Lecture 6: Examples of Carleson Measures in One Parameter

Our goal in this lecture is to get a sense of what types of functions generate Carleson measures. For simplicity we will work on the disc and in one-variable, since we can use more tools of complex analysis. However, much of what we say can be transferred to the case of several variables with more work. We point the interested reader to the necessary literature at the end of the notes.

1. Carleson Measures for $H^2(\mathbb{D})$

We first introduce the space $H^2(\mathbb{D})$. Let $f \in \text{Hol}(\mathbb{D})$, then we say that $f \in H^2(\mathbb{D})$ if

$$\sup_{0<r<1} \int_{\mathbb{T}} |f(re^{i\theta})|^2 dm(\theta) := \|f\|_{H^2(\mathbb{D})}^2 < \infty. \tag{1.1}$$

We now show other norms that can be used to study the functions in $H^2(\mathbb{D})$. First, recall that the Fourier transform of a function $f \in L^2(\mathbb{T})$ is given by

$$\hat{f}(n) = \int_{\mathbb{T}} f(e^{i\theta}) e^{-in\theta} dm(\theta).$$

Then, a simple computation shows that

$$\int_{\mathbb{T}} e^{i(n-m)\theta} dm(\theta) = \begin{cases} 1 & : \quad n = m \\ 0 & : \quad n \neq m \end{cases}$$

Using this, we see that for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ that

$$\begin{aligned} \|f\|_{H^2(\mathbb{D})}^2 &= \sup_{0<r<1} \int_{\mathbb{T}} |f(re^{i\theta})|^2 dm(\theta) \\ &= \sup_{0<r<1} \int_{\mathbb{T}} \left| \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \right|^2 dm(\theta) \\ &= \sup_{0<r<1} \sum_{n,m=0}^{\infty} a_n \overline{a_m} r^n r^m \int_{\mathbb{T}} e^{i(n-m)\theta} dm(\theta) \\ &= \sum_{n=0}^{\infty} |a_n|^2 = \|f\|_{H^2(\mathbb{D})}^2 . \end{aligned}$$

Note that this norm says that it is possible to study the behavior of the functions in $H^2(\mathbb{D})$ via their Fourier coefficients. An easy exercise is to prove the following exercise.

Exercise 1.1. *For $0 < r < 1$ and $z \in \mathbb{D}$ let $f_r(z) = f(rz)$. Suppose that $f \in H^2(\mathbb{D})$. Then, the sequence $\{f_r\}$ is Cauchy in $L^2(\mathbb{T})$.*

Now note that since $L^2(\mathbb{T})$ is a complete space, then we have an element $f^* \in L^2(\mathbb{T})$ given by $f^* = \lim_{r\to 1} f_r$ also in $L^2(\mathbb{T})$. Since $f^* \in L^2(\mathbb{T})$ we can compute the Fourier coefficients to be

$$\begin{aligned}\widehat{f^*}(n) &= \int_{\mathbb{T}} f^*(e^{i\theta})e^{-in\theta}dm(\theta)\\ &= \lim_{r\to 1}\int_{\mathbb{T}} f_r(e^{i\theta})e^{-in\theta}dm(\theta)\\ &= \begin{cases} a_n & : \ n\geq 0\\ 0 & : \ n<0\end{cases}.\end{aligned}$$

Note that the computations we have done thus far proves the following proposition.

Proposition 1.2. *Suppose that $f\in H^2(\mathbb{D})$ and $f^*(e^{i\theta})=\lim_{r\to 1} f(re^{i\theta})$ then*

$$\|f\|_{H^2(\mathbb{D})}^2=\sum_{n=0}^{\infty}|a_n|^2=\|f^*\|_{L^2(\mathbb{T})}^2.$$

The only fact that remains to complete the proof of this proposition is that

$$\sum_{n=0}^{\infty}|a_n|^2=\|f^*\|_{L^2(\mathbb{T})}^2$$

which holds by Parseval's Theorem. This also shows that the inner product on $H^2(\mathbb{D})$ will satisfy

$$\langle f,g\rangle_{H^2(\mathbb{D})}=\int_{\mathbb{T}} f^*(e^{i\theta})\overline{g^*(e^{i\theta})}dm(\theta)=\sum_{n=0}^{\infty}a_n\overline{b_n}$$

where we have $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $g(z)=\sum_{n=0}^{\infty}b_nz^n$.

