Lecture 8: Examples of Multi-Parameter Carleson Measures

Now that we know the story for Carleson measures in the multi-parameter setting is more complicated, our goal should be to attempt to gain some understanding of examples of Carleson measures. Again for simplicity we restrict to the case of two parameters, and will focus on the case of the bidisc $\mathbb{D}^2$ (or equivalently, the bi-upper half plane) since it contains all the main ideas necessary to understand the subtleties of the examples associated to Carleson measures in two parameters.

First, recall that given an open set $\Omega \subset \mathbb{T}^2$, we form the tent over Ω to be the union of the products of the rectangles $R = I \times J$ with $I, J \subset \mathbb{T}$ and I on the boundary of the first disc, J on the boundary of the second disc, and $R \subset \Omega$. Namely,

$$S(\Omega) = \bigcup_{R \subset \Omega} S(I) \times S(J).$$

Here $S(I)$ is the (one-parameter) tent over the interval I.

We now will focus on a special example of Carleson measures:

Theorem 0.1 (Chang [1], Fefferman [2]). *Let $f \in L^\infty(\mathbb{T}^2)$, and let $v(z,w)$ denote the multiple Poisson extension of f to the bidisc $\mathbb{D}^2$. Then for every open set $\Omega \subset \mathbb{T}^2$ we have*

$$\iint_{S(\Omega)} \left|\nabla_z \nabla_w v(z,w)\right|^2 \log\frac{1}{|z|} \log\frac{1}{|w|} dA(z) dA(w) \lesssim |\Omega|.$$

The theorem above is the core ingredient in understanding the duality of H^1 and BMO on the bidisc. The first proof of this fact was given by Chang in [1], with an alternate proof given by Fefferman in [2]. We will follow the approach by Fefferman in this lecture.

The corresponding statement on the bi-upper half plane is the following.

Theorem 0.2 (Chang [1], Fefferman [2]). *Let $f \in L^\infty(\mathbb{R}^2)$, and let $v(z_1, z_2)$ denote the multiple Poisson extension of f to the bidisc $\mathbb{R}^2_+ \times \mathbb{R}^2_+$. Then for every open set $\Omega \subset \mathbb{R}^2$ we have*

$$\iint_{S(\Omega)} \left|\nabla_1 \nabla_2 v(z_1, z_2)\right|^2 y_1 y_2 dA(z_1) dA(z_2) \lesssim |\Omega|.$$

Here $z_j = x_j + iy_j$.

We will focus on proving this Theorem now. As a point of strategy, in the proof below, we will seeking to prove estimates of the form

$$X \leq CY$$

where C is some absolute constant. Here X will be things like the left hand side of the estimate 0.2, and Y is the right hand side. Note that we can prove a weaker estimate,

$$X \leq AY + BX^{1/2} Y^{1/2}$$

and then will be able to deduce the corresponding estimates $X \leq C'Y$ for some worse constant C'. In some cases below we will be able to deduce the good estimate we are after, while in other cases, we should aim for the second worse estimate.

1. Proof of Theorem 0.2

For the rest of the proof going forward, we make the following assumptions and notational conventions. First, we suppose that $f \in L^\infty(\mathbb{R}^2)$ with $\|f\|_\infty = 10^{-10}$. As before v will denote the multiple Poisson extension of the function f, and u will denote the multiple Poisson extension of the function χ_Ω.

1.1. Preliminary Lemmas.

Now we collect some key lemmas.

