ENTIRE FUNCTIONS AND COMPLETENESS PROBLEMS

A. POLTORATSKI

Lecture 3

A version of the Heisenberg Uncertainty Principle formulated in terms of Harmonic Analysis claims that a non-zero measure (distribution) and its Fourier transform cannot be simultaneously small, see for instance [3]. This broad statement raises a multitude of deep mathematical questions, each corresponding to a particular sense of "smallness." It includes problems on completeness of exponentials and polynomials that we discussed earlier, inverse spectral problems for differential operators and Krein's canonical systems, classical problems in the theory of stationary Gaussian Processes, signal processing, etc. Many of such problems remain open to this day.

For instance, the Beurling-Malliavin Problem discussed in the last lecture fits into the general statement of the Uncertainty Principle in the following way. We consider functions with small support, i.e. square summable functions whose support is contained in a finite interval. We want to show that the Fourier transform of such a function cannot be small in the sense that it cannot have a large zero set (a sequence of external density larger than the length of the interval). In the opposite direction, if $\hat{f}$ vanishes on a sequence of large density, then the support of f cannot be contained in a small interval.

Our next goal is to discuss another area within the Uncertainty Principle that deals with Spectral Problems for differential operators and completeness problems for special functions. To do that we need some preparation in basic complex analysis. The topics we discuss here are covered in a number of textbooks such as [6, 10, 2, 1].

Herglotz functions

We say that a function f on $\mathbb{R}$ is Poisson-summable, and write $f \in L^1_\Pi$ if it is summable with respect to the Poisson measure Π,

$$d\Pi = dx/(1 + x^2).$$

We say that a complex measure μ on $\mathbb{R}$ is Poisson-finite if

$$\int \frac{d|\mu(x)|}{1+x^2}.$$

We denote the set of all Poisson-finite measures on $\mathbb{R}$ by $M_\Pi(\mathbb{R})$. We will also consider the set of measures $M_\Pi(\hat{\mathbb{R}})$ on $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Each measure from $M_\Pi(\hat{\mathbb{R}})$ has the form $\mu = \nu + c\delta_\infty$, where $\nu \in M_\Pi(\mathbb{R})$, $c \in \mathbb{C}$ and δ_∞ is the unit point mass at infinity.

For any $\mu \in M_\Pi(\mathbb{R})$ one can consider its Poisson integral in the upper half-plane $\mathbb{C}_+$,

$$P\mu(x+iy) = \frac{1}{\pi}\int \frac{y}{(x-t)^2+y^2} d\mu(t),$$

that defines a harmonic function in $\mathbb{C}_+$. If the measure μ is positive, then $P\mu$ defines a positive harmonic function in $\mathbb{C}_+$ (and a negative harmonic function in $\mathbb{C}_-$).

For $\mu \in M_\Pi(\hat{\mathbb{R}})$ the Poisson formula takes the form

$$P\mu(x+iy) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2} d\mu(t) + cy,$$

where the term cy is understood as the Poisson integral of the point mass at infinity.

The well-known representation theorem says that any positive harmonic function in $\mathbb{C}_+$ is a Poisson integral of a positive measure in $M_\Pi(\hat{\mathbb{R}})$. (Exercise: find a proof of that statement. Hint: for a positive harmonic function in the unit disk, show that L^1-norms of its restrictions on circles centered at the origin are bounded and take a weak limit of a subsequence as $r \to 1$.)

Moving on from harmonic to analytic functions, for any $\mu \in M_\Pi(\mathbb{R})$ the Schwarz integral

$$\mathrm{S}\mu(z) = \frac{1}{\pi i}\int \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu(t)$$

defines an analytic function in $\mathbb{C}_+$ (and in $\mathbb{C}_-$). If μ is positive, $S\mu$ has positive real part in $\mathbb{C}_+$. Analytic functions with positive real parts are called Herglotz functions (and the integral $S\mu$ is often called Herglotz integral).

For $\mu \in M_\Pi(\hat{\mathbb{R}})$,

$$S\mu(z) = \frac{1}{\pi i}\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu(t) - icz,$$

where, once again, the last term is understood as the Schwarz integral of the point mass at infinity. In the analytic case the representation statement is called the Herglotz Representation Theorem, which says that any Herglotz function F in $\mathbb{C}_+$ can be represented as

$$F = \mathrm{S}\mu + ib,$$

where $\mu > 0,\ \mu \in M_\Pi(\hat{\mathbb{R}}), b \in \mathbb{R}$.

