Lecture 8: Nevanlinna-Pick Interpolation in $H^\infty(\mathbb{D})$

Remark. Since this is a shorter week because of the Thanksgiving Holiday, the lecture will be shorter as well.

We next turn to studying another interpolation problem in $H^\infty(\mathbb{D})$. Given a finite set of distinct points $Z = \{z_1, \ldots, z_n\}$ (the *nodes*) and a finite set $W = \{w_1, \ldots, w_n\}$ (the *targets*), we want to characterize when it is possible to find a function $f \in H^\infty(\mathbb{D})$ with $\|f\|_\infty \leq 1$ and $f(z_j) = w_j$.

First, lets see that for this problem to have a solution, there is an obvious necessary condition. Let M_f denote the operator of multiplication by f acting on the Hardy space $H^2(\mathbb{D})$. Since $f \in H^\infty(\mathbb{D})$ we have that this operator is bounded and $\|M_f\| = \|M_f^*\| = \|f\|_\infty \leq 1$. Let $(a_1, \ldots, a_n) \in \mathbb{C}^n$ and consider the function

$$\sum_{j=1}^n a_j k_{z_j}(z)$$

where k_{z_j} is the reproducing kernel for $H^2(\mathbb{D})$. Then, we have

$$\left\| M_f^* \left(\sum_{j=1}^n a_j k_{z_j} \right) \right\|_{H^2(\mathbb{D})}^2 \leq \left\| \sum_{j=1}^n a_j k_{z_j} \right\|_{H^2(\mathbb{D})}^2$$

We then compute both the left hand side and the right hand side of the above inequality. Using that $M_f^* k_\lambda = \overline{f(\lambda)} k_\lambda$ for any $\lambda \in \mathbb{D}$ we see that

$$\begin{aligned}
\left\| M_f^* \left(\sum_{j=1}^n a_j k_{z_j} \right) \right\|_{H^2(\mathbb{D})}^2 &= \left\langle M_f^* \left(\sum_{j=1}^n a_j k_{z_j} \right), M_f^* \left(\sum_{j=1}^n a_j k_{z_j} \right) \right\rangle_{H^2(\mathbb{D})} \\
&= \left\langle \sum_{j=1}^n \overline{f(z_j)} a_j k_{z_j}, \sum_{j=1}^n \overline{f(z_j)} a_j k_{z_j} \right\rangle_{H^2(\mathbb{D})} \\
&= \left\langle \sum_{j=1}^n \overline{w_j} a_j k_{z_j}, \sum_{j=1}^n \overline{w_j} a_j k_{z_j} \right\rangle_{H^2(\mathbb{D})} \\
&= \sum_{j,k=1}^n \overline{w_j} w_k \left\langle k_{z_j}, k_{z_k} \right\rangle_{H^2(\mathbb{D})} a_j \overline{a_k} \\
&= \sum_{j,k=1}^n \frac{\overline{w_j} w_k}{1 - \overline{z_j} z_k} a_j \overline{a_k}.
\end{aligned}$$

Here we have used the fact that $f(z_j) = w_j$ for $j = 1, \ldots, n$. Similar computations show that

$$\left\| \sum_{j=1}^n a_j k_{z_j} \right\|_{H^2(\mathbb{D})}^2 = \sum_{j,k=1}^n \frac{1}{1 - \overline{z_j} z_k} a_j \overline{a_k}.$$

Rearrangment then gives that for all vectors $(a_1, \ldots, a_n) \in \mathbb{C}^n$ we have

$$0 \leq \sum_{j,k=1}^{n} \frac{1-\overline{w_j}w_k}{1-\overline{z_j}z_k} a_j \overline{a_k}.$$

This shows that to solve this interpolation problem a necessary condition is that the matrix

$$Q_n(Z,W) := \left(\frac{1-\overline{w_j}w_k}{1-\overline{z_j}z_k} \right)_{i,j}$$

is positive semi-definite. It turns out that this simple necessary condition is in fact sufficient. This is the content of the following Theorem

Theorem 0.1 (Nevanlinna-Pick Interpolation)**.** *Let $Z = \{z_1, \ldots, z_n\}$ be a collection of distinct points in $\mathbb{D}$ and let $W = \{w_1, \ldots, w_n\}$ be another collection of points in the disc. Then there exists a function $f \in H^\infty(\mathbb{D})$ of norm at most one with $f(z_j) = w_j$ for $j = 1, \ldots, n$ if and only if that matrix*

$$Q_n(Z,W) := \left(\frac{1-\overline{w_j}w_k}{1-\overline{z_j}z_k} \right)_{i,j} \geq 0$$

Moreover, when $Q_n(Z,W) \geq 0$ there is a Blaschke product of degree at most n that solves the problem.

