Lecture 4: Multi-Parameter Calderón–Zygmund Operators

As we saw in the last lecture, if we are given a Calderón–Zygmund operator T, that is represented by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y)f(y)dy$$

and the kernel K satisfies some growth and cancellation conditions, then for $1 < p < \infty$ we have that

$$\|Tf\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

Now we turn to the multi-parameter setting where we consider $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ and are interested in what a Calderón–Zygmund operator is, and if it is bounded. First, note that if the kernel is really a product kernel, namely,

$$K(x-y) = K_1(x_1-y_1)K_2(x_2-y_2)$$

and if each of the kernels $K_j(x_j, y_j)$ satisfy appropriate kernel estimates, then by a simple application of Fubini's Theorem we have that the operator

$$T(f)(x_1, x_2) = \text{p.v.} \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} K_1(x_1-y_1)K_2(x_2-y_2)f(y_1,y_2)dy_1dy_2$$

is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ when $1 < p < \infty$. So in particular we have that the operators H_1H_2, the composition of the Hilbert transform in the first and second variable, is a bounded operator on $L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2)$. A similar statement applies when considering iterations of Riesz transforms.

However, this argument needs to be modified when the kernel isn't of this product type. As soon as we move to the product setting in this regard, it turns out that that problem becomes more interesting. In this lecture we will look at a special case of when multi-parameter Calderón–Zygmund operators are bounded. To focus on ideas and avoid some of the technicalities associated with the objects being studied, we will restrict to the case of two-parameter Calderón–Zygmund operators.

1. Definitions and Statement of the Result

To motivate the definitions in the multi-parameter setting we recall the conditions we need on the kernel in the one-parameter result. We let $K(x)$ be a function on $\mathbb{R}^n$ and will suppose that

(i) $|K(x)| \lesssim |x|^{-n}$;
(ii) $\int_{a<|x|<b} K(x)dx = 0$ for $0 < a < b$;
(iii) $\int_{|x|>2|h|} |K(x+h) - K(x)|\, dx \lesssim 1$ for all $h \neq 0$.

Then using the tools developed in Lecture 3, one can show that the operator $T(f)(x) = f * K(x)$ is bounded on $L^p(\mathbb{R}^n)$ when $1 < p < \infty$.

Now to state the results when the kernel K is supposed to be multi-parameter, we consider a function $K(x,y)$ defined on $\mathbb{R}^n \times \mathbb{R}^m$ that satisfies the following list of conditions:

(a) $|K(x)| \lesssim |x|^{-n} |y|^{-m}$;

(b1) $\left|\int_{a<|x|<b} K(x,y)dx\right| \lesssim |y|^{-m}$ for $0<a<b$ for all $y \in \mathbb{R}^m$;

(b2) $\left|\int_{a<|y|<b} K(x,y)dy\right| \lesssim |x|^{-n}$ for $0<a<b$ for all $x \in \mathbb{R}^n$;

(b3) $\left|\int_{a<|y|<b}\int_{c<|x|<d} K(x,y)dxdy\right| \lesssim 1$ for $0<a<b$, $0<c<d$;

(c1) $\int_{|x|>2|h|} |K(x+h,y)-K(x,y)|\,dx \lesssim |y|^{-m}$ for all $h \neq 0$ and for all $y \in \mathbb{R}^m$;

(c2) $\int_{|y|>2|k|} |K(x,y+k)-K(x,y)|\,dy \lesssim |x|^{-n}$ for all $h \neq 0$ and for all $x \in \mathbb{R}^n$;

(c3) $\int_{|y|>2|k|}\int_{|x|>2|h|} |K(x+h,y+k)-K(x+h,y)-K(x,y+k)+K(x,y)|\,dxdy \lesssim 1$ for all $h \neq 0$ and $k \neq 0$;

(d1) If $G(x) = \int_{a<|y|<b} K(x,y)dy$, then

$$\int_{|x|>2|h|} |G(x+h)-G(x)|\,dx \lesssim 1$$

uniformly in a,b;

(d2) If $H(y) = \int_{a<|x|<b} K(x,y)dx$, then

$$\int_{|y|>2|h|} |H(y+h)-H(y)|\,dx \lesssim 1$$

uniformly in a,b.

These conditions that we impose may seem unnecessarily technical. However, to have a kernel be a "multi-parameter" kernel, we really will need information about how the functions behave when we restrict the number of variables, and how they the variables interact with each other.

