ENTIRE FUNCTIONS AND COMPLETENESS PROBLEMS

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Lecture 5

We continue to discuss applications of complex function theory to spectral problems for differential operators. In this lecture we will be able to reach recent results and enter an area of current research.

Abstract Hochstadt-Liberman problem

We will be considering the following problem concerning general meromorphic inner functions. In the next section we will explain its relation to Hochstadt-Liberman's theorem on the spectra of Schrödinger operators [8].

Let Φ and Ψ be meromorphic inner function and $\Theta = \Psi\Phi$. As usual, $\sigma(\Theta)$ denotes the (point) spectrum of Θ, $\{\Theta = 1\}$, see the last lecture. Recall that $\sigma(\Theta)$ may include ∞. We say that *the data* $[\Psi, \sigma(\Theta)]$ *determine* Θ if any inner function divisible by Ψ whose spectrum is $\sigma(\Theta)$ is equal to Θ. I.e. $[\Psi, \sigma(\Theta)]$ determine Θ if for any inner function $\tilde{\Phi}$,

$$\tilde{\Theta} = \Psi\tilde{\Phi}, \quad \sigma(\tilde{\Theta}) = \sigma(\Theta) \qquad \Rightarrow \qquad \Theta = \tilde{\Theta}.$$

Alternatively, we can say that Ψ and $\sigma(\Phi\Psi)$ determine Φ. Given Φ and Ψ, the problem is to decide if this is the case.

The set of Herglotz measures of inner functions $\tilde{\Theta}$ satisfying $\Psi|\tilde{\Theta}$ (Ψ divides Θ) and $\sigma(\tilde{\Theta}) = \sigma(\Theta)$ is convex, see Section 1.2 in [16]. We will refer to the dimension of this set as the dimension of the set of solutions.

Example. Suppose Θ is a *finite Blaschke product*. Then

$$[\Psi, \sigma(\Theta)] \quad \text{determine} \quad \Theta \qquad \Leftrightarrow \qquad 2\deg\Psi > \deg\Theta.$$

The proof is elementary; it also follows from the results below. As an illustration consider the simplest case $\Theta = b^2$, $\Psi = b$, where

$$b(z) = \frac{z-i}{z+i}.$$

Then $\sigma(\Theta) = \{0, \infty\}$, and the data $[\Psi, \sigma(\Theta)]$ does not determine Θ. In fact, the set of solutions is one-dimensional; the solutions are given by the formula

$$\tilde{\Phi}(z) = \frac{z - ia}{z + ia}, \qquad (a > 0).$$

Finding necessary and sufficient conditions for $[\Psi, \sigma(\Theta)]$ to determine Θ is an important problem of complex function theory. As we will see shortly, it appears in spectral theory for differential operators, as well as in other areas of analysis. In [16] several such conditions are formulated in terms of the Toeplitz kernels with symbol $U = \bar{\Phi}\Psi$. The theory of Toeplitz operators is a broad and important part of complex analysis that, due to lack of space, will not be covered in this introductory course. We refer an interested reader to [16, 17] for more information and further references.

The rough meaning of the conditions formulated in [16] is the following: for the data $[\Psi, \sigma(\Theta)]$ to determine Θ, the known factor Ψ of the inner function has to be "bigger" than the unknown factor Φ. Let us formulate one of the results of the 'Toeplitz approach' from [16]. This relatively simple example extends the original Hochstadt-Liberman theorem, which we will state in the next subsection.

Corollary. *Suppose $\Theta = \Psi^2$. Then the set of solutions is exactly one-dimensional: $\tilde{\Theta}$ satisfies $\Psi | \tilde{\Theta}$, $\sigma(\tilde{\Theta}) = \sigma(\Theta)$ iff*

$$\exists r \in (-1, 1), \qquad \tilde{\Theta} = \Psi \frac{r + \Psi}{1 + r\Psi}. \tag{0.1}$$

Spectral theory interpretation: Hochstadt-Liberman and Khodakovski theorems

Consider a Schrödinger operator $L = (q, \alpha, \beta)$ on (a, b), where $q \in L^1_{\text{loc}}(a, b)$ and α, β are selfadjoint boundary conditions at a and b respectively; the endpoints can be infinite and/or singular. We assume that L has compact resolvent. As usual, $\sigma(L)$ denotes the spectrum of L.

