ENTIRE FUNCTIONS AND COMPLETENESS PROBLEMS

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Lecture 4

In this lecture we continue our discussion of connections with spectral theory for differential operators, started in Lecture 3, in more detail. We will only discuss the case of Schrödinger operators although similar theories exist for general canonical systems. See [4] and [2] for the basics of the spectral theory. Present discussion is a shortened version of selected sections of our paper with N. Makarov [3].

Recall that the H^2-model space of an inner function Θ,

$$K_\Theta = H^2 \ominus \Theta H^2 = H^2 \cap \Theta \bar{H}^2,$$

is a Hilbert space with the Hilbert structure inherited from H^2. A function k_λ from a Hilbert space H of analytic functions in a complex domain is called a reproducing kernel corresponding to the point λ from the domain if for any $f \in H$,

$$< f, k_\lambda >_H = f(\lambda).$$

In the case of K_Θ the reproducing kernels are given by the formula

$$k_\lambda^\Theta(z) = \frac{1}{2\pi i} \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{\bar{\lambda} - z}, \qquad \lambda \in \mathbb{C}_+. \tag{0.1}$$

The system of all reproducing kernels is complete in K_Θ (why?). It follows that if Θ is meromorphic, then all elements of K_Θ are meromorphic, and one can extend (0.1) to all $\lambda \in \mathbb{R}$. The monograph [5] provides a comprehensive study of model spaces.

Weyl inner functions

As was discussed earlier, meromorphic inner functions appear in the theory of 2nd order selfadjoint differential operators. Let q be a real locally summable function on (a, b). We always assume that selfadjoint operators associated with the differential operation $u \mapsto -\ddot{u} + qu$ have *compact resolvent*. This will be the case, for instance, if q is from $L^p(a, b)$, as well as for most other 'reasonable' potentials. We suppose that a is a *regular* point (finite with q summable near a), but we allow b to be infinite and/or singular. Let us fix a selfadjoint boundary condition β at b; for example, β means $u \in L^2$ at b in the case when b is infinite. The Weyl-Titchmarsh m-function of $(q; b, \beta)$ evaluated at a,

$$m(\lambda) = m^a_{b,\beta}(\lambda), \qquad \lambda \in \mathbb{C},$$

is defined by the formula

$$m(\lambda) = \frac{\dot{u}_\lambda(a)}{u_\lambda(a)},$$

where $u_\lambda(\cdot)$ is a non-trivial solution of the Schrödinger equation satisfying the boundary condition at b. It is well-known that m is a Herglotz function, and therefore we can define the corresponding inner function $\Theta^a_{b,\beta}$ as

$$\Theta^a_{b,\beta} = \frac{m - i}{m + i}.$$

We call $\Theta^a_{b,\beta}$ the *Weyl* (or Weyl-Titchmarsh) *inner function* of q.

Similarly, if $b \in \mathbb{R}$ is a *regular* point and α is a selfadjoint boundary condition at $a \in [-\infty, b)$, we can consider the m-function of $(q; a, \alpha)$ evaluated at b,

$$m^b_{a,\alpha}(\lambda) = -\frac{\dot{u}_\lambda(b)}{u_\lambda(b)}$$

and define the corresponding Weyl inner function $\Theta^b_{a,\alpha}$. Note that the sign in the last formula has changed with the change of the endpoint.

Example. The Weyl inner functions of the potential $q \equiv 0$ on $[0, 1]$ with Dirichlet and, respectively, Neumann boundary conditions at $a = 0$ are

$$\Theta_D(\lambda) = \frac{\sqrt{\lambda}\cos\sqrt{\lambda} + i\sin\sqrt{\lambda}}{\sqrt{\lambda}\cos\sqrt{\lambda} - i\sin\sqrt{\lambda}}, \qquad \Theta_N(\lambda) = \frac{\sqrt{\lambda}\sin\sqrt{\lambda} - i\cos\sqrt{\lambda}}{\sqrt{\lambda}\sin\sqrt{\lambda} + i\cos\sqrt{\lambda}}. \tag{0.2}$$

(The m-functions are $m_D(\lambda) = -\sqrt{\lambda}\cot\sqrt{\lambda}$, and and $m_N(\lambda) = \sqrt{\lambda}\tan\sqrt{\lambda}$.)

Example. More generally, for $\nu \geq -1/2$ consider the potential

$$q(t) = \frac{\nu^2 - \frac{1}{4}}{t^2} \quad \text{on} \quad (0, 1),$$

and let the boundary condition α at $a = 0$ be satisfied by the solution

$$u_\lambda(t) = \sqrt{t}J_\nu(t\sqrt{\lambda})$$

of the Schrödinger equation. For example, if $\nu = -1/2$ then $\alpha = (N)$, and if $\nu = 1/2$ then $\alpha = (D)$, and we have the limit point case if $\nu \geq 1$.