Inner functions

By a theorem of Fatou, any bounded analytic function in $\mathbb{C}_+$ has non-tangential boundary values almost everywhere on $\mathbb{R}$, see for instance [6]. If those limits have absolute value 1 a. e. on $\mathbb{R}$, then such a function is called inner. I. e., inner functions in $\mathbb{C}_+$ are bounded analytic functions that are equal to 1 by the absolute value a. e. on the boundary.

An important example of an inner function in $\mathbb{C}_+$ is the exponential function

$$S^a(z) = e^{iaz},\ a > 0.$$

(Question: why should a be positive?) Another example is a Blaschke factor

$$b_\lambda = \frac{z-\lambda}{z-\bar{\lambda}}, \quad \lambda \in \mathbb{C}_+.$$

Obviously, any finite product of Blaschke factors will also define an inner function. Finally, if $\Lambda = \{\lambda_n\} \subset \mathbb{C}_+$ is a sequence satisfying the Blaschke condition

$$\sum \frac{\Im\lambda_n}{1+|\lambda_n|^2} < \infty,$$

Then the infinite product

$$B_\Lambda(z) = \prod \varepsilon_n b_{\lambda_n},$$

where the constants ε_n are chosen so that $\varepsilon_n b_{\lambda_n}(i) > 0$, converges normally in $\mathbb{C}_+$ and defines an inner function B_Λ called a Blaschke product. A theorem by Beurling says that any inner function I in $\mathbb{C}_+$ has the form

$$I(z) = C\exp(-S\mu)B_\Lambda,$$

where B_Λ is a Blaschke product, $\mu \in M_\Pi(\hat{\mathbb{R}})$ is a *positive singular* measure and C is a unimodular constant, see for instance [1, 6, 10, 2].

Meromorphic inner functions

An important subclass of inner functions in $\mathbb{C}_+$ consists of the so-called meromorphic inner functions. These are inner functions that can be extended into the whole complex plane meromorphically. Such inner functions play an important role in problems of the Uncertainty Principle.

The condition of existence of meromorphic extension into $\mathbb{C}_-$ immediately implies the following representation formula for a meromorphic inner function θ:

$$\theta(z) = Ce^{iaz}B_\Lambda,$$

where C is a unimodular constant, $a > 0$ and B_Λ is a Blaschke product corresponding to a *discrete* sequence Λ (i.e. Λ satisfies the Blaschke condition and has no finite accumulation points).

Spaces of analytic functions

Recall that the Hardy space $H^2 = H^2(\mathbb{C}_+)$ in the upper half-plane is defined as the space of all analytic functions f in the upper half-plane such that

$$||f||^2_{H^2} = \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2 dx < \infty.$$

By Fatou theorem, each function in H^2 is uniquely determined by its non-tangential boundary values on $\mathbb{R}$. If one identifies each function in H^2 with its boundary values on $\mathbb{R}$, the space becomes a closed subspace of $L^2(\mathbb{R})$ and a Hilbert space. Via this connection, H^2 inherits the inner product from $L^2(\mathbb{R})$:

$$< f, g >_{H^2} = < f, g >_{L^2} = \int_{\mathbb{R}} f(x)\bar{g}(x)dx.$$

It turns out that the norm for H^2 defined above coincides with the L^2-norm, see for instance [6].

An important role in the problems of Uncertainty Principle is played by the following collection of subspaces of H^2. If θ is an inner function in $\mathbb{C}_+$, denote by θH^2 the set of functions

$$\theta H^2 = \{\theta f |\ f \in H^2\}.$$

It is clear that θH^2 is a closed subspace of H^2 (that consists of all functions in H^2 that are divisible by θ). Hence, we can consider an orthogonal complement of θH^2 in H^2:

$$K_\theta = H^2 \ominus \theta H^2.$$

This is the definition of the so-called model space K_θ corresponding to the inner function θ.

As we can see from the definition, such a space K_θ can be constructed for any inner function θ in $\mathbb{C}_+$. Such spaces play fundamental role in the Functional Model Theory as the only invariant subspaces of the backward shift operator in H^2, see [9, 8].

The spaces K_θ may be viewed as generalizations of the classical Payley-Wiener spaces PW_a, see exercises below. They possess many intriguing properties and are still under investigation by analytic function theorists. Problems on sampling, interpolation or uniqueness in K_θ spaces serve as natural modern extensions of classical completeness problems discussed so far in this course. We hope to illustrate this with further examples in our future lectures.

