Lecture 2: Strong Maximal Functions and Covering Lemmas

We now turn our attention to a different type of maximal function. In Lecture 1 we looked at maximal functions associated to cubes or, equivalently balls. These geometric objects are in principle described by one piece of data, the side length or the radius. However, we could form maximal functions with respect to more general collections of sets and attempt to study what happens. In this lecture we turn our attention to the maximal operator formed with respect to rectangles. We now see a first difference between one parameter harmonic analysis and multi-parameter harmonic analysis.

1. Definitions and Basic Properties

Let $\mathcal{R}$ denote the collection of rectangles R in $\mathbb{R}^n$ with sides parallel to the coordinate axes. Then we define the strong maximal function to be the operator

$$M_S f(x) = \sup_{R \ni x} \frac{1}{|R|} \int_R |f(y)|\, dy.$$

Once again it is trivial to see that $\|M_S f\|_\infty \leq \|f\|_\infty$. Next, if we denote by M_j the one dimensional maximal function in the jth coordinate, then it isn't difficult to see that

$$M_S f \leq M_1 M_2 \cdots M_n f.$$

Thus, if we want to understand the behavior of $M_S : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ when $1 < p < \infty$, it turns out to be rather easy. Since the maximal function M_j will be bounded on $L^p(\mathbb{R})$ by the results in Lecture 1, we have that an iteration of them will be bounded on $L^p(\mathbb{R}^n)$ by applications of Fubini's Theorem along with the boundedness of the maximal functions M_j.

Note that this iteration argument will fail us when we turn to the L^1 behavior of M_S. Now we know by the results in Lecture 1, that each M_j has a weak $(1,1)$ bound. We then have for two iterations $M_1 M_2 f$ that

$$\|M_1 M_2 f\|_{L^{1,\infty}} \lesssim \|M_2 f\|_{L^1},$$

but at this stage we run into a problem since the maximal function need not preserve L^1. However, it is true that the one dimensional maximal functions take $L \log L$ into L^1. So we see that we should expect a weak-type estimate that is closer to $L \log L$ in flavor.

In this lecture we will give a direct proof (not via iteration) that the strong maximal function is bounded on $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$. Namely,

Theorem 1.1. *Let $1 < p \leq \infty$. Then there exists a constant $C(p,n)$ such that*

$$\|M_S f\|_{L^p(\mathbb{R}^n)} \leq C(p,n) \|f\|_{L^p(\mathbb{R}^n)}.$$

Of course the maximal function M_S has some appropriate weak-type estimates near $L^1(\mathbb{R}^n)$. We won't give the proof of this fact, but instead point out what the correct growth should be and give the more refined covering lemma that can be proved.

2. Estimates for the Strong Maximal Function

In this section we turn to understanding the proof of the covering lemma that will be useful to prove the L^p boundedness of the strong maximal function. We will accomplish this by proving a covering lemma and then proving that covering lemmas are in fact equivalent to estimates for the maximal function.

Theorem 2.1 (A Covering Lemma for the Strong Maximal Theorem)**.** *Suppose that $\{R_j\}_{j\in J}$ is a family of rectangles in $\mathbb{R}^n$ with sides parallel to the axes. Then there is a sequence of rectangles $\{\tilde{R}_k\} \subset \{R_j\}_{j\in J}$ that satisfy*

(a) $\left|\bigcup_{j\in J} R_j\right| \leq c(n) \left|\bigcup_k \tilde{R}_k\right|$

(b) $\left\|\left(\sum_k \chi_{\tilde{R}_k}\right)\right\|_{L^p(\mathbb{R}^n)}^p \leq C(n) \left|\bigcup_{j\in J} R_j\right|$

The main point behind (a) is that it is possible to select a sub-collection of the given rectangles so that the size of the collected set is still comparable to the original collection.

Proof. The proof is by induction on the dimension. Suppose that the Theorem is true for $n-1$, then we will show how to get the Lemma in the case of n. Note that since the Theorem is true for $n-1$ by Theorem 1.1 we have that the maximal operator $M_S^{n-1} : L^p(\mathbb{R}^{n-1}) \to L^p(\mathbb{R}^{n-1})$ is bounded when $1 < p < \infty$.

