ENTIRE FUNCTIONS AND COMPLETENESS PROBLEMS

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Lecture 1

A set of vectors in a Banach or Hilbert space is called complete if finite linear combinations of its vectors are dense in the corresponding space with respect to the standard topology generated by the norm. For example, any basis, or any set containing a basis, is a complete set. One of the first advanced examples of a complete set we see in the standard analysis course is the set of monomials $1, x, x^2, x^3, ...$ in the space $C([a, b])$ of all continuous functions on a closed interval with the standard supremum norm. The Weierstrauss theorem tells us that the set is complete in the space.

Completeness problems appear in many areas of analysis and its applications. For instance, a function on a subset of the real line may represent a wave and one may ask if it can be approximated, in a specified sense, by linear combinations of specially selected functions, often called harmonics. In modern terms, defining the criterion of approximation amounts to defining the norm in the space. Verifying wether any function in the space can be approximated by finite linear combinations of harmonics is equivalent to proving that harmonics are complete in the space. Problems of this kind gave name to a large and important part of mathematics, Harmonic Analysis.

The role of harmonics in the above set-up can be played by a number of different sets of functions, such as trigonometric functions, monomials, complex exponentials, or specials functions such as Bessel functions, Jacobi or Chebyshev polynomials or Airy functions originating from Physics. The most common choices for the Banach space are L^p spaces and spaces of continuous or smooth functions with various norms.

Let us start by formulating some of the classical completeness problems that will be discussed in this course.

Bernstein's problem

Let $W : \mathbb{R} \to [1, \infty)$ be a continuous function satisfying $x^n = o(W(x))$ for any $n \in \mathbb{N}$, as $x \to \pm\infty$. Denote by C_W the space of all continuous functions f on $\mathbb{R}$ such that $f/W \to 0$ as $x \to \pm\infty$ with the norm

$$||f||_W = \sup_{\mathbb{R}} \frac{|f|}{W}. \tag{0.1}$$

The famous weighted approximation problem posted by Sergei Bernstein in 1924 [2] asks to describe the weights W such that polynomials are dense in C_W.

Bernstein's problem can be viewed as a natural consequence of the Weierstrauss theorem. If instead of $C([a, b])$ one tries to consider $C(\mathbb{R})$ with the same supremum norm, one immediately runs into an obstacle: polynomials do not belong to such a space. Bernstein's weighted sup norm turns out to be the best way to remedy that situation. A slightly more general version of Bernstein's problem allows the weight W to be semicontinuous and take infinite values. This extension allows one to study polynomial approximation on arbitrary closed subsets of $\mathbb{R}$.

Throughout the 20th century Bernstein's problem was investigated by many prominent analysts including N. Akhiezer, L. de Branges, L. Carleson, T. Hall, P. Koosis, B. Levin, P. Malliavin, S. Mandelbrojt, S. Mergelyan, H. Pollard and M. Riesz. This activity continues to our day with more recent significant contributions by A. Bakan, M. Benedicks, A. Borichev, P. Koosis, M. Sodin and P. Yuditski, among others. Besides the natural beauty of the original question, such an extensive interest towards Bernstein's problem is generated by numerous links with adjacent fields, including its close relation with the moment problem.

Further information and references on the remarkable history of Bernstein's problem can be found in two classical surveys by Akhiezer [1] and Mergelyan [13], a recent one by Lubinsky [12], or in the first volume of Koosis' book [8].

A simple if and only if solution in Bernstein's problem most likely does not exist. We plan to discuss some of the classical conditions along with recent progress in these lectures.

The Beurling-Malliavin problem

Let Λ be a subset of the complex plane. Denote by E_Λ the set of complex exponential functions with 'frequencies' from Λ:

$$\mathcal{E}_\Lambda = \{\exp(2\pi i\lambda t)|\ \lambda \in \Lambda\}.$$

The most standard example of such a set of functions is

$$E_{\mathbb{Z}} = \{e^{2\pi i n}\}_{n\in\mathbb{Z}},$$

which forms an orthonormal basis in $L^2([0,1])$. A natural extension of this important example is the following question.

Consider the case when $\Lambda = \{\lambda_n\}$ is a general sequence of complex numbers. Under what conditions on the frequencies Λ will the system of exponentials $\mathcal{E}_\Lambda$ be complete in $L^2(0,a)$?