Lemma 1.1. *Let u denote the multiple Poisson integral of χ_Ω. The for some constant $\delta > 0$ we have*

$$u(z_1, z_2) \geq \delta \quad \forall z \in S(\Omega).$$

Proof. The idea behind this proof is exactly what happens in one variable. We can estimate the multiple Poisson kernel $P_{y_1}(x_1)P_{y_2}(x_2)$ from below by

$$P_{y_1}(x_1)P_{y_2}(x_2) \gtrsim \frac{1}{2y_1}\chi_{[-cy_1,cy_1]}(x_1)\frac{1}{2y_2}\chi_{[-cy_2,cy_2]}(x_2).$$

If we have $(z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) \in S(\Omega)$, then for some intervals I_j with $|I_j| \gtrsim y_j$ we have $(x_1, x_2) \in I_1 \times I_2 \subset \Omega$. This then allows us to conclude that

$$u(z_1, z_2) \gtrsim \chi_\Omega * \left(\frac{1}{2y_1}\chi_{[-cy_1,cy_1]}(x_1)\frac{1}{2y_2}\chi_{[-cy_2,cy_2]}(x_2)\right) \gtrsim \delta > 0$$

since for appropriate choice of constants we will have that $[x_1-cy_1, x_1+cy_1]\times[x_2-cy_2, x_2+cy_2]$ have some fixed fraction of its area contained in Ω. □

Lemma 1.2. *For u the multiple Poisson integral of χ_Ω, we have*

$$\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \left(|\nabla_1\nabla_2 u|^2 + |\nabla_1 u|^2 |\nabla_2 u|^2\right) y_1 y_2 dA(z_1)dA(z_2) \lesssim |\Omega|$$

Proof. The idea behind this Lemma is very simple. We will apply Green's formula to the function $u^4 + u^2$ and then just compute. Some of the terms we are interested in will immediately appear when computing. We will be left with many remainder terms, but each of these remainder terms can be hidden back on the other side. Let's now carry out this strategy.

Lets first apply Green's formula to find that

$$\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \Delta_1\Delta_2(u^4+u^2) y_1 y_2 dA(z_1)dA(z_2) = \int_{\mathbb{R}^2} (\chi_\Omega^4 + \chi_\Omega^2)dx = 2|\Omega|$$

Next compute $\Delta_1\Delta_2(u^4 + u^2)$. Basic calculus lets us get that this can be written as

$$\Delta_1\Delta_2(u^4+u^2) = (2+12u^2)|\nabla_1\nabla_2 u|^2 + 24|\nabla_1 u|^2|\nabla_2 u|^2 + \text{Remainder}.$$

Now, consider the quantity

$$G = \int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \left((2+12u^2)|\nabla_1\nabla_2 u|^2 + 24|\nabla_1 u|^2|\nabla_2 u|^2\right) y_1 y_2 dA(z_1)dA(z_2). \tag{1.1}$$

We claim that the remainder terms can be handled by some multiple of G. There are two types of typical terms that we need to consider, those that arise from computing $\Delta_1\Delta_2 u^2$ and those that come from $\Delta_1\Delta_2 u^4$. Typical terms that arise when computing $\Delta_1\Delta_2 u^2$ are like

$$\partial_1\overline{\partial}_1\partial_2 u\overline{\partial}_2 u \quad \text{or} \quad \partial_1\overline{\partial}_2 u\partial_2\overline{\partial}_1 u.$$

Other terms of this type similar in nature and can be handled by the same arguments as we now give. For the first of these terms, we actually have 0 since we have taken the multiple Poisson extension. The second term is clearly controlled by terms like those appearing as the first term in (1.1). More precisely, we have

$$\left|\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \partial_1\overline{\partial}_2 u\partial_2\overline{\partial}_1 u y_1y_2 dA(z_1)dA(z_2)\right| \leq \int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \left|\nabla_1\nabla_2 u\right|^2 y_1y_2 dA(z_1)dA(z_2)$$

by Cauchy-Schwarz.

Handling the terms that arise from $\Delta_1\Delta_2 u^4$ are similar. Here, a typical term that needs to be estimated is

$$\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} u\partial_1 u\partial_2 u\overline{\partial}_1\overline{\partial}_2 u y_1y_2 dA(z_1)dA(z_2).$$

But, this term is easily controlled by

$$\left(\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \left|\nabla_1 u\right|^2\left|\nabla_2 u\right|^2 y_1y_2 dA(z_1)dA(z_2)\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} u^2\left|\nabla_1\nabla_2 u\right|^2 y_1y_2 dA(z_1)dA(z_2)\right)^{\frac{1}{2}}$$

which is in turn controlled by the term (1.1).