Spectral theory

Consider the Schrödinger equation

$$-\ddot{u} + qu = \lambda u \tag{0.1}$$

on some interval (a, b) and assume that the potential $q(t)$ is locally integrable and a is a regular point, i.e. a is finite and q is L^1 at a. Let us fix some selfadjoint boundary condition at b and consider the Weyl m-function

$$m(\lambda) = \frac{\dot{u}_\lambda(a)}{u_\lambda(a)}, \qquad \lambda \notin \mathbb{R},$$

where $u_\lambda(t)$ is any non-trivial solution of (0.1) satisfying the boundary condition. We will deal only with the *compact resolvent* case, which is equivalent to saying that m extends to a meromorphic function. It is well known that m defines a Herglotz function in $\mathbb{C}_+$. Thus we can define the meromorphic inner function

$$\Theta = \frac{m - i}{m + i},$$

(see exercises) which we call the *Weyl inner function* associated with the potential and the fixed boundary condition at b. The transformation

$$f(t) \mapsto F(\lambda) = \int_a^b f(t) \frac{u_\lambda(t)}{\dot{u}_\lambda(a) + i u_\lambda(a)}\, dt \qquad (0.2)$$

identifies $L^2(a,b)$ with the model space K_Θ in the same way as the classical Fourier transform (times S^a) identifies $L^2(-a,a)$ with PW_a. I. e., it is a unitary operator between the two spaces. This allows us to interprete the completeness problem for families of solutions $\{u_\lambda : \lambda \in \Lambda\}$ as a problem of uniqueness sets in the model space of Θ.

For some special choices of the potential $q(t)$ the families of solutions u_{λ_n} can become families of special functions, such as Bessel, Jacobi or Airy functions, see for instance [7]. Completeness problems of this type, particularly problems involving families of special functions, are well-known in the literature, see e.g. [4]. More on this in future lectures.

Exercises and notes

1) Prove the Herglotz Representation Theorem. Show that the theorem fails for non-positive harmonic functions.

2) Prove the Beurling Theorem on the representation formula for inner functions. You may use the fact that for any bounded analytic functions its zeros satisfy the Blaschke condition.

3) Show that any meromorphic inner function has the simplified representation formula given above.

4) Verify that the formula used to define the Weyl inner function,

$$\Theta = \frac{m-i}{m+i},$$

establishes a one-to-one correspondence between inner and Herglotz functions.

5) An alternative definition of the Hardy space is that H^2 is equal to the image of $L^2(\mathbb{R}_+)$ under the Fourier transform. The norm and the inner product in H^2 are then defined to make the Fourier transform a unitary operator $\mathcal{F} : L^2(\mathbb{R}_+) \to H^2$. Can you verify the equivalence of this definition to our definition?

6) Show that $\mathcal{F}(L^2([0,a]))$ is a K_θ space with $\theta = S^{2\pi a}$. This shows that the Payley-Wiener spaces are particular cases of K_θ:

$$S^{a\pi} PW_a = K_{S^{2\pi a}}.$$

The classical completeness problems, that are equivalent to uniqueness problems in PW_a, as was discussed in the previous lecture, now become particular cases of uniqueness problems in K_θ. For general θ, most of such problems remain unsolved.

References

[1] J. A. Cima, A. L. Matheson, W. T. Ross, *The Cauchy transform*, Mathematical Surveys and Monographs, vol. 125, American Mathematical Society, Providence, RI, 2006. MR 2215991 (2006m:30003)

[2] Garnett, J. *Bounded analytic functions.* Academic Press, New York, 1981

[3] Havin, V., Jöricke, B. *The uncertainty principle in harmonic analysis.* Springer-Verlag, Berlin, 1994.

[4] Higgins, J.R. *Completeness and basis properties of sets of special functions.* Cambridge Univ. Press, Cambridge, 1977

[5] Koosis, P. *The logarithmic integral, Vol. I & II,* Cambridge Univ. Press, Cambridge, 1988

[6] Koosis, P. *Introduction to H^p spaces.* Cambridge Univ. Press, Cambridge, 1980

[7] Makarov, N., Poltoratski, A. *Meromorphic inner functions, Toeplitz kernels, and the uncertainty principle,* in *Perspectives in Analysis*, Springer Verlag, Berlin, 2005, 185–252

[8] N. K. Nikolski, *Operators, functions, and systems: an easy reading. Vol. 2. Model operators and systems.* Mathematical Surveys and Monographs **93**. Amer. Math. Soc., Providence, RI, 2002. xiv+439 pp. ISBN: 0-8218-2876-2. MR 1892647 (2003i:47001b)

[9] N. K. Nikolski, *Treatsie on the shift operator*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **273**. Springer-Verlag, Berlin, 1986. xii+491 pp. ISBN: 3-540-15021-8. MR 0827223 (87i:47042)

[10] W. Rudin, *Real and Complex Analysis*, McGraw-Hill Publishing, 1970.