Suppose $a < c < b$. We will write q_- for the restriction of q to (a, c) and q_+ for the restriction of q to (c, b). We say that the data $(q_-, \alpha, \sigma(L))$ *determines* L if for any other Schrödinger operator $\tilde{L} = (\tilde{q}, \tilde{\alpha}, \tilde{\beta})$,

$$q_- = q, \ \alpha = \tilde{\alpha}, \ \sigma(\tilde{L}) = \sigma(L) \qquad \Rightarrow \qquad \tilde{q}_+ = q_+, \ \tilde{\beta} = \beta.$$

Let Θ_- denote the Weyl inner function of (q_-, α) computed at c and Θ_+ the Weyl inner function of (q_+, β) computed at c.

Lemma. $\sigma(L) = \sigma(\Theta_-\Theta_+)$.

Proof. The equation $\Theta_-(\lambda)\Theta_+(\lambda) = 1$ is equivalent to the statement

$$m_+(\lambda) + m_-(\lambda) = 0 \quad \text{or} \quad m_-(\lambda) = m_+(\lambda) = \infty$$

for the corresponding m-functions. The latter means that we have the matching

$$\frac{\dot{u}_-(c,\lambda)}{u_-(c,\lambda)} = \frac{\dot{u}_+(c,\lambda)}{u_+(c,\lambda)}$$

for any two non-trivial solutions $u_-(\cdot,\lambda)$ and $u_+(\cdot,\lambda)$ of the Schrödinger equation with boundary conditions α and β respectively, which is possible if and only if λ is an eigenvalue of L. □

Corollary. *$(q_-, \alpha, \sigma(L))$ determine L if the data $(\Theta_-, \sigma(\Theta_-\Theta_+))$ determine Θ_+.*

Here we rely on the fundamental uniqueness theorem of Borg and Marchenko [1], [14]: the m-function (and therefore the Weyl inner function) determines both the potential and the boundary condition.

Remark. We would have an "iff" statement if we considered the problem in some class of canonical systems with a one-to-one correspondence between the systems and inner functions such as the class of Krein's "strings", see [2], [3]. The effective characterization of inner functions of Schrödinger operators is an open problem, so we will use our general results to state only sufficient conditions for Schrödinger operators. To obtain necessary condition one has to use more specific techniques of the Schrödinger operator theory, see [1], [9].

Let us apply the above corollary to the situation described at the end of the last subsection.

Example 1. *Let L be a Schrödinger operator on $\mathbb{R}$ with compact resolvent and limit point boundary conditions at $\pm\infty$. Suppose the potential $q(x)$ is an even function:*

$$q(-x) = q(x), \qquad (x > 0).$$

Then $q|_{\mathbb{R}_-}$ and $\sigma(L)$ determine L.

Proof. By Everitt's theorem [4] (see [16] for a more detailed discussion of that result), all the inner functions $(r+\Psi)/(1+r+\Psi)$ in (0.1) with $r \neq 0$ are not Weyl inner functions corresponding to a Schrödinger operator. $\square$

This result is a special case of Khodakovski's theorem [11], where only the equality $q(-x) \leq q(x)$ for $x > 0$ is assumed. The full version of Khodakovski's theorem requires a slightly different approach which we describe in the next subsection. Similarly, we derive the following statement (if follows from our previous discussion and from the remark at the end of Section 2.5 in [16]).

Proposition. *Let L be as above, and let $\tilde{L}$ be another Schrödinger operator on $(-\infty, b)$, $b \geq 0$. If*

$$q = \tilde{q} \quad \text{on} \quad \mathbb{R}_- \qquad \text{and} \qquad \sigma(\tilde{L}) \subset \sigma(L),$$

then either $\tilde{L} = L$ or $b = 0$ and $\tilde{L}$ is the operator with potential q_- and Dirichlet or Neumann condition at 0.

Example 2. *Let L be a regular selfadjoint Schrödinger operator on $[a, b]$ with non-Dirichlet boundary conditions α and β at a and b respectively. If $c = (a+b)/2$, then $(q_-, \alpha, \sigma(L))$ determine L.*

The statement is also true if one or both boundary conditions are Dirichlet, see next subsection. This is a stronger version of the Hochstadt-Liberman theorem [8], see also [5] which states that if *both* L and $\tilde{L}$ are regular, and $\tilde{q}_- = q_-$, $\tilde{\alpha} = \alpha$, $\sigma(\tilde{L}) = \sigma(L)$, then $\tilde{L} = L$. We *do not* require $\tilde{L}$ to be regular (recall that regular = summable potential). Also, we can replace $\sigma(\tilde{L}) = \sigma(L)$ with $\sigma(\tilde{L}) \subset \sigma(L)$.