As a class of examples of kernels that satisfy these conditions, one can take a kernel $K(x,y)$ that is smooth away from the "fat diagonal" $\{(x,y) : x=0 \text{ or } y=0\}$ and satisfies

$$|K(x,y)| \lesssim \frac{1}{|x|^n\,|y|^m},$$

$$|\nabla_x K(x,y)| \lesssim \frac{1}{|x|^{n+1}\,|y|^m} \quad |\nabla_y K(x,y)| \lesssim \frac{1}{|x|^n\,|y|^{m+1}}$$

and

$$|\nabla_x\nabla_y K(x,y)| \lesssim \frac{1}{|x|^{n+1}\,|y|^{m+1}},$$

and

$$\int_{r<|x|<R} K(x,y)dx = \int_{r<|y|<R} K(x,y)dy = 0 \quad 0<r<R.$$

In particular this handles kernels more general than the products of Riesz kernels.

Observe that what we have done is expanded the conditions for the standard Calderón–Zygmund kernels and considered appropriate size and cancellation conditions uniformly in each variable.

We can now prove the following Theorem

Theorem 1.1 (Fefferman, [2]). *Let K be a kernel that satisfies* (a), (b1), (b2), (c1), (c2), (c3) *above. Then the operator $Tf(x) := f * K(x)$ is bounded on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$, i.e.,*

$$\|Tf\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)} \lesssim \|f\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}.$$

Proof. Our strategy will be to show that under the hypotheses above, we have that the Fourier transform of K is bounded on all of $\mathbb{R}^n \times \mathbb{R}^m$. If we demonstrate this, then we have

$$\|Tf\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)} = \left\|\hat{K}\hat{f}\right\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)} \lesssim \|f\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}.$$

We now turn to computing $\hat{K}$. The idea will be to split the domain $\mathbb{R}^n \times \mathbb{R}^m$ into four pieces. One each piece, we will use some of the properties of the kernel to obtain the desired estiamte. Now by dilation invariance of the kernel and the estimates in question, we can suppose that $|\eta| = |\xi| = 1$. Note that by the definition of the Fourier transform we have, for some $R > 10$,

$$\begin{aligned}
\hat{K}(\eta,\xi) &= \int_{\mathbb{R}^n\times\mathbb{R}^m} K(x,y)e^{i(\eta x+\xi y)}dxdy \\
&= \int_{|x|<R}\int_{|y|<R} + \int_{|x|<R}\int_{|y|\geq R} + \int_{|x|\geq R}\int_{|y|<R} + \int_{|x|\geq R}\int_{|y|\geq R} \left(K(x,y)e^{i(\eta x+\xi y)}\right) dxdy \\
&= I + II + III + IV.
\end{aligned}$$

Note that II and III are symmetric with respect to the variables x and y and so if suffices to prove the boundedness of one of the desired integrals since the other will follow by the same argument. We thus need to show the boundedness of the following integrals:

$$\begin{aligned}
I &= \int_{|x|<R}\int_{|y|<R} K(x,y)e^{i(\eta x+\xi y)}dxdy \\
III &= \int_{|x|>R}\int_{|y|\leq R} K(x,y)e^{i(\eta x+\xi y)}dxdy \\
IV &= \int_{|x|\geq R}\int_{|y|\geq R} K(x,y)e^{i(\eta x+\xi y)}dxdy
\end{aligned}$$

Lets consider the term III. We can write this as

$$\begin{aligned}
III &= \int_{|x|>R}\int_{|y|\leq R} K(x,y)e^{i(\eta x+\xi y)}dxdy \\
&= \int_{|x|>R}\int_{|y|\leq R} K(x,y)e^{i\eta x}\left(1+e^{i\xi y}-1\right)dxdy \\
&= III_1 + III_2.
\end{aligned}$$

Consider now term III_2. Notice that for each fixed y, we have

$$\begin{aligned} III_2 &= \int_{|y|\leq R} \left(e^{i\xi y}-1\right)\left(\int_{|x|>R} K(x,y)e^{i\eta x}dx\right)dy \\ &= \int_{|y|\leq R} \left(e^{i\xi y}-1\right)\left(\int_{|x|>R} e^{i\eta x}\left(K(x,y)-K(x+\pi\eta,y)\right)dx\right)dy \\ &\quad + \int_{|y|\leq R} \left(e^{i\xi y}-1\right)\left(\int_{|x|>R} e^{i\eta x}K(x+\pi\eta,y)dx\right)dy \end{aligned}$$

Consider the first term that appears above, for the inner integral, we use condition (c1) to conclude that

$$\int_{|x|>R} |K(x,y)-K(x+\pi\eta,y)|\,dx \lesssim |y|^{-m}$$

Couple this with the following estimate to see that

$$\begin{aligned} \int_{|y|\leq R} \left(e^{i\xi y}-1\right)\left(\int_{|x|>R} e^{i\eta x}\left(K(x,y)-K(x+\pi\eta,y)\right)dx\right)dy &\lesssim \int_{|y|\leq R} |y|^{-m}\left|e^{i\xi y}-1\right|dy \\ &\lesssim 1, \end{aligned}$$

with the last estimate following by direct computations.