Now instead of the function u, we consider the function $10^{-10}u$. Using these estimates from above one can now show that

$$G \leq C\left|\Omega\right| + 10^{-8}G$$

which then gives the claim in the Lemma. □

Lemma 1.3. *For u the multiple Poisson integral of χ_Ω, and for v the multiple Poisson integral of f we have*

$$\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \left|\partial_1 u\partial_2 v\right|^2 y_1y_2 dA(z_1)dA(z_2) \lesssim \left|\Omega\right|.$$

Proof. One starts from the function $\partial_1 u\partial_1 uv^2$ and computes Δ_2 of this function. Again, calculus gives that

$$\Delta_2(\partial_1 u\partial_1 uv^2) = 2\left|\partial_1 u\partial_2 v\right| + \text{Remainder}.$$

Thus, one is left showing that all the remainder terms, when estimated, contribute a constant times either $|\Omega|$ or

$$\left|\Omega\right|^{\frac{1}{2}}\left(\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \left|\partial_1 u\partial_2 v\right|^2 y_1y_2 dA(z_1)dA(z_2)\right)^{\frac{1}{2}}.$$

These are rather straightforward applications of Lemma 1.2 or using one-dimensional arguments. We leave the computations to the reader. □

Note that we would also have a symmetric version of this Lemma where the roles of the derivatives falling on u and v had been swapped. We don't state this as an explicit lemma, but just note that we have flexibility when applying Lemma 1.3.

1.2. **Proof Proper of Theorem 0.2.** Now to prove the main result we will use a similar strategy as what appeared when proving Lemma 1.2. We will apply Green's formula to a certain function. When we go to the boundary, this function will obviously produce the desired estimate of $|\Omega|$, and then we will be left with computing terms on the domain $\mathbb{R}^2_+ \times \mathbb{R}^2_+$. Some of these terms will be directly controllable and produce the terms we want to estimate, while the others will be remainder terms and we will have to show that each of these remainder terms contributes at most $|\Omega|$ or $|\Omega|^{\frac{1}{2}}$ times another quantity that we can control. We now proceed to carry out this strategy.

First take a function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(x) \geq 0$ for all $x \in \mathbb{R}$, φ is non-decreasing, $\varphi(x) = 1$ if $x \geq \delta$, $\varphi(x) = 0$ for $x \leq \frac{\delta}{2}$, and $\left|\varphi^{(j)}(x)\right|^2 \lesssim |\varphi|$ for $j = 1, 2, 3, 4$.

Let $\psi(x) = x^2 + x^4$, and consider the function $\varphi(u)\psi(v)$. First, by Green's Theorem we have

$$\int_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} \Delta_1 \Delta_2 (\varphi(u)\psi(v)) y_1 y_2 dA(z_1) dA(z_2) = \int_{\mathbb{R}^2} \varphi(u)\psi(v) = \int_{\Omega} v^2 + v^4 dx_1 dx_2 \lesssim |\Omega| .$$

Here we have used the fact that on the boundary $\varphi(u) = \varphi(\chi_\Omega)$, and that $v = f$ on the boundary, and so is bounded. Now we proceed to compute

$$\Delta_1 \Delta_2 (\varphi(u)\psi(v)) = \sum_{k,j=0}^{4} \varphi^{(k)} \psi^{(j)} I_{kj}(u, v).$$

Here I_{kj} is an expression involving derivatives of the functions u and v (and some of the $I_{kj} = 0$).

Consider the terms corresponding to $j = 2, 4, k = 0$ and compute that

$$I_{02} + I_{04} = (2 + 12v^2) \left|\nabla_1 \nabla_2 v\right|^2 + 24 \left|\nabla_1 v\right|^2 \left|\nabla_2 v\right|^2$$

and so we have the corresponding integrals related to these terms $I_{02} + I_{04}$

$$M = \int_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} \left((2 + 12v^2) \left|\nabla_1 \nabla_2 v\right|^2 + 24 \left|\nabla_1 v\right|^2 \left|\nabla_2 v\right|^2\right) \varphi y_1 y_2 dA(z_1) dA(z_2).$$

It is clear that we can show that $M \lesssim |\Omega|$, then we will have a proof of our Theorem. Our plan now is to show that every term that arises when we computed the expression from Green's Formula can be controlled by $M^{\frac{1}{2}} |\Omega|^{\frac{1}{2}}$ or by $|\Omega|$.

