ENTIRE FUNCTIONS AND COMPLETENESS PROBLEMS

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Lecture 2

Let us recall one of the classical completeness problems discussed in the first lecture. Let $\Lambda = \{\lambda_n\}$ be a sequence of distinct points in the complex plane and let

$$E_\Lambda = \{e^{i2\pi\lambda_n x}\}$$

be a sequence of complex exponential functions on $\mathbb{R}$ with frequencies from Λ. We ask under what conditions on Λ will E_Λ be complete in $L^2([0,a])$.

In this general form the problem does not have a reasonable answer. To formulate the results, including the famous Beruling-Malliavin theorem mentioned in the first lecture, we will have to refine the formulation of the problem. Recall that for any complex sequence Λ its radius of completeness is defined as

$$R(\Lambda) = \sup\{a \mid E_\Lambda \text{ is complete in } L^2(0,a)\}.$$

A more realistic goal is to find a formula for $R(\Lambda)$ for an arbitrary Λ. That goal was accomplished by Beruling and Malliavin and we will state their result in this lecture.

It is well-known in the theory of completeness that the general problem can be easily reduced to the case of real sequences Λ. More precisely, if Λ is a general complex sequence then E_Λ is complete in $L^2([0,a])$ if and only if $E_{\Lambda'}$ is complete in $L^2([0,a])$, where Λ' is the real sequence defined as $\lambda'_n = 1/\Re\frac{1}{\lambda_n}$, see for instance [1]. Also, as will be explained below, one can always assume that Λ is a discrete sequence, i.e. has no finite accumulation points.

One of the main tools in the theory of completeness of complex exponentials is the Fourier transform. Recall that if $f \in L^2(\mathbb{R})$ then its Fourier transform, $\hat{f}$ is defined as

$$\hat{f}(z) = \int_{\mathbb{R}} e^{-2\pi i x} f(x)dx.$$

The Payley-Wiener theorem says that if $\operatorname{supp} f \subset [-a,a]$ then $\hat{f}(z)$ is an entire function of exponential type at most $2\pi a$, i.e.

$$|\hat{f}(z)| \leq \text{const} e^{2\pi a|z|},$$

and $\hat{f}(x) \in L^2(\mathbb{R})$. By Parseval's theorem

$$||f||_{L^2([-a,a])} = ||\hat{f}||_{L^2(\mathbb{R})}.$$

Moreover, every entire function of exponential type at most $2\pi a$ that belongs to $L^2(\mathbb{R})$ is the Fourier transform of a function from $L^2([-a,a])$. The image of $L^2([-a,a])$ under the Fourier transform is the so-called Payley-Wiener space of entire functions, PW_a. More directly, PW_a is a space of entire functions of exponential type at most $2\pi a$ that belong to $L^2(\mathbb{R})$.

A system of vectors in a Hilbert space is incomplete if and only if there exists a non-zero vector orthogonal to all of the vectors of the system. In particular, a system of complex exponentials E_Λ is incomplete in $L^2([0,a])$ if and only if there exists a non-zero $f \in L^2([0,a])$ such that $f \perp e^{i2\pi\lambda_n x}$ for all $\lambda_n \in \Lambda$. By the definition of the Fourier transform, the last condition is equivalent to the condition that $\hat{f}(\lambda_n) = 0$ for all $\lambda_n \in \Lambda$.

Thus, via the Fourier transform and the Payley-Wiener theorem, the completeness problem we are discussing becomes a problem of complex analysis. Namely, E_Λ is incomplete in $L^2([0,2a)])$ if and only if Λ is a zero set of a non-zero function from PW_a.

Let us point out one immediate consequence of this connection: If Λ has a finite accumulation point then $R(\Lambda) = \infty$. Indeed, if there exists a finite $a > 0$ such that E_Λ is incomplete in $L^2([0,a])$ then there exits a non-zero $f \in L^2([0,a]), f \perp E_\Lambda$. Then the entire function $\hat{f}$ is non-zero and vanishes on Λ. But non-zero entire functions cannot vanish on sets with finite accumulation points.