Example: Bessel inner functions

This is an extension of the previous example. We want to show that the Hochstadt-Liberman phenomenon occurs not only for regular potentials.

We consider the Bessel inner functions Θ_ν, $\nu \geq -1/2$, see the last lecture.

Applying our methods we get the following result, see [16].

Theorem. *Let L be the Schrödinger operator with potential $q(t) = 2t^{-2}$ on $[0,2]$ and with Dirichlet boundary condition at $t = 2$. Then $q|_{(0,1)}$ and the spectrum $\sigma(L)$ determine L in the class of Schrödinger operators.*

Defining sets of inner functions

Let Φ be a meromorphic inner function, $\Phi = e^{i\phi}$ on $\mathbb{R}$ for a smooth real function ϕ. Let $\Lambda \subset \mathbb{R}$. We say that Λ is a *defining set* for Φ if for any other meromorphic inner function $\tilde{\Phi}$, $\tilde{\Phi} = e^{i\tilde{\phi}}$,

$$\tilde{\phi} = \phi \quad \text{on} \quad \Lambda \qquad \Rightarrow \qquad \Phi \equiv \tilde{\Phi}.$$

In this definition we tacitly assume $\phi(\pm\infty) = \pm\infty$. In the "one-sided" case, say if $\phi(-\infty) > -\infty$ and $\phi(+\infty) = +\infty$, one should modify the definition in an obvious way. (An important and well-known property of arguments of meromorphic inner functions on $\mathbb{R}$ is that they are always monotonnically growing. Why?)

One can extend this definition to divisors. For instance, if all points in $\Lambda \subset \sigma(\Phi)$ are double, then the equality $\tilde{\Phi} = \Phi$ on Λ means that the spectral measures of the inner functions coincide on Λ.

Let us mention several special cases.

(a) *Two spectra problem.*

Let Φ be a meromorphic inner function. Then a meromorphic inner function $\tilde{\Phi}$ satisfies $\{\tilde{\Phi} = 1\} = \{\Phi = 1\}$ and $\{\tilde{\Phi} = -1\} = \{\Phi = -1\}$ iff

$$\tilde{\Phi} = \frac{\Phi - c}{1 - c\Phi}, \qquad c \in (-1, 1). \tag{0.2}$$

This corresponds to the case

$$\Lambda = \{\Phi = 1\} \cup \{\Phi = -1\}.$$

The meaning of the statement is that Λ is defining for Φ with deficiency one (in the case $\phi(\pm\infty) = \pm\infty$, to be accurate). Various related statements are of course well-known, see e.g. [1].

The easiest way to see this is to use Krein's shift construction: since

$$\Re\left[\frac{1}{\pi i}\log\frac{\tilde{\Phi}+1}{\tilde{\Phi}-1}\right] = \chi_e \qquad \text{on} \quad \mathbb{R},$$

where $e = \{\Im\Phi > 0\}$, we have

$$\frac{1}{\pi i}\log\frac{\tilde{\Phi}+1}{\tilde{\Phi}-1} = \mathcal{S}\chi_e + \text{const.} \qquad \square$$

This argument also shows that *given any two intertwining discrete sets* $\Lambda_\pm$ *of real numbers there is a meromorphic inner function* Φ *such that*

$$\{\Phi = \pm 1\} = \Lambda_\pm$$

(see Exercises at the end).

Let us also mention that the statement (0.2) can be derived from the twin inner function theorem, see Section 2.8 in [16].

(b) *General mixed data spectral problem.* The Hochstadt-Liberman problem for inner functions that we discussed above can be viewed as a special case of the defining sets problem. It is easy to see that if (assuming $\arg\Theta(\pm\infty) = \pm\infty$) $\Theta = \Psi\Phi$ and $\Lambda = \sigma(\Theta)$, then

$$(\Psi, \sigma(\Theta)) \quad \text{determine} \quad \Theta \qquad \Leftrightarrow \qquad \Lambda \quad \text{is defining for} \quad \Phi.$$

This can be generalized in the following way. Let $\Theta = \Psi\Phi$ be a given meromorphic inner function and let $\{\lambda_n\}$ be the set of its eigenvalues numbered in the increasing order. Given $M \subset \mathbf{Z}$ we denote

$$\sigma_M(\Theta) = \{\lambda_n : \ n \in M\}.$$

The question is whether the factor Ψ and the partial spectrum $\sigma_M(\Theta)$ determine Θ, i.e. whether

$$\tilde{\Theta} = \Psi\tilde{\Phi}, \quad \tilde{\lambda}_n = \lambda_n \ (n \in M) \qquad \Rightarrow \qquad \tilde{\Theta} \equiv \Theta.$$

Once again, this is equivalent (assuming $\phi(\pm\infty) = \pm\infty$) to saying that $\Lambda = \sigma_M(\Theta)$ is a defining set for Φ. The spectral theory meaning was explained in [16] and the partial spectral problem for Schrödinger operators and Jacobi matrices appeared in several publications, e.g. [5], [6].

