# Lecture 7: Multi-Parameter Carleson Measures

In the last lecture we introduced Carleson measures in one parameter. Let $Q \subset \mathbb{R}^n$ be a cube and let

$$T\,(Q) = \{(x, y) : x \in Q, 0 \leq y \leq \ell(Q)\}$$

Recall that a measure $\mu$ is a Carleson measure if for all cubes $Q \subset \mathbb{R}^n$ we have a constant $C$ such that

$$\mu\,(T\,(Q)) \leq C\,|Q|\,. \tag{0.1}$$

Now, note that we also have another way by which we can test is a measure is Carleson. For simplicity, we focus on the case $n = 1$. Let $\Omega \subset \mathbb{R}$ be an arbitrary open set. Then it is easy to see that a measure is Carleson if and only if for all open sets $\Omega \subset \mathbb{R}$ we have

$$\mu\,(T\,(\Omega)) \leq C\,|\Omega|\,. \tag{0.2}$$

It is clear that if (0.2) holds, then so does (0.1). Now, suppose that (0.1) holds. Since $\Omega$ is a open set in $\mathbb{R}$, there exists disjoint open intervals $\{I_k\}$ such that $\Omega = \cup I_k$. Consider now $T\,(\Omega)$, and note

$$\begin{aligned} \mu\,(T\,(\Omega)) &\leq \mu\,(T\,(\cup I_k)) \\ &\leq \sum_k \mu(T(I_k)) \\ &\leq C \sum_k |I_k| = C\,|\Omega|\,. \end{aligned}$$

A similar argument applies when $\Omega \subset \mathbb{R}^n$ is open. In this case, we use the Whitney Decomposition of an open set. The interested reader can see [2] for the details of this decomposition of an open set in $\mathbb{R}^n$.

Thus we have two possible ways by which to study Carleson measures in one-parameter. We can either form the tent over arbitrary open sets, or we can form the tent over cubes. This suggests that we have two competing definitions of Carleson measures in the multi-parameter setting. For simplicity, we will focus on the case of when $\Omega \subset \mathbb{R}^2$ is open. One possible candidate is:

$$\mu\,(T\,(R)) \leq C\,|R| \quad \forall R = I \times J \tag{0.3}$$

Here the tent over the rectangle $R$ is given by $T(R) = T(I) \times T(J)$. The other competing definition is the following:

$$\mu\,(T\,(\Omega)) \leq C\,|\Omega| \quad \forall \Omega \subset \mathbb{R}^2. \tag{0.4}$$

Again, it is immediate that (0.4) implies (0.3). But, it is not so clear that (0.3) should (or could) imply (0.4). One reason to be suspicious about the equivalence between (0.3) and (0.4) is that there isn't a "good" way to decompose an open set into disjoint rectangles, and so the idea of proof above appears to be problematic.

Our goal in this lecture is to gain an understanding of which is the right notion of Carleson measure in the multi-parameter setting.

## 1. Preliminaries

Note that in the last lecture we (essentially) proved the following Theorem.

**Theorem 1.1** (Carleson's Embedding Theorem)**.** *Let $f \in L^p(\mathbb{T})$ and let $u(z)$ denote the Poisson integral of $f$. If $\mu$ is a positive measure on the disc $\mathbb{D}$, then the following are equivalent*

(a) *$\mu$ is a Carleson measure, i.e., $\mu(T(I)) \leq C(\mu)\,|I|$ for any $I \subset \mathbb{T}$;*
(b) *For $1 < p < \infty$ and for all $f \in L^p(\mathbb{T})$, $u(z) \in L^p(\mathbb{D};\mu)$;*
(c) *For $1 < p < \infty$*

$$\int_{\mathbb{D}} |u(z)|^p \, d\mu(z) \leq C(p) \int_{\mathbb{T}} |f(t)|^p \, dt \quad \forall f \in L^p(\mathbb{T}).$$

It was proved explicitly in the case of $\mathbb{R}$ and the upper half plane, but without much work, the proof we gave can be modified to handle the case in the Theorem above.