A study of zero sets of PW_a-functions can give many more results on completeness of complex exponentials. This idea was first used by Payley and Wiener themselves and later perfected by Levinson in his classical book [2] of 1940, that remains one of the best books in the area. One of the first fundamental results on completeness obtained this way is the following theorem mentioned in the last lecture.

Theorem 1 (Paley and Wiener, 1934)**.**

$$R(\Lambda) \geqslant \bar{D}(\Lambda) = \limsup_{x\to\infty} \frac{n\#(\Lambda \cap (0,x))}{x}.$$

The idea of the proof is simple. One needs to show that if $F \in PW_a$ then the sequence of its real positive zeros N cannot satisfy $\bar{D}(N) > a$. This can be done using standard methods of complex analysis, such as Jensen's inequality. (To complete the details of the proof is a good exercise.)

This theorem started a long and intensive hunt for the formula for $R(\Lambda)$ in terms of densities. The upper density $\bar{D}(\Lambda)$, or its various derivations, proved to be insufficient for that goal, as follows from the following historic examples.

Example 1 (Kahane, 1959)**.** *There exits* $\Lambda \subset \mathbb{R}$ *such that* $\bar{D}(\Lambda) = 0$ *but* $R(\Lambda) = \infty$*.*

In other words, even a very 'thin' sequence of frequencies, in terms of upper density, can generate a sequence of exponentials that will be complete in L^2 on any finite interval.

In Kahane's example the sequence had large clusters (multiplicities) of points. An immediate questions that followed naturally from his construction was if such clustering is necessary for such a sequence. In particular it was still unclear if a separated sequence can produce a similar example. A sequence $\Lambda = \{\lambda_n\}$ is separated if $|\lambda_n - \lambda_k| > c > 0$ for some c and all $n \neq k$. The new question was answered much quicker with the following example by Koosis, ruining the remaining hopes for the use of the upper density.

Example 2 (Koosis, 1960)**.** *There exists* $\Lambda \subset \mathbb{Z}$ *such that* $\bar{D}(\Lambda) = 0$ *but* $R(\Lambda) = 1$*.*

Notice, that since $\Lambda \subset \mathbb{Z}$, $R(\Lambda) \leq R(\mathbb{Z}) = 1$. Hence, Koosis' example gives a subsequence, much thinner than $\mathbb{Z}$, that has maximal possible radius of completeness.

Nevertheless, it turned out that the equation in the Payley-Wiener result will hold if one replaces the upper density with a more delicate density found in the early sixties by Beurling and Malliavin. We now pass to the definition of the Beurling-Malliavin (effective) density of a real discrete sequence and a dual density that will be used later in the course.

If $\{I_n\}$ is a sequence of disjoint intervals on $\mathbb{R}$, we call it short if

$$\sum \frac{|I_n|^2}{1 + \operatorname{dist}^2(0, I_n)} < \infty$$

and long otherwise.

Let us point out some simple examples and properties of long (short) sequences of intervals. If $|I_n| < C$ then $\{I_n\}$ is short:

$$\sum \asymp \sum \frac{1}{n^2}.$$

Also, $I_n = (n^a, (n+1)^a)$ is short for any $a > 0$:

$$\sum \asymp \sum \frac{1}{n^2}.$$

At the same time, a subsequence of diadic intervals, $I_n = (2^{n_k}, 2^{n_{k+1}})$, is long for any $n_k \nearrow \infty$, no mater how rare:

$$\sum \asymp 1 + 1 + 1 + ...$$

If Λ is a sequence of real points define its exterior BM density (effective BM density) as

$$D^*(\Lambda) = \sup\{d \mid \exists \text{ long } \{I_n\} \text{ such that } \#(\Lambda \cap I_n) \geqslant d|I_n|\}, \ \forall n\}$$

Let us point out the following obvious properties of the new density:

$$D^*(\Lambda) \geq \bar{D}(\Lambda)$$

and

$$D^*(\mathbb{Z}) = D^*(\mathbb{N}) = 1, \quad D^*(C\mathbb{Z}) = D^*(C\mathbb{N}) = C^{-1}.$$

A dual definition is used to introduce the interior BM density:

$$D_*(\Lambda) = \inf\{d \mid \exists \text{ long } \{I_n\} \text{ such that } \#(\Lambda \cap I_n) \leqslant d|I_n|\}, \ \forall n\}.$$

We postpone the discussion of the interior density until future lectures.

