the definition of defining sets), then we can estimate the argument of $\tilde\Phi\Phi$ in terms of the data (Φ, Λ), so we can apply our results concerning uniqueness sets.

Defining sets of regular operators

We now consider the defining sets problem in some restricted classes of inner functions. We will use the spectral theory language. For $r \geq 1$ let $\text{Schr}(L^r, D)$ denote the class of selfadjoint Schrödinger operators on $[0,1]$ with an L^r potential and Dirichlet boundary condition at 0.

We say that $\Lambda \subset \mathbb{R}$ is a *defining set for the class* $\text{Schr}(L^r, D)$ if for any two operators in $\text{Schr}(L^r, D)$ with potentials q and $\tilde q$, the equality $\tilde\Theta = \Theta$ on Λ implies $\tilde q \equiv q$, where $\tilde\Theta$ and Θ are the corresponding Weyl inner functions.

We have a similar definition for the classes $\mathrm{Schr}(L^r, N)$ of Schrödinger operators with Neumann boundary condition at 0.

Let Θ_D denote the standard inner function, i.e. the Weyl inner function in the case $q \equiv 0$, see previous lectures. The following statement follows immediately from Lemma 3.9 in [2]:

Λ is defining in the class $\mathrm{Schr}(L^1, D)$ if Λ is a uniqueness set of $K^\infty_{\Theta_D^2}$.

This sufficient condition is not optimal because for regular operators, the function $\tilde{\Phi} - \Phi$ (see the statement of Lemma 3.9) has some extra smoothness at infinity as follows from the standard asymptotic formulae (see the end of this section), which are getting more precise if we require more regularity of the potential, in particular if we consider the case $q \in L^r$ with $r > 1$.

A theorem of Horváth

In a paper published in Annals in 2005 [1] Horváth gives a description of defining sets for Schrödinger operators in terms of completeness problem for exponential functions. The same result was independently found in our joint work with N. Makarov and presented (by N. Makarov) at a conference in Stockholm in 2003, devoted to the 75th birthday of L. Carleson (see also our article [2] in the proceedings of the conference).

Below is a selection of Horváth' results. We use the following notation: $\sqrt{\Lambda} = \{z : z^2 \in \Lambda\}$, and $\sqrt{\Lambda} \cup \{*, *\}$ means $\sqrt{\Lambda}$ plus any two points. Recall that by E_Λ we denote the system of exponentials

$$\{e^{2\pi i\lambda} |\ \lambda \in \Lambda\}.$$

Theorem.

(i) Λ is defining in the class $\mathrm{Schr}(L^r, D)$ iff $E_{\sqrt{\Lambda}\cup\{*,*\}}$ is complete in $L^r(-2, 2)$;

(ii) Λ is defining in $\mathrm{Schr}(L^r, N)$ if $E_{\sqrt{\Lambda}}$ is complete in $L^r(-2, 2)$.

(In the second case, the "only if" part of Horváth' theorem comes with some additional condition.)

Let us explain how to prove the "if" parts of these statements using our methods, see [2]. For example, (ii) in the case $r = 2$ can be equivalently reformulated as

Proposition. Λ *is defining in the class* $\mathrm{Schr}(L^2, D)$ *if* $\sqrt{\Lambda} \cup \{*, *\}$ *is a uniqueness set of* PW_2*.*

Proof. Let $q, \tilde{q} \in L^2(0,1)$. Without loss of generality we will assume that the corresponding Schrödinger operators with boundary conditions (D) at 0 and (N) at 1 are positive. Otherwise, we simply add a large positive constant a to both potentials, and using the transformation

$$F(z) \mapsto F(\sqrt{z^2 + a^2})$$

for even entire functions we observe that $\sqrt{\Lambda}$ is a uniqueness set iff $\sqrt{\Lambda + a}$ is.

It is well-known that if m is a Herglotz function such that

$$0 < m < +\infty \qquad \text{on} \quad \mathbb{R}_-,$$

then $m^*(\lambda) = \lambda m(\lambda^2)$ is again a Herglotz function. If

$$\Theta = (m - i)/(m + i),$$

then the inner function corresponding to m^* is

$$\Theta^*(z) = \frac{(z+1)\Theta(z^2) + (z-1)}{(z-1)\Theta(z^2) + (z+1)}$$

We call Θ^* the square root transform of Θ.