Based on this, and as an attempt to understand the multi-parameter Carleson measures, we should attempt to study functions with are harmonic in each variable on the bidisc $\mathbb{D}^2$. And, we should attempt to understand the measures $\mu$ such that we have

$$\int_{\mathbb{D}^2} |u(z,w)|^p \, d\mu(z,w) \leq C(p) \int_{\mathbb{T}^2} |f(t,s)|^p \, dtds \quad \forall f \in L^p(\mathbb{T}^2), \tag{1.1}$$

where $u$ is the corresponding extension. It is obvious that a simple necessary condition is the following estimate

$$\mu(T(R)) \leq C\,|R|\,.$$

This is nothing other then (0.3). To see that (1.1) implies estimate (0.3) holds, simply consider functions $u(z,w) = f(z)g(w)$ where $f$ and $g$ are harmonic.

One of our main results from this lecture will be the following Theorem of Carleson.

**Theorem 1.2** (Carleson, [1])**.** *There exists a measure $\mu$ such that (0.3) holds, but (1.1) fails for any $p \geq 1$.*

We first reduce the result we want to prove to a covering problem. Let $\mathcal{R} = \{R\}$ be a collection of rectangles with the property that for any rectangle $Q$ we have

$$\sum_{R \subset Q : R \in \mathcal{R}} |R| \leq A\,|Q|\,. \tag{1.2}$$

Now suppose that (0.3) implies (1.1). Then we claim that for the collection of rectangles in $\mathcal{R}$ that we have

$$\sum_{R \in \mathcal{R}} |R| \lesssim AC(p) \left| \bigcup_{R \in \mathcal{R}} R \right| \tag{1.3}$$

where the implied constant is numerical, and $A$ is the constant appearing in (1.2) and $C(p)$ is the constant appearing in (1.1).

We need to construct our (problem) measure $\mu$ now. Let $a = \frac{1}{100A}$ and for each rectangle $R \in \mathcal{R}$, we form the following point in $\mathbb{D}^2$,

$$(z_R, w_R) = ((1-|I|)c(I), (1-|J|)c(J))$$

where $c(I)$ is the center of the corresponding interval $I$. Then, define the measure $\mu$ as

$$\mu = \sum_{R \in \mathcal{R}} a\,|R|\,\delta_{(z_R, w_R)}.$$

Namely, $\mu$ is a weighted sum of Dirac functions at the points constructed. Then using (1.2) we can easily see that for any rectangle $S \subset \mathbb{T}^2$ that we have

$$\mu\,(T(S)) \lesssim |S|\,.$$

Now, let $f(s,t) = \chi_{\bigcup_{\mathcal{R}} R}(s,t)$, and $u(z,w)$ the corresponding bi-harmonic function. Then using (1.1) we find that

$$\begin{aligned} C(p)\left|\bigcup_{R \in \mathcal{R}} R\right| &= C(p)\int_{\mathbb{T}^2} |f(s,t)|^p\,dsdt \\ &\geq \int_{\mathbb{D}^2} |u(z,w)|^p\,d\mu(z,w) \\ &= a\sum_{R \in \mathcal{R}} |R|\,u(z_R, w_R)^p \end{aligned}$$

Next, note that $u(z_R, w_R) \gtrsim 1$ for some absolute constant. To see this, one uses the Poisson representation for the function $u(z_R, w_R)$, the choice of point $(z_R, w_R)$, and obvious estimates. This is much as the same estimates in the case of one-parameter, and so we leave them to the reader. So we have that

$$\sum_{R \in \mathcal{R}} |R| \lesssim AC(p)\left|\bigcup_{R \in \mathcal{R}} R\right|$$

Since we are after a contradiction, the idea now becomes clear. We need to construct a collection of rectangles $\mathcal{R}$ such that $\sum_{R \in \mathcal{R}} |R|$ is large, say,

$$\sum_{R \in \mathcal{R}} |R| = 1$$

and

$$\left|\bigcup_{R \in \mathcal{R}} R\right|$$

can be made as small as we wish.