J_ν is of course the standard notation for the Bessel function of order ν. Since

$$u_\lambda(1) = J_\nu(\sqrt{\lambda}), \qquad \dot{u}_\lambda(1) = \frac{1}{2}J_\nu(\sqrt{\lambda}) + \sqrt{\lambda}J_\nu'(\sqrt{\lambda}),$$

the corresponding Weyl inner function is

$$\Theta_\nu(\lambda) = \frac{\sqrt{\lambda}J_\nu'(\sqrt{\lambda}) + (1/2+i)J_\nu(\sqrt{\lambda})}{\sqrt{\lambda}J_\nu'(\sqrt{\lambda}) + (1/2-i)J_\nu(\sqrt{\lambda})}. \tag{0.3}$$

In particular, we have $\Theta_{-1/2} = \Theta_N$ and $\Theta_{1/2} = \Theta_D$.

We will discuss the Bessel example further in our lectures.

Modified Fourier transform

Let $\Theta = \Theta^a_{b,\beta}$ be the Weyl-Titchmarsh inner function of a potential q defined in the previous section. We will construct a unitary operator $L^2(a,b) \to K_\Theta$, which is a modification of the Weyl-Titchmarsh Fourier transform. We modify the usual construction so that the case of a singular (i.e. non-regular) endpoint b could be included.

For every $z \in \mathbb{C}$ we choose a non-trivial solution $u_z(t)$ of the Schrödinger equation satisfying the boundary condition β. (For real z such a solution exists because of the compact resolvent assumption). If $z \in \mathbb{C}_+ \cup \mathbb{R}$, then the solution

$$w_z(t) = \frac{u_z(t)}{\dot{u}_z(a) + iu_z(a)}$$

does not depend on the choice of u_z, and $w_z \in L^2(a,b)$. The transform $\mathcal{W}$ is defined as follows:

$$\mathcal{W}: \; f(t) \mapsto F(z) = \int_a^b f(t)w_z(t)dt, \quad (z \in \mathbb{C}_+ \cup \mathbb{R}). \tag{0.4}$$

To state the main result we introduce the *dual reproducing kernel* of the model space K_Θ. For $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$ we define

$$k^*_\lambda(z) = \frac{1}{2\pi i}\frac{\Theta(z) - \Theta(\lambda)}{z - \lambda}, \qquad (z \in \mathbb{C}_+ \cup \mathbb{R}), \tag{0.5}$$

so we have

$$\bar{\Theta}k^\Theta_\lambda = \overline{k^*_\lambda} \qquad \text{on} \quad \mathbb{R},$$

and $k^*_\lambda \in K_\Theta$. Note that if $\lambda \in \mathbb{R}$, then $k^*_\lambda = \text{const}\; k^\Theta_\lambda$.

Theorem 1. [3] *The modified Fourier transform $\mathcal{W}$ is (up to a factor $\sqrt{\pi}$) a unitary operator $L^2(a,b) \to K_\Theta$. Furthermore, we have*

$$\mathcal{W}w_\lambda = \pi k^*_\lambda, \quad \mathcal{W}\bar{w}_\lambda = \pi k_\lambda \qquad (\lambda \in \mathbb{C}_+ \cup \mathbb{R}). \tag{0.6}$$

Proof. The formulae (0.6) follow from the Lagrange identity

$$(z-\lambda)\int_a^b u_\lambda u_z = u_\lambda(a)\dot u_z(a) - \dot u_\lambda(a)u_z(a).$$

(The Wronskian at b is zero because the two solutions satisfy the same boundary conditions.) The rest is straightforward:

$$(\bar w_\lambda, \bar w_\mu)_{L^2} = \int_a^b w_\mu \bar w_\lambda = \mathcal{W}\bar w_\lambda(\mu) = \pi k_\lambda(\mu) = \pi(k_\lambda, k_\mu)_{K_\Theta},$$

etc. □

As was discussed in the last lecture, a meromorphic Herglotz function is a meromorphic function m such that

$$\Im m > 0 \quad \text{in } \mathbb{C}_+, \qquad m(\bar z) = \overline{m(z)}.$$

One can establish a 1-to-1 correspondence between meromorphic inner and Herglotz functions by means of the equations