Let Θ^* and $\tilde{\Theta}^*(z)$ be the square root transforms of Θ and $\tilde{\Theta}$, the Weyl functions taken with sign minus, see Section 1.8 in [2]. From the standard asymptotic formula for solutions of a regular Schrödinger equation we obtain

$$\frac{\Theta^*}{S^2} = \frac{\bar{H}}{H} \quad \text{on} \quad \mathbb{R}, \qquad H^{\pm 1} \in H^\infty, \tag{0.2}$$

and

$$x[\Theta^*(x) - \tilde{\Theta}^*(x)] \in L^2(\mathbb{R}). \tag{0.3}$$

(For convenience we reproduce the standard argument at the end of the proof .)

If $\tilde{\Theta} = \Theta$ on Λ, then since $\tilde{\Theta}^*(0) = \Theta^*(0)$, we have

$$\Theta^* = \tilde{\Theta}^* \qquad \text{on} \quad \{0\} \cup \sqrt{\Lambda},$$

where we regard Θ^* and $\tilde{\Theta}^*$ as meromorphic functions in the whole plane. By (0.3),

$$(z-1)(\Theta^* - \tilde{\Theta}^*) \in K_{\Theta^* \tilde{\Theta}^*},$$

so $\sqrt{\Lambda} \cup \{0, 1\}$ is a zero set of some $K_{\Theta^* \tilde{\Theta}^*}$-function, and therefore by (0.2) a zero set of some function in K_{S^4} or PW_2 . (For zeros in $\mathbf{C}_-$ we can use the argument with dual reproducing kernels as in Section 3.1. of [2].) □

Proof of (0.2)–(0.3). If $s>0$, then the solution $u_s(t)$ of the IVP

$$-\ddot{u}+qu=s^2u,\qquad u(0)=0,\quad \dot{u}(0)=1,$$

satisfies the integral equation

$$u_s(x)=\sin sx+\frac{1}{s}\int_0^x \cos s(x-t)\ q(t)\ u_s(t)\ dt.$$

Iterating, we have

$$u_s(1)=\sin s+\frac{F(s)}{s}+\frac{R(s)}{s^2},$$

where

$$F(s)=\int_0^1 \cos s(1-t)\ \sin st\ q(t)\ dt,$$

and

$$R(s)=\int_0^1 \cos s(1-x)\ q(x)\ dx\int_0^x \cos s(x-t)\ q(t)\ u_s(t)\ dt.$$

We have an elementary a priori bound

$$|u_s(t)|\le C,\qquad (s>0,\ t\in[0,1]),$$

so

$$\forall s,\quad |R(s)|\le \text{const}.$$

On the other hand, F is basically the Fourier transform of a function on $(-1,1)$, and

$$q\in L^2\quad\Rightarrow\quad F\in L^2(\mathbb{R}).$$

We also get the corresponding estimates of $\dot{u}_s(1)$. The resulting estimates of Θ imply both statements. □

Exercises

1) Prove the lemma at the end of the first subsection.

2) Formulate an equivalent version of Horváth' theorem for $r=2$ replacing conditions of completeness of exponentials with conditions that the sequence is a uniqueness set in a space of entire functions.

3) Using the Beurling-Malliavin theorem and Horváth' theorem, formulate an if and only if condition for a sequence of real points to be a defining set of a Schrödinger operator on an interval $[0,1-\varepsilon]$ for any

$0 < \varepsilon < 1$, with an L^2 potential and Dirichlet (Neumann) boundary condition at 0.

4) Horváth's paper [1] contains a list of classical theorems by Ambarzumian, Borg, Levinson, Hoschtad-Liberman, as well as more recent results by Gesztesy-Simon and by del Rio-Gesztesy-Simon that follow from the last theorem. It is a good exercise to try to deduce those statements from Horváth' result.

References

[1] Horváth M. *Inverse spectral problems and closed exponential systems,* Annals of Math, Volume 162 (2005), Issue 2.

[2] Makarov, N., Poltoratski, A. *Meromorphic inner functions, Toeplitz kernels, and the uncertainty principle,* in *Perspectives in Analysis*, Springer Verlag, Berlin, 2005, 185–252