First note that if we compute I_{03}, then we will have that the term $I_{03} \leq \frac{24}{10^{10}}(I_{02} + I_{04})$. We have that

$$I_{03} = \overline{\partial}_1 v \partial_1 \overline{\partial}_2 v \partial_2 v + \overline{\partial}_1 v \overline{\partial}_2 v \partial_1 \partial_2 v + \partial_1 v \overline{\partial}_1 \overline{\partial}_2 v \partial_2 v + \partial_1 v \overline{\partial}_2 v \overline{\partial}_1 \partial_2 v + \partial_1 v \overline{\partial}_1 v \overline{\partial}_2 v \partial_2 v.$$

Once you see this, it becomes obvious that

$$\left| \int_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} 24 v I_{03} y_1 y_2 \varphi dA(z_1) dA(z_2) \right| \leq \frac{24}{10^{10}} M$$

The terms I_{00} and I_{01} are easier (no contribution at all actually). Using this estimate we then see that

$$\left(1 - \frac{24}{10^{10}}\right) M \leq |\Omega| + \sum_{k=1}^{4} \sum_{j=0}^{4} \left| \int_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} \varphi^{(k)} \psi^{(j)} I_{kj}(u, v) y_1 y_2 dA(z_1) dA(z_2) \right| .$$

The plan is now to show that each of the terms on the right hand side is controlled by the desired quantities. We won't show this for all the terms, but will give some representative terms that provide the main ideas. To obtain these estimates we will have to use Lemmas 1.2 and 1.3.

First, consider the case of $j = 0$ and $2 \leq k \leq 4$ (again we get no contribution from the $k = 1$ term. Since we have computed the term I_{03}, for concreteness lets consider the term I_{30}. In this case, we are attempting to control terms that look like

$$\left| \int_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} \varphi^{(3)}(u)\psi(v)\overline{\partial}_1 u \partial_1 \overline{\partial}_2 u \partial_2 u y_1 y_2 dA(z_1) dA(z_2) \right|$$

And it is easy to see by Cauchy-Schwarz and using obvious estimates on $\psi(v)$ and φ that this term above is controlled by

$$\left(\int_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} \left|\overline{\partial}_1 u\right|^2 \left|\overline{\partial}_2 u\right|^2 y_1 y_2 dA(z_1) dA(z_2) \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} \left|\partial_1 \overline{\partial}_2 u\right|^2 y_1 y_2 dA(z_1) dA(z_2) \right)^{\frac{1}{2}} \lesssim |\Omega| \, .$$

Where the last inequality follows from Lemma 1.2. All the other terms in I_{30} are handled identically. The terms corresponding to I_{20} and I_{40} are also handled directly by Lemma 1.2.

It is easy to see that for terms that arise in I_{12}, you are left estimating integrals that look like:

$$\int_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} \varphi^{(2)}(4v^3 + 2v) \left(\overline{\partial}_2 u \partial_1 \partial_2 u \overline{\partial}_1 v\right) y_1 y_2 dA(z_1) dA(z_2)$$

However, it is very easy to see that terms like this can be easily controlled using Hölder's and then applicaiton of Lemma 1.2 and one application of Lemma 1.3. Thus, terms like this contribute at most $|\Omega|$. There are other terms that can arise when computing I_{12}, and they look like

$$\int_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} \varphi^{(2)}(4v^3 + 2v) \left(\overline{\partial}_2 u \partial_1 \partial_2 v \overline{\partial}_1 u\right) y_1 y_2 dA(z_1) dA(z_2)$$

But this term can then easily be controlled by Hölder and Lemma 1.2 to arrive at an estimate of the form $M^{\frac{1}{2}} |\Omega|^{\frac{1}{2}}$. The full details are left to the reader.