Our main theorem from the last lecture can be translated to the following beautiful result for $H^2(\mathbb{D})$.

Theorem 1.3 (Carleson Embedding Theorem)**.** *Let μ be a non-negative measure in $\mathbb{D}$. Then the following are equivalent:*

(i) *The embedding operator $\mathcal{J}:L^2(\mathbb{T})\to L^2(\mathbb{D},\mu)$, with $\mathcal{J}(f)(z)=f(z)$, is bounded.*
(ii) $C(\mu)^2:=\sup_{z\in\mathbb{D}}\left\|\mathcal{J}\tilde{k}_z\right\|_{L^2(\mu)}^2=\sup_{z\in\mathbb{D}}\|P_z\|_{L^1(\mu)}<\infty$*, where* $\tilde{k}_z(\xi)=\frac{(1-|z|^2)^{1/2}}{(1-\xi\overline{z})}$*, the reproducing kernel for the Hardy space $H^2(\mathbb{D})$.*
(iii) $I(\mu)=\sup\left\{\frac{1}{r}\mu(\mathbb{D}\bigcap Q(\xi,r)):r>0,\xi\in\mathbb{T}\right\}<\infty$*, where $Q(\xi,r)$ is a ball measured with respect to the non-isotropic metric associated to $\mathbb{D}$.*

Moreover, the following inequalities hold

$$C(\mu)\leq\|\mathcal{J}\|\leq 4C(\mu)$$

and

$$32I(\mu)\leq C(\mu)^2\leq 32I(\mu)$$

The equivalence between (i) and (iii) is the main content of the theorem from the last lecture. The equivalence between (ii) and (iii) is a good exercise for the reader to verify.

2. Littlewood-Paley Identities and $H^2(\mathbb{D})$

We now show how it is possible to obtain another norm on $H^2(\mathbb{D})$ using information about the function on the disc $\mathbb{D}$. This equivalent norm will prove useful when we we study the space of Carleson measures for $H^2(\mathbb{D})$ since it will allow us to generate a natural family of examples of functions which generate Carleson measures. Also, this new norm will allow us to place the Hardy space in a scale of Besov-Sobolev spaces.

First, we begin by recalling Green's formula in the case of the unit disc $\mathbb{D}$ and its boundary $\mathbb{T}$. Then Green's formula takes the form:

$$\int_{\mathbb{T}} u(\xi)dm(\xi) - u(0) = \int_{\mathbb{D}} \Delta u(z) \log \frac{1}{|z|} dA(z)$$

Note that we can move the point 0 to any other point $z \in \mathbb{D}$ by a Möbius map of the form $\varphi_z(w) = \frac{w-z}{1-\bar{z}w}$.

Exercise 2.1. *Work out Green's formula for the point $z \in \mathbb{D}$.*

We will begin with a function $g \in L^1(\mathbb{T})$ and, as usual, let $g(z)$ denote the Poisson extension of the function g. The gradient of a function g is given by $\nabla g = (\partial_x g, \partial_y g)$ and we have

$$|\nabla g(z)|^2 = |\partial_x g(x,y)|^2 + |\partial_y g(x,y)|^2 .$$

In the case when g happens to be an analytic function we have that

$$|\nabla g(z)|^2 = |\partial g(z)|^2 = |g'(z)|^2 .$$

Lemma 2.2 (Littlewood-Paley Identity)**.** *Suppose that $g \in L^1(\mathbb{T})$ and if $g(0) = \int_{\mathbb{T}} g dm$ then*

$$2\int_{\mathbb{D}} |\nabla g(z)|^2 \log \frac{1}{|z|} dA(z) = \int_{\mathbb{T}} |g - g(0)|^2 \, dm = \int_{\mathbb{T}} |g|^2 \, dm - |g(0)|^2 .$$