(c) *A version for spectral measures.* Given a meromorphic inner function Θ and a factor $\Psi|\Theta$, and also given a part of the spectrum $\Lambda = \sigma_M(\Theta)$, the question is whether there is another inner function $\tilde{\Theta} \neq \Theta$ such that $\Psi|\tilde{\Theta}$ and the spectral measures $\mu = \mu_\Theta$ and $\tilde{\mu} = \mu_{\tilde{\Theta}}$ coincide on Λ:

$$\tilde{\lambda}_n = \lambda_n, \quad \tilde{\mu}\{\lambda_n\} = \tilde{\mu}\{\lambda_n\}, \qquad (n \in M).$$

Claim: If $\Theta = \Psi\Phi$, then Ψ and the spectral measure on $\Lambda = \sigma_M(\Theta)$ determine Θ iff the divisor $2\chi_\Lambda$ is defining for Φ.

Indeed, if $\tilde\Theta = \Psi\tilde\Phi$, and

$$\arg\tilde\Theta(\lambda_n) = \arg\tilde\Theta(\lambda_n) = 2\pi n, \qquad (n \in M),$$

then

$$\arg\tilde\Phi = \arg\Phi \qquad \text{on} \quad \Lambda.$$

The relation

$$\mu\{\lambda\} = \tilde\mu\{\lambda\}, \qquad \lambda \in \Lambda$$

then implies $\tilde\Theta'(\lambda) = \Theta'(\lambda)$, so

$$\Psi'(\lambda)\tilde\Phi(\lambda) + \Psi(\lambda)\tilde\Phi'(\lambda) = \Psi'(\lambda)\Phi(\lambda) + \Psi(\lambda)\Phi'(\lambda), \qquad (\lambda \in \Lambda),$$

and

$$(\arg\tilde\Phi)' = (\arg\Phi)' \qquad \text{on} \quad \Lambda. \qquad \square$$

Again, the spectral theory interpretation is the same as above: we know some part of a differential operator and some part of its spectral measure and we want to know if this information determines the operator uniquely.

As usual we can consider the problem in a restricted class of inner functions. Here is the simplest example.

Example. Let $\Theta = \Psi\Phi$ be a finite Blaschke product. Then Ψ and $\Lambda \subset \sigma(\Theta)$ determine Θ iff $\#\Lambda > 2\deg\ \Phi$ in the class of Blaschke products of a fixed degree. Similarly, Ψ and the spectral measure on Λ determine Θ iff $\#\Lambda > \deg\ \Phi$. This extends in an obvious way to the cases where only Φ or Ψ has a finite degree. These facts follow for instance from the statements in the next section, also cf. [6].

Exercises

Note: these exercises are difficult, but if you can do it, you are about ready to start your own research in this area. You may want to consult [16] or further references given there if you need help. Even if you cannot finish, try to proceed as far as you can in each exercise.

1) Prove that for any two alternating discrete sequences $\Lambda_+,\ \Lambda_-$ on $\mathbb{R}$ there exists a unique meromorphic inner function Φ such that

$$\{\Phi = \pm 1\} = \Lambda_\pm.$$

Consider the case, when one of the sequences contains infinity. What does this statement mean for Schrödinger operators and their spectra?

2) Consider the following simplified case of the H-L theroem. Let L be a regular Schroödinger operator on an interval $[0, 2]$. It is well-known

that then its spectrum $\sigma(L)$ is a discrete sequence on the real line. Suppose that $\tilde{L}$ is another regular Schrödinger operator on $[0, 2]$ such that $\tilde{q} = q$ on $[0, 1]$, the boundary conditions for L and $\tilde{L}$ coinside and $\sigma(\tilde{L})$ contains every other point from $\sigma(L)$, plus at least one more point from $\sigma(L)$. Try to prove that then $L = \tilde{L}$, i.e. $q = \tilde{q}$ on the whole interval. First translate the problem into the language of Weyl inner functions. Then find a way to apply complex analytic methods to show that $\Theta = \tilde{\Theta}$.

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