Now for the remaining error term, we make the following observation geometric observation. For the choice of R and η, we have that, after a change of variables, that the inner integral in the second remaining term is given by

$$\int_S K(x,y)e^{ix\eta}dx$$

where $S \subset \{x : R-4 < |x| < R+4\}$. So to estimate this term, we now use condition (a), and then we have that

$$\begin{aligned} \int_{|y|\leq R} \left(e^{i\xi y}-1\right)\left(\int_{|x|>R} e^{i\eta x}K(x+\pi\eta,y)dx\right)dy &\lesssim \int_{|y|\leq R} |y|^{-m}\left|e^{i\xi y}-1\right|dy \int_S |x|^{-n}\,dx \\ &\lesssim 1. \end{aligned}$$

Next, turn to term III_1, then note that

$$III_1 = \int_{|x|>R} e^{i\eta x}\int_{|y|\leq R} K(x,y)dydx$$

The idea is similar to what appeared above, so we just sketch more of the computations. We can write the term III_1 as

$$|III_1| \leq \int_{|x|>R} |G(x+\pi\eta)-G(x)|\,dx + \int_S |G(x)|\,dx.$$

Here S is the same set that appears above. This is then finite since the first term above can be controlled by the condition appearing in (d1). The second term can be controlled since $|G(x)| \lesssim |x|^{-n}$ by (b2), and we have already computed that the integral of $|x|^{-n}$ over S is finite. This then completes the estimate of term III.

The estimate for II is identical to what we did for III, but we just interchanged the roles of x and y, and then use the appropriate hypotheses on the kernel. It also turns out that all the techniques we used in estimating term III can be used to handle the terms in I and

IV as well. Since the proof is essentially the same in each of the remaining cases, we simply point out the necessary steps and leave the computations and verification to the reader.

To handle the term I, note that we can write

$$e^{ix\eta}e^{iy\xi} = (e^{ix\eta} - 1)(e^{iy\xi} - 1) + (e^{iy\xi} - 1) + (e^{ix\eta} - 1) + 1$$

This implies that we can write term I as

$$\begin{aligned}
\int_{|x|<R}\int_{|y|<R} K(x,y)e^{i(\eta x+\xi y)}dxdy &= \int_{|x|<R}\int_{|y|<R} K(x,y)(e^{ix\eta} - 1)(e^{iy\xi} - 1)dxdy \\
&\quad + \int_{|x|<R}\int_{|y|<R} K(x,y)(e^{iy\xi} - 1)dxdy \\
&\quad + \int_{|x|<R}\int_{|y|<R} K(x,y)(e^{ix\eta} - 1)dxdy \\
&\quad + \int_{|x|<R}\int_{|y|<R} K(x,y)dxdy \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Form term I_1 use the fact that $|e^{ix\eta} - 1| \lesssim |x|$ and $\left|e^{iy\xi} - 1\right| \lesssim |y|$ along with the kernel estimates $|K(x,y)| \lesssim |x|^{-n} |y|^{-m}$ to obtain

$$|I_1| \lesssim \int_{|x|<R}\int_{|y|<R} |x|^{-n+1} |y|^{-m+1} dxdy \lesssim 1.$$

The terms I_2 and I_3 one uses ideas from above. One can see that in this case, we will use the fact that $|e^{ix\eta} - 1| \lesssim |x|$ and $\left|e^{iy\xi} - 1\right| \lesssim |y|$ and then apply estimates from (b1), (b2), (c1), and (c2), (d1), (d2) to conclude that the terms are finite. The term I_4 is also handled in a similar fashion, except now one uses the estimates in (c3) and the kernel estimates in (a).

Finally, to handle term IV one simply writes

$$e^{i(x\eta+y\xi)} = -e^{i\left(x+\frac{\pi}{2}\eta\right)\eta}e^{i\left(y+\frac{\pi}{2}\xi\right)\xi}$$

and then split the integral to be estimated into pieces for which the hypotheses can be applied.

□

2. Extensions of The Main Result

It is also possible to prove an L^p version of the above Theorem. We won't state the result at this point since the conditions to impose on the kernel are slightly different than what we have already been discussing. See the paper by Fefferman, [2], for the exact hypothesis needed. The method of proof is to incorporate the square function into the mix. Since we haven't introduced the multi-parameter square function at this point, we will just take this as a fact for now. We unfortunately can't apply the techniques of proving a weak-type $(1,1)$ estimate (since it isn't true) and then interpolating. However, if we impose slightly stronger conditions on the kernel, then it is possible to prove the following Theorem.