An alternative definition of the two densities, that proves to be more convenient in some of the applications, can be given as follows.

For a discrete sequence $\Lambda \subset \mathbb{R}$ we denote by $n_\Lambda(x)$ its counting function, i.e. the step function on $\mathbb{R}$, that is constant between any two points of Λ, jumps up by 1 at each point of Λ and is equal to 0 at 0. We say that Λ is *a-regular* if

$$\int \frac{|n_\Lambda(x) - ax|}{1 + x^2} < \infty.$$

On other words, a sequence is a-regular if it is close to the arithmetical progression $a^{-1}\mathbb{Z}$, in the sense that the difference between their counting functions is summable with respect to the Poisson weight $dx/(1+x^2)$.

The exterior density, introduced above, can be equivalently defined as

$$D^*(\Lambda) := \inf\{a \mid \exists\ a\text{-regular supsequence } \ \Lambda' \supset \Lambda\}.$$

Similarly, for the interior density we have

$$D_*(\Lambda) := \sup\{a \mid \exists\ a\text{-regular subsequence } \ \Lambda' \subset \Lambda\}.$$

This definition shows why the terms 'exterior' and 'interior' were used by Beurling and Malliavin in the names of their densities.

Now we are ready to formulate one of the deepest theorems of the 20th century Harmonic Analysis.

Theorem 2 (Beurling and Malliavin, around 1961)**.** *Let Λ be a discrete real sequence. Then*

$$R(\Lambda) = D^*(\Lambda).$$

(The family $\mathcal{E}_\Lambda = \{e^{i\lambda z},\ \lambda \in \Lambda\}$ is complete in L^2 on any interval of length less than $2\pi D^*(\Lambda)$ and incomplete on any interval of length more than $2\pi D^*(\Lambda)$.)

As was mentioned at the beginning of the lecture, the formula extends to the general case $\Lambda \subset \mathbb{C}$ as follows. If Λ satisfies the *Blaschke* condition

$$\text{(B)},\ \sum_{\lambda\in\Lambda} \left|\Im\ \lambda^{-1}\right| < \infty,$$

then $R(\Lambda) = \pi D_{\text{eff}}(\Lambda^*)$, where

$$\Lambda^* = \left\{\lambda^* \middle|\ \lambda^* := \left[\Re\ \lambda^{-1}\right]^{-1},\ \lambda \in \Lambda\right\},$$

and $R(\Lambda) = \infty$ if $\Lambda \notin$ (B).

Exercises.

1) Prove equivalence of the two definitions of BM densities.

2) If $D^*(\Lambda_1) = a$ and $D^*(\Lambda_2) = b$, what can be said about $D^*(\Lambda_1 \cap \Lambda_2)$, $D^*(\Lambda_1 \cup \Lambda_2)$, $D^*(\Lambda_1 \setminus \Lambda_2)$? The same question about D_*.

3) Using the Beurling-Malliavin theorem, produce real sequences that satisfy the conditions of Kahane's and Koosis' examples.

4) If $0 \le a \le \infty$, $\varepsilon > 0$ and Λ is a sequence such that $R(\Lambda) = a$, show that the sequence of complex exponentials E_Λ contains infinitely many

disjoint subsequences such that each of the subsequences is complete in $L^2([0, a-\varepsilon])$.

References

[1] Koosis, P. *The logarithmic integral, Vol. I & II,* Cambridge Univ. Press, Cambridge, 1988

[2] Levinson, N. *Gap and density theorems,* AMS Colloquium Publications, 26 (1940)