Before we prove this theorem, we make a couple observations about this Theorem in some simple cases. In the case $n = 1$ we need a function $f(z_1) = w_1$ with norm at most one. This is always possible by choosing a Blaschke product since the automorphism group of the disc $\mathbb{D}$ is transitive so for appropriate choices of $a = a(z_1, w_1)$ and $\varphi = \varphi(z_1, w_1)$ we will have

$$\varphi_a(z) = e^{i\varphi} \frac{z-a}{1-\overline{a}z}$$

will satisfy $\varphi(z_1) = w_1$.

In the case when $n = 2$ we first note that $Q_n(Z,W) \geq 0$ if and only if $1 \geq |w_1|$ and $\det Q_n(Z,W) \geq 0$. A simple computation gives that

$$\det Q_n(Z,W) = \frac{1-|w_1|^2}{1-|z_1|^2} \frac{1-|w_2|^2}{1-|z_2|^2} - \frac{1-\overline{w_1}w_2}{1-\overline{z_1}z_2} \frac{1-\overline{w_2}w_1}{1-\overline{z_2}z_1} \geq 0$$

which we can rearrange to be

$$\frac{(1-|z_1|^2)(1-|z_2|^2)}{|1-\overline{z_1}z_2|^2} \leq \frac{(1-|w_1|^2)(1-|w_2|^2)}{|1-\overline{w_1}w_2|^2},$$

and can further be simplified to

$$\left| \frac{w_1 - w_2}{1-\overline{w_2}w_1} \right| \leq \left| \frac{z_1 - z_2}{1-\overline{z_2}z_1} \right|.$$

Recall the following Lemma

Lemma 0.2 (Invariant Schwarz Lemma)**.** *Let $f \in H^\infty(\mathbb{D})$ with $\|f\|_\infty \leq 1$. Then*

$$\left| \frac{f(z) - f(z_0)}{1-\overline{f(z_0)}f(z)} \right| \leq \left| \frac{z - z_0}{1-\overline{z_0}z} \right| \qquad z \neq z_0$$

Using this Lemma we then see that in the case when $n = 2$ the problem reduces to something well known and studied.

Exercise 0.3. *Prove this Lemma. Hint: Think about the proof of the usual Schwarz Lemma, and then connect it with Möbius maps appropriately.*

In the proof below, given a vector $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ we will let

$$Q_n(Z,W)(a) = \sum_{j,k=1}^{n} \frac{1-\overline{w_j}w_k}{1-\overline{z_j}z_k} a_k \overline{a_j}$$

Proof. The argument given before the statement of Theorem 0.1 gives the necessity of the matrix condition.

It only remains to prove the sufficiency of the condition on the matrix $Q_n(Z,W)$. The proof will proceed by induction. The case $n=1$ is trivial by the above discussion, so suppose that the result is true for $n-1$ points. Clearly we must have that $|w_j| \leq 1$ for $j = 1, \ldots, n$. For simplicity, we will suppose that $|w_j| < 1$ and leave the remaining case as an exercise since it is easier.

We first show that it is possible to reduce to the case when $Z = \{z_1, \ldots, z_{n-1}, 0\}$ and $W = \{w_1, \ldots, w_{n-1}, 0\}$. Move the point z_n and w_n via Möbius maps. In doing this, we must move the remaining points in the disc as well. This gives us a new collection of points

$$z_j' = \frac{z_j - z_n}{1-\overline{z_n}z_j} \quad w_j' = \frac{w_j - w_n}{1-\overline{w_n}w_j} \quad 1 \leq j \leq n.$$

It will be possible to find a function f of norm at most one and satisfying $f(z_j) = w_j$ if and only if

$$g(z) = \frac{f\left(\frac{z+z_n}{1+\overline{z_n}z}\right) - w_n}{1-\overline{w_n}f\left(\frac{z+z_n}{1+\overline{z_n}z}\right)}$$

is of norm at most one and solves $g(z_j') = w_j'$ for $j = 1, \ldots, n$. Also we will have f a Blaschke product of degree at most n if and only if g is a Blaschke product of degree at most $n-1$. Consider the resulting quadratic form $Q_n(Z', W')$. We will see that it is very closely related to the quadratic form for $Q_n(Z,W)$. A simple computation gives

$$\frac{1-\overline{z_k'}z_j'}{1-\overline{z_k}z_j} = \frac{1-|z_n|^2}{(1-\overline{z_n}z_j)(1-\overline{z_k}z_n)} := \alpha_j \overline{\alpha_k}.$$