## 2. Construction of the Counterexample

We are going to follow the general idea as illustrated by Carleson, [1], and as further explained by Tao, [3]. First, define a *quilt* to be a finite collection of rectangles $\mathcal{R}$ in the unit square $[0,1] \times [0,1] := [0,1]^2$ such that

$$\sum_{R \in \mathcal{R}} |R| = 1 \tag{2.1}$$

and

$$\sum_{R \in \mathcal{R}\colon R \subset Q} |R| \leq |Q| \tag{2.2}$$

for every dyadic cube $Q \subset [0,1]^2$ (a dyadic rectangle $Q = I \times J$ is one for which the sides $I$, $J$ are dyadic intervals). The first condition is saying that our quilt has "large" area in the unit square, while the second condition is related to (0.3). Also note that quilts clearly exist since as one can easily check $\{[0,1]^2\}$ is a quilt. Define the *area* of our quilt to be

$$\left| \bigcup_{R \in \mathcal{R}} R \right|$$

Now, we show that it is possible to find quilts with area as small as we wish.

**Lemma 2.1.** *There exist quilts of arbitrarily small area. Namely, for any $\epsilon > 0$ there exists a quilt with area less than $\epsilon$.*

Note now that Lemma 2.1 then gives us the desired counter example to Theorem 1.2

Here is the idea we are aiming for in the proof below. We have a quilt of rectangles $\mathcal{R}$ that have desired area. The plan is to take each rectangle in the collection and "shrink" it on one of its sides, and then place "copies" of these rectangles back inside the unit square, but well separated. This shrinking and coping will be done in both the horizontal and vertical directions. The shrinking and copying should be done in a manner that allows for the collections to be disjoint, which will be useful in proving certain estimates on the desired sets. For the construction of the new quilt, it will (more or less) obvious that we still have the covering property, (2.1) and the "embedding" property (2.2). Showing that the area of the new quilt is the desired amount is the slightly more complicated part. The fact that the resulting area is small will be due to the new quilt having lots of overlap. Here we have tried to follow the ideas of the note by Tao, [3]. The note by Tao is doing a fantastic job of extracting the key ideas from the more complicated paper by Carleson [1].

A very useful exercise to consider in the proof of this Lemma is the following example. One should start with the following quilt:

$$\mathcal{R} = \left\{ \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right], \left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, 1\right], \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right] \times \left[\frac{1}{2}, 1\right] \right\}.$$

If one performs the "scaling" and "copying", it will be obvious that these versions will have a certain disjointness property. It will also be clear from this example that the area of the new quilt is what appears in the claim of the Lemma.

*Proof.* Given a quilt $\mathcal{R}$ with area $\alpha$ we will construct a new quilt $\mathcal{R}'$ that has area $\alpha - \frac{\alpha^2}{4}$. Then to construct any quilt with small enough area, we simply will start with the quilt $[0,1]^2$ and iterate the necessary construction as often as necessary to obtain the lemma.

Fix the finite collection $\mathcal{R}$ and the area of the quilt $\alpha$ (think of this as an iterated version of the unit quilt). Let $N \geq 1$ be a sufficiently large integer so that every rectangle in $\mathcal{R}$ has length and width at least $2^{-N}$. For each integer $0 \leq j \leq 2^N - 1$, define affine transformations

$$\begin{aligned} A_j^x(x,y) &= \left( \frac{j}{2^N} + \frac{x}{2^{N+1}}, y \right) \\ A_j^y(x,y) &= \left( x, \frac{j}{2^N} + \frac{y}{2^{N+1}} \right) \end{aligned}$$