For a rectangle $S \subset \mathbb{R}^n$, let $\hat{S}$ denote the rectangle obtained by keeping all the sides of S the same except the nth side, which we triple in size. Given the collection $\{R_j\}$, first relabel if necessary so that the R_j are ordered so that their nth sidelengths decrease. We now give a selection procedure to find the desired collection of rectangles. Take $\tilde{R}_1 = R_1$ and let $\tilde{R}_2$ be the first rectangle R_k such that

$$\left|R_k \cap \tilde{R}_1\right| < \frac{1}{2}\left|R_k\right|.$$

Suppose have now chosen the rectangles $\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_{j-1}$. We select $\tilde{R}_j$ to be the first rectangle occurring after $\tilde{R}_{j-1}$ so that

$$\left|R_k \cap \left(\bigcup_{l<j} \hat{\tilde{R}}_l\right)\right| < \frac{1}{2}\left|R_k\right|.$$

Choose any point x inside a rectangle R_i that is **not** one of the selected rectangles $\tilde{R}_j$. Slice a rectangle R on the nth coordinate by a hyperplane that is perpendicular to the nth axis at a height given by the nth coordinate of x. Note that these produce $n-1$ dimensional rectangles T. And,

$$\left|T_i \cap \left(\bigcup_j \hat{\tilde{T}}_j\right)\right|_{n-1} \geq \frac{1}{2}\left|T_i\right|_{n-1} \tag{2.1}$$

where we use the sub-script to denote $n-1$ dimensional Lebesgue measure. (Note that we had given an incorrect argument at this point before. It wasn't such a simple contradiction argument as I had indicated). We write $R = T \times I$ for the rectangles. To see this last inequality note that if R_i is not a selected rectangle, the we must have

$$\left| R_i \cap \left(\bigcup_j \tilde{\hat{R}}_j \right) \right|_{n-1} \geq \frac{1}{2} |R_i| = \frac{1}{2} |T_i|_{n-1} |I_i| .$$

Note that we haven't used the condition on the decreasing side lengths yet. Here we now exploit this fact to conclude that

$$\left| T_i \cap \left(\bigcup_j \tilde{\hat{T}}_j \right) \right|_{n-1} \geq \frac{1}{2} |T_i|_{n-1} .$$

Now note that (2.1) implies that

$$M_S^{n-1} \left(\chi_{\bigcup \tilde{\hat{T}}_j} \right)(x) \geq \frac{1}{2} \tag{2.2}$$

and so we have for all points $x \in \cup T_i$ that

$$\bigcup \hat{T}_i \subset \left\{ M_S^{n-1} \left(\chi_{\bigcup \tilde{\hat{T}}_j} \right)(x) \geq \frac{1}{2} \right\} .$$

Since for points in selected rectangles, this containment is obvious, while for points in the non-selected rectangles, (2.2) gives the containment. This then implies

$$\begin{aligned} \left| \bigcup T_i \right|_{n-1} &\leq \left| \left\{ M_S^{n-1} \left(\chi_{\bigcup \tilde{T}_j} \right)(x) \geq \frac{1}{2} \right\} \right| \\ &\leq C \int_{\mathbb{R}^{n-1}} \chi_{\bigcup \tilde{\hat{T}}_j}(x) dx \\ &= C \left| \bigcup \tilde{\hat{T}}_j \right|_{n-1} . \end{aligned}$$

Here we have used the induction hypothesis that $M_S^{n-1} : L^p(\mathbb{R}^{n-1}) \to L^p(\mathbb{R}^{n-1})$ is bounded when $1 < p < \infty$ coupled with Chebyshev's Inequality for p. Finally, integrate this inequality over the nth coordinate to arrive at

$$\left| \bigcup R_i \right| \leq C \left| \bigcup \tilde{R}_j \right|$$

which proves (i) in the Covering Lemma.

Now, we turn to (ii). We want to show that the function $\chi_{\bigcup \tilde{R}}$ belongs to $L^p(\mathbb{R}^n)$. To accomplish this, we will use a duality argument and test it against $f \in L^{p'}(\mathbb{R}^n)$ and show we have good estimates.

With this in mind, note that if we define the sets

$$\tilde{E}_j = \tilde{\hat{T}}_j \setminus \bigcup_{l<j} \tilde{\hat{T}}_l$$

then (2.1) gives that

$$|E_j|_{n-1} > \frac{1}{2} \left| \hat{T}_j \right|_{n-1} .$$

Choose $f \in L^{p'}(\mathbb{R}^{n-1})$ of norm at most 1 (here we identify $\mathbb{R}^{n-1}$ with the first $n-1$ coordinates). Then we have

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} \sum_j \chi_{\tilde{T}_j}(x) f(x)dx &= \sum_j \int_{\tilde{T}_j} f(x)dx \\
&= \sum_j \frac{\left|\tilde{T}_j\right|}{\left|\tilde{T}_j\right|} \int_{\tilde{T}_j} f(x)dx \\
&\leq 2\sum_j |E_j| \frac{1}{\left|\tilde{T}_j\right|} \int_{\tilde{T}_j} f(x)dx \\
&\leq 2\sum_j |E_j|\, M_S^{n-1} f(x).
\end{aligned}$$