Finally, we consider the expressions that arise when studying $I_{13} + I_{12}$, namely. Now in the integral

$$\int_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} \varphi'(\psi^{(3)} I_{13}(u, v) + \psi^{(2)} I_{12}(u, v)) y_1 y_2 dA(z_1) dA(z_2) \tag{1.2}$$

we collect the terms for which we have u appearing with 2 derivatives, and v appearing with either 2 derivatives, or two copies of v, each with one derivative, call this sum H. Now it is easy to see that for terms that appear in this sum, we have

$$|H| \lesssim M^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} \, .$$

The two derivatives of u are controlled directly by Lemma 1.2, and the term involving the derivatives of v are estimated as term M.

In in (1.2) collect all the terms that have u appears by its first derivative $\overline{\partial}_2 u$ and call the term $K_{\overline{2}}$. Similarly collect the terms in (1.2) that have u appearing by $\overline{\partial}_1 u$, $\partial_1 u$ and $\partial_2 u$ and call these terms $K_{\overline{1}}$, K_1 and K_2 respectively. It then suffices to show that for each of these terms we have

$$|K_j| \lesssim |\Omega|^{\frac{1}{2}} |M|^{\frac{1}{2}} \quad \forall j = 1, 2, \overline{1}, \overline{2}.$$

These estimates will all follow from Lemma 1.2 and 1.3. To carry out this program is lots of computations with derivatives. Do so, we find the following expression (assuming I didn't make a mistake!)

$$I_{13} = \partial_1 v \overline{\partial}_1 v \partial_2 v \overline{\partial}_2 u + \partial_1 v \overline{\partial}_1 v \partial_2 u \overline{\partial}_2 v + \partial_1 v \overline{\partial}_1 u \partial_2 v \overline{\partial}_2 u + \partial_1 u \overline{\partial}_1 v \partial_2 v \overline{\partial}_2 v.$$

For expressions like this, it is easy to see how to use Lemma 1.2 and 1.3 along with the properties of φ and ψ to obtain the estimate $|\Omega|^{\frac{1}{2}} M^{\frac{1}{2}}$.

The term I_{12} can be computed similarly (a good exercise in derivative computation) and here we see that we have the terms corresponding to the second derivatives of u, and then expressions like:

$$\partial_1 v \overline{\partial}_1 \overline{\partial}_2 v \partial_u$$

and permutations of the various derivatives. In all these cases we can apply Lemma 1.2 and 1.3 as well.

2. Extension of the Main Result in this Lecture

Using the main ideas from this lecture it is possible to extend the work we have done and obtain the following characterization of the dual of $H^1(\mathbb{D}^2)$ obtained by Chang:

Theorem 2.1 (Chang, [1]). *Let f denote a function on $\mathbb{T}^2$ whose multiple Poisson integral on the bidisc $\mathbb{D}^2$ is denoted by u. For f in the dual of $H^1(\mathbb{T}^2)$ we have the following Carleson measure condition satisfied:*

$$\frac{1}{|\Omega|} \int_{S(\Omega)} |\nabla_1 \nabla_2 u(z_1, z_2)|^2 \log \frac{1}{|z_1|} \log \frac{1}{|z_2|} dA(z_1) dA(z_2) \lesssim 1$$

where $\Omega \subset \mathbb{T}^2$ is an arbitrary open set, $S(\Omega)$ is the tent over the open set, and M is independent of Ω.

There is of course more work in obtaining this Theorem, but the interested reader now has some of the necessary tools by which to understand more of what is going on for this Carleson measure condition.

References

[1] Sun-Yung A. Chang, *Carleson measure on the bi-disc*, Ann. of Math. (2) **109** (1979), no. 3, 613–620. ↑1, 6

[2] R. Fefferman, *Bounded mean oscillation on the polydisk*, Ann. of Math. (2) **110** (1979), no. 2, 395–406. ↑1