Proof. With out loss of generality we may assume that $g(0) = 0$, since we can reduce to this case by considering the function $\tilde{g} = g - g(0)$. We will apply Green's Theorem with the function $u = |g|^2$. Since $g(0) = 0$ we have that $u(0) = |g(0)|^2 = 0$. Now observe that

$$\begin{aligned}
\partial\overline{\partial} \, |g(z)|^2 &= \partial \left(\overline{\partial} g \overline{g} + g \overline{\partial} \overline{g} \right) \\
&= \partial\overline{\partial} g \overline{g} + g \partial\overline{\partial}\overline{g} + \overline{\partial} g \partial \overline{g} + \partial g \overline{\partial}\overline{g} \\
&= \overline{\partial} g \partial \overline{g} + \partial g \overline{\partial}\overline{g} = |\partial g|^2 + \left|\overline{\partial} g\right|^2 \\
&= \frac{1}{2} |\nabla g(z)|^2 .
\end{aligned}$$

Here the last equality follows from the definitions of the operators ∂ and $\overline{\partial}$. Using this we see that

$$\Delta |g(z)|^2 = 2 |\nabla g(z)|^2 .$$

Substituting into Green's formula we have

$$\int_{\mathbb{T}} |g(\xi)|^2 \, dm = \int_{\mathbb{D}} \Delta(|g(z)|^2) \log \frac{1}{|z|} dA(z) = 2 \int_{\mathbb{D}} |\nabla g(z)|^2 \log \frac{1}{|z|} dA(z)$$

proving the Lemma. ∎

Using this lemma, we have another way to compute the norm of a function in $H^2(\mathbb{D})$.

Proposition 2.3. *Suppose that $f \in H^2(\mathbb{D})$ then we have*

$$\|f\|^2_{H^2(\mathbb{D})} = |g(0)|^2 + 2\int_{\mathbb{D}} |g'(z)|^2 \log\frac{1}{|z|} dA(z).$$

The proof of this follows by simple rearrangement of the above Lemma.

We give a slightly different way to see the resulting norm that in some cases is easier to use. More importantly for us, it will allow us to place the Hardy space in a scale of analytic function spaces that are very interesting.

Lemma 2.4. *If $g \in L^1(\mathbb{T})$ then*

$$\int_{\mathbb{D}} |\nabla g(z)|^2 (1-|z|^2) dA(z) \leq 2\int_{\mathbb{D}} |\nabla g(z)|^2 \log\frac{1}{|z|} dA(z) \leq C\int_{\mathbb{D}} |\nabla g(z)|^2 (1-|z|^2) dA(z).$$

Proof. First note that $1-t \leq 2\log\frac{1}{t}$ if $0 \leq t < 1$. So we have that

$$\int_{\mathbb{D}} |\nabla g(z)|^2 (1-|z|^2) dA(z) \leq 2\int_{\mathbb{D}} |\nabla g(z)|^2 \log\frac{1}{|z|} dA(z).$$

To prove the alternate inequality, first, suppose that the integral on the right hand side is finite and then normalize it so that

$$\int_{\mathbb{D}} |\nabla g(z)|^2 (1-|z|^2) dA(z) = 1.$$

Now, if $|z| > \frac{1}{4}$ then we have that $\log\frac{1}{|z|} \leq C(1-|z|^2)$, and so we then have that

$$\int_{\frac{1}{4}\leq|z|\leq 1} |\nabla g(z)|^2 \log\frac{1}{|z|} dA(z) \leq C\int_{\mathbb{D}} |\nabla g(z)|^2 (1-|z|^2) dA(z).$$

In the case when $|z| < \frac{1}{4}$ we exploit the subharmonicity of $|\nabla g(z)|$. By subharmonicity we have that

$$\begin{aligned} |\nabla g(z)|^2 &\leq 16\int_{|\xi-z|<\frac{1}{4}} |\nabla g(\xi)|^2 dA(\xi) \\ &\leq 32\int_{|\xi|<\frac{1}{2}} |\nabla g(\xi)|^2 (1-|\xi|^2) dA(\xi) = 32 \end{aligned}$$