$$m = i\frac{1+\Theta}{1-\Theta}, \qquad \Theta = \frac{m-i}{m+i}. \tag{0.7}$$

Meromorphic Herglotz functions (and therefore inner functions) can be described by parameters (b, c, μ) in the Herglotz representation

$$m(z) = bz + c + iS\mu, \tag{0.8}$$

where $b \geq 0$, $c \in \mathbb{R}$, and μ is a positive discrete measure on $\mathbb{R}$ satisfying

$$\int \frac{d\mu(t)}{1+t^2} < \infty.$$

It is convenient to interprete the number πb as a point mass of μ at infinity. In the case $m = m_\Theta$, see (0.7), we call this extended measure μ_Θ the *spectral* (or *Herglotz) measure* of Θ. By definition, the (point) *spectrum* of Θ is the set

$$\sigma(\Theta) = \operatorname{supp}\, \mu_\Theta = \ \{\Theta = 1\} \text{ or } \{\Theta = 1\} \cup \{\infty\},$$

and by residue calculus we have

$$\mu_\Theta(t) = \frac{2\pi}{|\Theta'(t)|}, \qquad t \in \sigma(\Theta). \tag{0.9}$$

The following *equivalent* conditions are necessary and sufficient for $\mu_\Theta(\infty) \neq 0$, see e.g. [6]:

(i) $\Theta - 1 \in H^2$; (ii) $\Theta(\infty) = 1,\ \exists\Theta'(\infty)$; (iii) $\sum \Im\lambda < \infty$.

In (ii), $\Theta(\infty)$ and $\Theta'(\infty)$ mean the angular limit and angular derivative at infinity:

$$\Theta(\infty) = \lim_{y\to+\infty} \Theta(iy), \qquad \Theta'(\infty) = \lim_{y\to+\infty} y^2\Theta'(iy),$$

and in (iii) we also require that the singular factor is trivial.

Note that Weyl inner functions of Schrödinger operators have no point masses at infinity, so if $\Theta = \Theta^a_{b,\beta}$, then

$$\sigma(\Theta) = \sigma(q, D, \beta), \qquad \sigma(-\Theta) = \sigma(q, N, \beta).$$

Here $\sigma(q, D, \beta)$ means the spectrum of the Schrödinger operator with potential q, Dirichlet boundary condition at a, and boundary condition β at b. More generally, for $\alpha \in \mathbb{R}$ let α denote the following selfadjoint boundary condition at a regular endpoint a:

$$\cos\frac{\alpha}{2}u(a) + \sin\frac{\alpha}{2}\dot{u}(a) = 0. \tag{0.10}$$

Then

$$\sigma(e^{-i\alpha}\Theta) = \sigma(q, \alpha, \beta).$$

The spectral measure of the Schrödinger operator (q, α, β) is the Herglotz measure of the inner function $e^{-i\alpha}\Theta$ (this can be viewed as a definition of the spectral measure of a Schrödinger operator).

(Note: for those readers familiar with Clark theory, see for instance [1], the Herglotz measure of $e^{-i\alpha}\Theta$ is the Clark measure σ_α corresponding to Θ.)

Exercises

1) Let H be a Hilbert space of analytic functions in a complex domain Ω such that the linear functional of point evaluation $f \mapsto f(\lambda)$ is bounded with respect to the norm of H for any $\lambda \in \Omega$. Prove that H has a full system of reproducing kernels k_λ, i.e. that k_λ exists for any $\lambda \in \Omega$.

2) If H is as above, show that reproducing kernels are complete in H.

3) Using the formula for the reproducing kernel of a K_Θ space and the connection between K_Θ and PW_a discussed in the previous lecture, find the formula for a reproducing kernel for PW_a. Find the same formula directly by calculating the Fourier transform of a restriction of $e^{i\lambda z}$ on $(-a, a)$.

4) Calculate the Weyl inner functions for the free Laplacian ($q = 0$), given in the example above, by hand.

5) Let Θ be a fixed inner function. Prove the following simple properties of Herglotz measures σ_α for $e^{-i\alpha}\Theta$ $|\alpha| = 1$, (a.k.a. Clark measures).

a) All σ_α are singular (with respect to the Lebesgue measure on the line).

b) All σ_α are mutually singular, i.e. $\sigma_\alpha \perp \sigma_\beta$ for $\alpha \neq \beta$.

c) σ_α is supported on a set where the non-tangential limits of Θ are equal to α (note that the set is not generally closed, so it is not the closed support of σ_α).

Think about the corresponding statements for spectral measures for differential operators, that follow from a)-c) and the above discussion. For these and further properties of σ_α see for instance [1].

References

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