The last inequality holds for any $x \in \tilde{\tilde{T}}_j$. This last sum can be recovered as in integral over the desired sets. In particular, we end up with

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} \sum_j \chi_{\tilde{\tilde{T}}_j}(x) f(x)dx &\lesssim \int_{\bigcup_k \tilde{\tilde{T}}_k} M_S^{n-1} f(x)dx \\
&\leq \left\|M_S^{n-1} f\right\|_{L^{p'}(\mathbb{R}^{n-1})} \left|\bigcup \tilde{\tilde{T}}_k\right|^{\frac{1}{p}} \\
&\leq \|f\|_{L^{p'}(\mathbb{R}^{n-1})} \left|\bigcup \tilde{\tilde{T}}_k\right|^{\frac{1}{p}} \\
&\leq \left|\bigcup \tilde{\tilde{T}}_k\right|^{\frac{1}{p}}.
\end{aligned}$$

By duality, this last inequality gives that

$$\left\|\sum_j \chi_{\tilde{\tilde{T}}_j}\right\|_{L^p(\mathbb{R}^{n-1})} \lesssim \left|\bigcup \tilde{\tilde{T}}_k\right|^{\frac{1}{p}}.$$

Now raise this inequality to the p power and integrate over the remaining coordinate to arrive at the statement in (ii). □

A collection of rectangles has the covering property V_q, $1 \leq q \leq \infty$ if there exists constants $C_1 < \infty$ and $C_2 > 0$ so that given any family $\{R_j\}_J$ we can find a sequence $\{\tilde{R}_k\} \subset \{R_j\}_J$ such that

(a) $\left|\bigcup_{j\in J} R_j\right| \leq C_2(n) \left|\bigcup_k \tilde{R}_k\right|$

(b) $\left\|\left(\sum_k \chi_{\tilde{R}_k}\right)\right\|_{L^q(\mathbb{R}^n)}^q \leq C_1(n) \left|\bigcup_{j\in J} R_j\right|$.

Theorem 2.2 (Cordoba-Fefferman, [3]). *The maximal operator M_S is of weak-type (p,p) if and only if the collection of rectangles has the covering property V_q where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p \leq \infty$.*

It is easy to see how Theorem 2.2 and Theorem 2.1 can be used to prove Theorem 1.1.

Exercise 2.3. *Prove Theorem 1.1 by applying Theorem 2.2 and Theorem 2.1.*

Proof of Theorem 2.2. One direction is immediate. Suppose the collection of rectangles has the covering property V_q. Fix $\lambda > 0$ and consider the set

$$E_\lambda = \{x \in \mathbb{R}^n : M_S f(x) > \lambda\}.$$

Then we have that $E_\lambda = \bigcup_j R_j$ where for each rectangle R_j we have

$$\lambda < \frac{1}{|R_j|} \int_{R_j} |f(y)|\, dy.$$

Now apply the definition of V_q property to select a subcollection $\tilde{R}_k$. Then we have

$$\begin{aligned}
|E_\lambda| &= \left|\bigcup_j R_j\right| \\
&\leq C_2 \left|\bigcup_k \tilde{R}_k\right| \\
&\leq \frac{C_2}{\lambda} \sum_k \int_{\tilde{R}_k} |f(y)|\, dy \\
&= \frac{C_2}{\lambda} \int_{\mathbb{R}^n} \sum_k \chi_{\tilde{R}_k}(y)\, |f(y)|\, dy \\
&\leq \frac{C_2}{\lambda} \left\| \sum_k \chi_{\tilde{R}_k} \right\|_{L^q(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} \\
&\leq \frac{C_1 C_2}{\lambda} \left|\bigcup_j R_j\right|^{\frac{1}{q}} \|f\|_{L^p(\mathbb{R}^n)} \\
&= \frac{C_1 C_2}{\lambda} |E_\lambda|^{\frac{1}{q}} \|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

This then gives that

$$|E_\lambda| \leq \frac{C}{\lambda^p} \|f\|_{L^p(\mathbb{R}^n)}^p$$

which is the desired weak-type estimate.