For the last inequality, we have used that for $|z| < \frac{1}{4}$ and $|\xi - z| < \frac{1}{4}$ that $|\xi| < \frac{1}{2}$. We then use the fact that when $|\xi| < \frac{1}{2}$ that $1-|\xi|^2 \geq \frac{3}{4} \geq \frac{1}{2}$. Using this, we see that

$$\int_{|z|<\frac{1}{4}} |\nabla g(z)|^2 \log\frac{1}{|z|} dA(z) \leq C\int_{|z|<\frac{1}{4}} \log\frac{1}{|z|} dA(z) = C.$$

Combining the estimates we have obtained when $|z| \geq \frac{1}{4}$ and when $|z| \leq \frac{1}{4}$ then gives that

$$2\int_{\mathbb{D}} |\nabla g(z)|^2 \log\frac{1}{|z|} dA(z) \leq C\int_{\mathbb{D}} |\nabla g(z)|^2 (1-|z|^2) dA(z).$$

□

Again, by rearrangement of the above Lemma we have another equivalent norm on the space $H^2(\mathbb{D})$

$$|g(0)|^2 + \int_{\mathbb{D}} |g'(z)|^2 (1-|z|^2) dA(z) \leq \|f\|^2_{H^2(\mathbb{D})} \leq C\left(|g(0)|^2 + \int_{\mathbb{D}} |g'(z)|^2 (1-|z|^2) dA(z)\right).$$

Exercise 2.5. *Give an alternate proof of the above equivalent norm on $H^2(\mathbb{D})$ using Fourier series. Doing this, you can obtain a better (in fact sharp) estimate of the constant C.*

3. Examples of Carleson Measures

We now want to collect a couple of different families of Carleson measures that frequently appear. The first is a well known lemma due to Uchiyama.

Lemma 3.1 (Uchiyama's Lemma)**.** *Let φ be a non-negative, bounded, subharmonic function. Then for any $f \in H^2(\mathbb{D})$*

$$\int_{\mathbb{D}} \widetilde{\Delta}\varphi(z)\,|f(z)|^2\,d\mu(z) \leq e\|\varphi\|_\infty \|f\|_2^2.$$

Here $d\mu = \frac{2}{\pi}\log\frac{1}{|z|}dA(z)$, and $\widetilde{\Delta} = \frac{1}{4}\Delta = \partial\overline{\partial}$.

Proof. Because of homogeneity, we can assume without loss of generality that $\|\varphi\|_\infty = 1$. Direct computation shows that

$$\widetilde{\Delta}\big(e^{\varphi(z)}\,|f(z)|^2\big) = e^{\varphi}\widetilde{\Delta}\varphi\,|f|^2 + e^{\varphi}\,|\partial\varphi f + \partial f|^2 \geq \widetilde{\Delta}\varphi\,|f|^2\,.$$

Then Green's formula implies

$$\begin{aligned}
\int_{\mathbb{D}} \widetilde{\Delta}\varphi\,|f(z)|^2\,d\mu(z) &\leq \int_{\mathbb{D}} \widetilde{\Delta}\big(e^{\varphi}\,|f|^2\big)\,d\mu(z) \\
&= \int_{\mathbb{T}} e^{\varphi(\xi)}\,|f(\xi)|^2\,dm(\xi) - e^{\varphi(0)}\,|f(0)|^2 \\
&\leq e\int_{\mathbb{T}} |f(\xi)|^2\,dm(\xi) = e\,\|f\|_{H^2}^2\,.
\end{aligned}$$

□

Remark 3.2. It is easy to see, that the above Lemma implies the embedding

$$\int_{\mathbb{D}} |f(z)|^2\,d\mu(z) \leq C\int_{\mathbb{T}} |f(\xi)|^2\,dm(\xi)$$

(with $C = e$) for all analytic functions f. Using the function $4/(2-\varphi)$ instead of e^{φ} it is possible to get the embedding for harmonic functions with the constant $C = 4$. We suspect the constants e and 4 are the best possible for the analytic and harmonic embedding respectively, though this is still an open question. However, it is known that 4 is the best constant in the dyadic (martingale) Carleson Embedding Theorem.