We then define a new quilt $\mathcal{R}'$ from our old quilt using these transformations. Set

$$\mathcal{R}' = \bigcup_{s\in\{x,y\}} \bigcup_{j=1}^{2^N} A_j^s(\mathcal{R})$$

Lets first note that (2.1) holds for the collection $\mathcal{R}'$. If $R = I \times J = (s_1, s_2) \times (t_1, t_2)$ then we see that

$$A_j^x(R) = \left(\frac{j}{2^N} + \frac{s_1}{2^{N+1}}, \frac{j}{2^N} + \frac{s_2}{2^{N+1}}\right)\times(t_1,t_2) \text{ and } A_j^y(R) = (s_1,s_2)\times\left(\frac{j}{2^N} + \frac{t_1}{2^{N+1}}, \frac{j}{2^N} + \frac{t_2}{2^{N+1}}\right)$$

Note that this implies that the families of rectangles $A_j^s(R)$ are disjoint. In fact, one can show that we have $A_j^s(\mathcal{R})$ being disjoint in $j$ for fixed $s$. Essentially this reduces to the fact that each rectangle is translated by some amount, and the side lengths of the resulting rectangle are decreased by a smaller amount. Also note that

$$\left|A_j^s(R)\right| = 2^{-N-1}|R| \quad \forall 0 \le j \le 2^N - 1 \text{ and } s \in \{x,y\}.$$

From this last equality we have that

$$\sum_{A_j^s(R)\in\mathcal{R}'} \left|A_j^s(R)\right| = \sum_{R\in\mathcal{R}} |R| = 1$$

and so our new quilt satisfies (2.1). Before we show that the quilt satisfies the conditions of (2.2), we next show that this procedure decreases the area of our quilts.

Since the collections $A_j^x(\mathcal{R})$ are disjoint, we have

$$\left|\bigcup_{j=0}^{2^N-1} \bigcup_{R\in A_j^x(\mathcal{R})} R\right| = \sum_{j=0}^{2^N-1} \left|\bigcup_{R\in A_j^x(\mathcal{R})} R\right| = \sum_{j=0}^{2^N-1} 2^{-N-1}\alpha = \frac{\alpha}{2}.$$

Similarly, we have

$$\left|\bigcup_{j=0}^{2^N-1} \bigcup_{R\in A_j^y(\mathcal{R})} R\right| = \sum_{j=0}^{2^N-1} \left|\bigcup_{R\in A_j^y(\mathcal{R})} R\right| = \frac{\alpha}{2}.$$

So, it suffices to show that

$$\left|\left(\bigcup_{j=0}^{2^N-1} \bigcup_{R\in A_j^x(\mathcal{R})} R\right) \bigcap \left(\bigcup_{k=0}^{2^N-1} \bigcup_{R\in A_k^y(\mathcal{R})} R\right)\right| = \frac{\alpha^2}{4}. \tag{2.3}$$

and then note by the inclusion-exclusion principle that

$$\left|\bigcup_{R\in\mathcal{R}'} R\right| = \alpha - \frac{\alpha^2}{4}.$$

Now to show (2.3), let $I_j = [\frac{j}{2^N}, \frac{j+\frac{1}{2}}{2^N})$. Then, we can first write the set we are studying as

$$\left|\left(\bigcup_{j=0}^{2^N-1} \bigcup_{R\in A_j^x(\mathcal{R})} R\right) \bigcap \left(\bigcup_{k=0}^{2^N-1} \bigcup_{R\in A_k^y(\mathcal{R})} R\right)\right| = \sum_{j=0}^{2^N-1}\sum_{k=0}^{2^N-1} \left|\left(I_j \times I_k \cap \bigcup_{R\in A_j^x(\mathcal{R})} R\right) \bigcap \left(I_j \times I_k \cap \bigcup_{R\in A_k^y(\mathcal{R})} R\right)\right|.$$