This natural question occupied analysts for many decades. The history if its solution can be started from a theorem by Paley and Wiener (1935) that says that if a real sequence Λ has upper density greater than a, i.e.

$$\limsup_{x\to\infty} \frac{\#(\Lambda \cup (0,x))}{x} > a,$$

then E_Λ is complete in $L^2(0,a)$.

An immediate question is wether the statement of the theorem can be reversed. Examples by Levinson, Kahane and Koosis showed that no converse statement can be formulated using the primitive upper density, utilized by Payley and Wiener.

A solution to that problem was obtained by Beurling and Malliavin in a series of papers in the early 1960's. Instead of the upper density, they defined the so-called effective density of a sequence that allows one to replace an inequality in the Payley-Wiener result with an equation (after some additional reductions, that will be discussed in the future lectures.)

To obtain the formula for the 'completeness radius' of a sequence of exponentials Beurling and Malliavin proved three intermediate results that are now known as the first BM theorem, the little multiplier theorem and the big multiplier theorem. Each of these results has independent value and usage. Together these three theorems and the final result, the second BM theorem, form the so-called Beurling-Malliavin theory, that is considered to be one of the deepest parts of the 20's century Harmonic Analysis. Modern treatments of the BM theory can be found in [7, 8] along with further references.

The Type Problem

Consider a family $\mathcal{E}_a = \mathcal{E}_{[0,a]}$ of exponential functions whose frequencies belong to the interval from 0 to a. If μ is a finite positive measure on $\mathbb{R}$ we denote by T_μ its exponential type that is defined as

$$T_\mu = \inf\{\ a > 0 \mid \mathcal{E}_a \text{ is complete in } L^2(\mu)\ \} \tag{0.2}$$

if the set of such a is non-empty and as infinity otherwise. The type problem asks to calculate T_μ in terms of μ.

This question first appears in the work of Wiener, Kolmogorov and Krein in the context of stationary Gaussian processes that play an important role in Probability Theory (see [9, 10] or the book by Dym and McKean [5]). If μ is a spectral measure of a stationary Gaussian process, completeness of $\mathcal{E}_a$ in $L^2(\mu)$ is equivalent to the property that the process at any time is determined by the data for the time period from 0 to a. Hence the type of the measure is the minimal length of the period of observation necessary to predict the rest of the process. Since any even measure is a spectral measure of a stationary Gaussian process, and vice versa, this reformulation is practically equivalent. Important connections with spectral theory of second order differential operators were studied by Gelfand and Levitan [6] and Krein [10, 11].

For more on the history and connections of the type problem see, for instance, a note by Dym [4] or a recent paper by Borichev and Sodin [3].

Questions and Exercises

Let $\Lambda = \{\lambda_n\}$ be a real sequence. Define its radius of completeness as

$$R(\Lambda) = \sup\{a \mid \mathcal{E}_\Lambda \text{ is complete in } L^2(0,a)\}.$$

Main example:

$$R(\mathbb{Z}) = 1$$

(since $\mathcal{E}_\mathbb{Z}$ is an orthonormal basis of $L^2(0,1)$). Now let us consider one half of $\mathbb{Z}$, the set $2\mathbb{Z}$ of all even integers. Show that $R(2\mathbb{Z})$ is $1/2$. More generally, if we take $1/n$-th of $\mathbb{Z}$, $n\mathbb{Z}$, then the radius decreases correspondingly: $R(n\mathbb{Z}) = \frac{1}{n}$.

Question: Let us take one-half of $\mathbb{Z}$ in a different way and see what happens. Find $R(\mathbb{N})$, the radius of completeness of the set of positive integers (use the Payley-Wiener theorem; the answer may surprise you).

Question: May $R(\Lambda)$ change if one deletes finitely many points from Λ? Give an example of an infinite sequence $A \subset \mathbb{Z}$ such that $R(\mathbb{Z}\setminus A) = R(\mathbb{Z}) = 1$.

Question: Can A from the previous example have positive upper density?

Question: Does a complete set have to contain a basis? A basis in a Banach space is a set B of vectors such that every vector in the space can be uniquely represented as a series of constant multiples of vectors from B, converging in the norm. (Hint: consider, for instance, the set of monomials in $C([0,1])$. Try to describe the set of functions representable as uniformly convergent power series.)

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