For the converse, suppose that M_S satisfies the weak-type estimates. Given a family of rectangles $\{R_j\}$, we assume the existence of a subsequence $\{\tilde{R}_k\}$ such that

$$\left|\bigcup_j R_J\right| \leq C \left|\bigcup_k \tilde{R}_k\right|$$

and with the following disjointness property:

$$\left|\tilde{R}_k \cap \bigcup_{j<k} \tilde{R}_j\right| \leq \frac{1}{2} \left|\tilde{R}_k\right| \quad \forall k.$$

Then for this sequence we will also have that

$$\left\| \sum_k \chi_{\tilde{R}_k} \right\|_{L^q(\mathbb{R}^n)} \leq C \left|\bigcup_j R_j\right|^{\frac{1}{q}}.$$

Based on these sets, define $E_k = \tilde{R}_k \setminus \bigcup_{j<k} \tilde{R}_j$ and a related operator

$$Tf(x) = \sum_k \left(\frac{1}{\left|\tilde{R}_k\right|} \int_{\tilde{R}_k} f(y)dy \right) \chi_{E_k}(x).$$

We clearly have $|Tf(x)| \leq Mf(x)$ and for its adjoint

$$T^* f(x) = \sum_k \left(\frac{1}{\left|\tilde{R}_k\right|} \int_{E_k} f(y)dy \right) \chi_{\tilde{R}_k}(x)$$

we have $T^* \left(\chi_{\bigcup_k \tilde{R}_k} \right) \geq \frac{1}{2} \sum_k \chi_{\tilde{R}_k}$.

Since M_S satisfies a weak-type estimate, then T also satisfies the corresponding weak-type estimate. If we consider the action of the dual operator T^*, then it is also bounded between certain function spaces and we have

$$\left\| \sum_k \chi_{\tilde{R}_k} \right\|_{L^q(\mathbb{R}^n)} \leq C \left\| T^* \left(\chi_{\bigcup_k \tilde{R}_k} \right) \right\|_{L^q(\mathbb{R}^n)} \leq C \left| \bigcup_k \tilde{R}_k \right|^{\frac{1}{q}}.$$

It only remains to show how we can select the sequence of rectangles that has the additional disjointness property. This idea is similar to what we did in the proof of the covering lemma.

To see this, we set $\tilde{R}_1 = R_1$. Suppose that we have selected $\tilde{R}_2, \ldots, \tilde{R}_n$. To select $\tilde{R}_{n+1}$ look for the first rectangle in the sequence $\{R_j\}$ after $\tilde{R}_n$ with the property that

$$\left| \tilde{R}'_{n+1} \cap \left(\bigcup_{j \leq n} \tilde{R}_j \right) \right| < \frac{1}{2} \left| \tilde{R}_{n+1} \right|.$$

Now we claim that

$$\bigcup_j R_j \subset \left\{ x \in \mathbb{R}^n : M_S \left(\chi_{\bigcup \tilde{R}_k} \right)(x) \geq \frac{1}{2} \right\}.$$

Note that the claim implies that

$$\begin{aligned} \left| \bigcup_j R_j \right| &\leq \left| \left\{ x \in \mathbb{R}^n : M_S \left(\chi_{\bigcup \tilde{R}_k} \right)(x) \geq \frac{1}{2} \right\} \right| \\ &\lesssim \left\| \chi_{\bigcup \tilde{R}_k} \right\|_{L^p(\mathbb{R}^n)}^p \\ &= \left| \bigcup_k \tilde{R}_k \right|. \end{aligned}$$

□

3. The Strong Maximal Theorem and Behavior Near L^1

Being more careful in the selection procedure of the covering lemma above, one can prove the following better covering result.

Theorem 3.1 (Cordoba, Fefferman)**.** *Suppose that* $\{R_j\}_{j\in J}$ *is a family of rectangles in* $\mathbb{R}^n$ *with sides parallel to the axes. Then there is a sequence of rectangles* $\{\tilde{R}_k\} \subset \{R_j\}_{j\in J}$ *that satisfy*

(a) $\left|\bigcup_{j\in J} R_j\right| \leq c(n)\left|\bigcup_k \tilde{R}_k\right|$

(b) $\left\|\exp\left(\sum_k \chi_{\tilde{R}_k}\right)^{\frac{1}{n-1}}\right\|_{L^1(\mathbb{R}^n)} \leq C(n)\left|\bigcup_{j\in J} R_j\right|$

If one uses this covering Lemma in conjunction with the methods of this Lecture we arrive at the weak-type estimates associated to the strong maximal function.

Theorem 3.2. *The operator* M_S *is bounded from the Orlicz space* $L(\log L)^{n-1}(\mathbb{R}^n)$ *to* $L^{1,\infty}(\mathbb{R}^n)$*. Namely, for any* $f \in L(\log L)^{n-1}$ *and any* $\lambda > 0$

$$|\{x \in \mathbb{R}^n : M_s f(x) > \lambda\}| \leq \frac{C}{\lambda}\int_{\mathbb{R}^n} |f(x)|\left(1 + \left(\log\frac{|f(x)|}{\lambda}\right)^{n-1}\right) dx$$

Exercise 3.3. *Carry out the details to arrive at the above Theorem.*

References

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