This decomposition follows from the scaling and copying we have done. Now observe that we can recognize this last expression as

$$\left|\left(\bigcup_{j=0}^{2^N-1}\bigcup_{R\in A_j^x(\mathcal{R})} R\right)\bigcap\left(\bigcup_{k=0}^{2^N-1}\bigcup_{R\in A_k^y(\mathcal{R})} R\right)\right| = \sum_{j=0}^{2^N-1}\sum_{k=0}^{2^N-1}\left|E_{j,k}\times I_k\cap I_j\times F_{j,k}\right|.$$

Here the sets $E_{j,k}$ is a collection of intervals in the $x$ direction, with a similar statement applying to the sets $F_{j,k}$. It turns out that we have

$$\sum_{j=0}^{2^N-1}|F_{j,k}| = \sum_{k=0}^{2^N-1}|E_{j,k}| = \frac{\alpha}{2}.$$

(Drawing pictures at this point is very helpful in seeing that this it true, and is the only way I could convince myself; the example suggested before this Lemma is very helpful!) Once we have this, then we arrive at

$$\left|\left(\bigcup_{j=0}^{2^N-1}\bigcup_{R\in A_j^x(\mathcal{R})} R\right)\bigcap\left(\bigcup_{k=0}^{2^N-1}\bigcup_{R\in A_k^y(\mathcal{R})} R\right)\right| = \sum_{j=0}^{2^N-1}\sum_{k=0}^{2^N-1}|E_{j,k}|\,|F_{j,k}| = \frac{\alpha^2}{4},$$

and then we have the desired area of the new quilt:

$$\left|\bigcup_{R\in\mathcal{R}'} R\right| = \alpha - \frac{\alpha^2}{4}.$$

Finally, we turn to showing (2.2) for our collection $\mathcal{R}'$. Let $Q = I\times J$ be any dyadic rectangle with sides $|I| = 2^{-k_1}$ and $|J| = 2^{-k_2}$. First, suppose that $k_1 > N$. Then, if the sum appearing in (2.2) empty, then we trivially have (2.2) holding. If the sum is non-empty, then we have that at most one of the summands $A_j^x(\mathcal{R})$ can contribute. However, if this is true, then since the collection $\mathcal{R}$ satisfies (2.2), then we have that the collection $A_j^x(\mathcal{R})$ satisfies the condition as well, but with a larger constant, $2^N$ (just scale the rectangles appearing in (2.2) but for the collection $\mathcal{R}$). The case of $k_2 > N$ is handled identically.

It remains to address the case when $k_1, k_2 \leq N$. In this case,the first side of $Q$ has length $2^{-k_1} > 2^{-N}$, and so will cover more of the corresponding collections $A_j^x(\mathcal{R})$. In fact, there can be at most $2^{N-k_1}$ collections from $A_j^x(\mathcal{R})$ that can contribute to the sum. By rescaling, we can easily see that each of these collections can contribute at most $2^{-N-1-k_2}$. We then have the desired estimate for (2.2) in this case. The case of $A_j^y(\mathcal{R})$ is similar.

The careful reader will note that we really only focused on dyadic rectangles. However, if $Q$ is an arbitrary rectangle then it is possible to cover $Q$ by dyadic rectangles $T_i$ with

$$Q\subset\bigcup_{i=1}^{4} T_i$$

where each rectangle $T_i$ is dyadic, and $\frac{1}{4}|Q| \leq |T_i| \leq |Q|$. Since all the rectangles in $\mathcal{R}'$ are dyadic, using this last observation, we have that result for all rectangles, with a worse constant.

$\square$

## References


[1] Lennart Carleson, *A Counter Example for Measures Bounded on $H^p$ for the Bi-Disc* (1974). ↑2, 3, 4

[2] Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy; Monographs in Harmonic Analysis, III. ↑1

[3] Terrance Tao, *Dyadic Product $H^1$, BMO and Carleson's Counterexample* (1999), available at http://www.math.ucla.edu/~tao/preprints/Expository/product.dvi. ↑3, 4