3.1. **BMO and Carleson Measures.**
Recall that a function $\varphi \in BMO(\mathbb{T})$ if

$$\|\varphi\|_{BMO}^2 = \sup_{z\in\mathbb{D}} |\varphi|^2(z) - |\varphi(z)|^2 < \infty,$$

where $\varphi(z)$ denotes the harmonic extension of φ to $\mathbb{D}$, and $|\varphi|^2(z)$ denotes the harmonic extension of $|\varphi(\xi)|^2$. This is typically called the Garsia norm of the function and is one of many useful norms on this space. Note that the expression on the right hand side of the definition of BMO is always non-negative since we are integrating against a probability measure and a simple application of Cauchy-Schwarz.

Exercise 3.3. *Show that the following norm is an equivalent expression for the norm of a function on BMO. Let $I \subset \mathbb{T}$ be an interval, and let*

$$\varphi_I = \frac{1}{|I|}\int_I \varphi(\xi)dm(\xi).$$

Then an equivalent norm on BMO is given by

$$\|\varphi\|_{BMO}^2 = \sup_{I\subset\mathbb{T}} \frac{1}{|I|}\int_I \left|\varphi(\xi)-\varphi_I\right|^2 dm(\xi). \tag{3.1}$$

Note that we have the following identity holding

$$\int_{\mathbb{T}} \left|\varphi(\xi)-\varphi(z)\right|^2 P_z(\xi)dm(\xi) = \left|\varphi\right|^2(z) - \left|\varphi(z)\right|^2. \tag{3.2}$$

If we apply the conformally invariant version of Green's Theorem to the left hand side of (3.2) then we obtain

$$\left|\varphi\right|^2(z) - \left|\varphi(z)\right|^2 = \int_{\mathbb{D}} \left|\nabla\varphi(w)\right|^2 \log\left|\frac{1-\overline{z}w}{w-z}\right| dA(w) \tag{3.3}$$

Exercise 3.4. *Derive (3.3) from Green's formula for harmonic functions. Hint: First consider the case when $z=0$ and then make a suitable change of variables.*

Using the relationship $\log\frac{1}{t}\approx 1-t$ (which arose when we studied the equivalent norm on $H^2(\mathbb{D})$ and the identity

$$1-\frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2} = \left|\frac{z-w}{1-\overline{z}w}\right|^2$$

we see that

$$\int_{\mathbb{D}} \left|\nabla\varphi(w)\right|^2 \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2} dA(w) \leq \left|\varphi\right|^2(z) - \left|\varphi(z)\right|^2 \leq C\int_{\mathbb{D}} \left|\nabla\varphi(w)\right|^2 \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2} dA(w).$$

These computations then prove the following nice characterization of BMO functions in terms of Carleson measures.

Theorem 3.5. *Suppose that $\varphi\in L^1(\mathbb{T})$. Then $\varphi\in BMO(\mathbb{T})$ if and only if*

$$\left|\nabla\varphi(w)\right|^2(1-|w|^2)dA(w)$$

is a $H^2(\mathbb{D})$ Carleson measure with Carleson measure. Moreover,

$$\|\varphi\|_{BMO} \approx \left\|\left|\nabla\varphi(w)\right|^2(1-|w|^2)dA(w)\right\|_{CM}$$

There are versions of this Theorem that hold true in several variables as well. Namely a function belongs to $BMO(\mathbb{R}^n)$ if

$$\|\varphi\|_{BMO}^2 = \sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|}\int_Q \left|\varphi(\xi)-\varphi_Q\right|^2 d\xi.$$

Let Φ be a smooth function with integral $\int\Phi = 0$. Then one can prove the following theorem

Theorem 3.6 (Stein, [2])**.** *Suppose that $f\in BMO(\mathbb{R}^n)$ and let*

$$d\mu(x,t) = \left|f*\Phi_t(x)\right|\frac{dxdt}{t}.$$

Then $d\mu$ is a Carleson measure.

A converse to this Theorem holds as well, and we point the reader to Stein's book for more details.

References

[1] John B. Garnett, *Bounded analytic functions*, 1st ed., Graduate Texts in Mathematics, vol. 236, Springer, New York, 2007. ↑

[2] Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy; Monographs in Harmonic Analysis, III. ↑6