Theorem 2.1 (Fefferman, [2])**.** *Suppose that $K(x,y)$ is defined on $\mathbb{R}^2$ and is smooth away from the coordinate axes and satisfies*

(a) $\left|\partial_x^\alpha \partial_y^\beta K(x)\right| \lesssim |x|^{-\alpha-1} |y|^{-\beta-1}$ *for all $\alpha, \beta \geq 0$;*

(b1) $\left|\int_{c<|y|<d}\int_{a<|x|<b} K(x,y)dxdy\right| \lesssim 1$ *for* $0<a<b$ *and* $0<c<d$*;*

(b2) $\left|\int_{a<|x|<b} K(x,y)dx\right| \lesssim |y|^{-1}$ *for* $0<a<b$ *for all* $y\in\mathbb{R}$*;*

(b3) $\left|\int_{a<|y|<b} K(x,y)dy\right| \lesssim |x|^{-1}$ *for* $0<a<b$ *for all* $x\in\mathbb{R}$*.*

(c1) *If* $G(x)=\int_{a<|y|<b} K(x,y)dy$ *then* $G(x)$ *is smooth and*

$$|\partial_x^\gamma G| \lesssim |x|^{-\gamma-1}$$

uniformly in a,b*;*

(c1) *If* $H(y)=\int_{a<|x|<b} K(x,y)dx$ *then* $H(y)$ *is smooth and*

$$\left|\partial_y^\gamma H\right| \lesssim |y|^{-\gamma-1}$$

uniformly in a,b*.*

Then if $Tf(x,y)=f*K(x,y)$ *we have*

$$T : L\left(\log^+ L\right)(\mathbb{R}^2) \to L^{1,\infty}(\mathbb{R}^2).$$

The idea of proof is similar to what appears in the proof of Theorem 1.1. One notes that the hypotheses of this theorem allow one to prove $\hat{K}(\xi,\eta), \xi\hat{K}(\xi,\eta), \eta\hat{K}(\xi,\eta) \in L^\infty(\mathbb{R}^2)$. This allows one to apply a version of the Marcinkiewicz Multiplier Theorem to conclude that the operator $T(f)=f*K$ is bounded for all $1<p<\infty$.

Another more general extension of this result is given by Fefferman and Stein in [1]. Suppose that $K(x,y)$, $(x,y)\in\mathbb{R}^n\times\mathbb{R}^m$, is locally integrable away from the cross $\{x=0\}\cup\{y=0\}$.

(a) $\left|\int_{a_1<|x|<a_2}\right|\int_{b_1<|y|<b_2} K(x,y)dxdy \lesssim 1$ for all $0<a_1<a_2$ and $0<b_1<b_2$;

(b) Let $K_1(x)=\int_{b_1\le|y|\le b_2} K(x,y)dy=0$ then

$$\int_{|x|\le r} |x|\,|K_1(x)|\,dx \lesssim r$$

and

$$\int_{|x|\ge 2|h|} |K_1(x+h)-K_1(x)|\,dx \lesssim 1;$$

A similar condition holding for $K_2(y)=\int_{a_1<|x|<a_2} K(x,y)dx$.

(c) $\int_{|y|<r_2}\int_{|x|<r_1} |x|\,|y|\,|K(x,y)|\,dxdy \lesssim r_1r_2$ for all $0<r_1,r_2<\infty$;

(d1) $\int_{|y|<r_2}\int_{|x|\ge2|h|} |y|\,|K(x+h,y)-K(x,y)|\,dxdy \lesssim r_2$ for all $0<r_2<\infty$ and $h\neq0$;

(d2) $\int_{|x|<r_1}\int_{|y|\ge2|k|} |x|\,|K(x,y+k)-K(x,y)|\,dxdy \lesssim r_1$ for all $0<r_1<\infty$ and $k\neq0$;

(d3) $\int_{|x|\ge2|h|}\int_{|y|\ge2|k|} |K(x+h,y+k)-K(x+h,y)-K(x,y+k)+K(x,y)|\,dxdy \lesssim 1$ for all $h\neq0$ and $k\neq0$.

Theorem 2.2 (Fefferman and Stein, [1])**.** *Let* K *be a kernel that satisfies the above conditions, then we have that the operator* $Tf=f*K$ *satisfies*

$$T : L^2(\mathbb{R}^n\times\mathbb{R}^m) \to L^2(\mathbb{R}^n\times\mathbb{R}^m)$$

with norm depending on the implied constants in the hypotheses imposed upon the kernel.

The idea of proof is to again show that the function $\hat{K} \in L^\infty(\mathbb{R}^n \times \mathbb{R}^m)$. We leave this to the interested reader to check since the main ideas how to split the integral can be seen in Theorem 1.1.

References

[1] Robert Fefferman and Elias M. Stein, *Singular integrals on product spaces*, Adv. in Math. **45** (1982), no. 2, 117–143. ↑6

[2] Robert Fefferman, *Singular integrals on product domains*, Bull. Amer. Math. Soc. (N.S.) **4** (1981), no. 2, 195–201. ↑3, 5