Indeed, we have

$$\begin{aligned} \frac{1-\overline{z_k'}z_j'}{1-\overline{z_k}z_j} &= \frac{1}{1-\overline{z_k}z_j}\left(1 - \frac{z_j - z_n}{1-\overline{z_n}z_j}\frac{\overline{z_k}-\overline{z_n}}{1-\overline{z_k}z_n}\right) \\ &= \frac{(1-\overline{z_n}z_j)(1-\overline{z_k}z_n) - (z_j - z_n)(\overline{z_k}-\overline{z_n})}{(1-\overline{z_k}z_j)(1-\overline{z_n}z_j)(1-\overline{z_n}z_k)} \\ &= \frac{1-|z_n|^2}{(1-\overline{z_n}z_j)(1-\overline{z_k}z_n)}. \end{aligned}$$

Similarly, we have

$$\frac{1-\overline{w_k'}w_j'}{1-\overline{w_k}w_j} = \frac{1-|w_n|^2}{(1-\overline{w_n}w_j)(1-\overline{w_k}w_n)} := \beta_j \overline{\beta_k}.$$

So we have
$$\frac{1-\overline{w_j'}w_k'}{1-\overline{z_j'}z_k'} = \frac{1-\overline{w_j}w_k}{1-\overline{z_j}z_k}\frac{\beta_j}{\alpha_j}\frac{\overline{\beta_k}}{\overline{\alpha_k}}.$$
Thus, we have that
$$\sum_{j,k=1}^{n}\frac{1-\overline{w_j'}w_k'}{1-\overline{z_j'}z_k'}a_j\overline{a_k} = \sum_{j,k=1}^{n}\frac{1-\overline{w_j}w_k}{1-\overline{z_j}z_k}\frac{\beta_j}{\alpha_j}a_j\frac{\overline{\beta_k}}{\overline{\alpha_k}}\overline{a_k}.$$
Thus, $Q_n(Z',W')\geq 0$ if and only if $Q_n(Z,W)\geq 0$.

So we can suppose that the $Z=\{z_1,\ldots,z_{n-1},0\}$ and $W=\{w_1,\ldots,w_{n-1},0\}$. There will be a function $f\in H^\infty(\mathbb{D})$ of norm at most 1 such that $f(0)=0$ and $f(z_j)=w_j$ for $1\leq j\leq n-1$ if and only if there is a function $g=\frac{f}{z}\in H^\infty(\mathbb{D})$ such that
$$g(z_j)=\frac{w_j}{z_j}\quad 1\leq j\leq n-1.$$
Again, if we can find a Blachke product g of at most degree $n-1$, then the f will be a Blaschke product of degree at most n. By induction, we can find this function g if the matrix
$$\tilde{Q}_n(Z,W)=\left(\frac{1-\frac{w_j}{z_j}\frac{\overline{w_k}}{\overline{z_k}}}{1-\overline{z_k}z_j}\right)_{i,j}\geq 0.$$
We have thus reduced the proof of the Theorem to showing that
$$Q_n(Z,W)\geq 0 \Leftrightarrow \tilde{Q}_n(Z,W)\geq 0$$
under the assumption that $z_n=w_n=0$.

Because we have that $z_n=w_n=0$ a computation gives that
$$Q_n(Z,W)(a)=\left|a_n\right|^2+2\mathrm{Re}\sum_{j=1}^{n-1}\overline{a_j}a_n+\sum_{j,k=1}^{n-1}\frac{1-\overline{w_k}w_j}{1-\overline{z_k}z_j}a_j\overline{a_k}.$$
We complete the square to obtain
$$Q_n(Z,W)(a)=\left|\sum_{j=1}^{n}a_j\right|^2+\sum_{j,k=1}^{n-1}\left(\frac{1-\overline{w_k}w_j}{1-\overline{z_k}z_j}-1\right)a_j\overline{a_k}.$$
However,
$$\frac{1-\overline{w_k}w_j}{1-\overline{z_k}z_j}-1=\frac{1-\frac{w_j}{z_j}\frac{\overline{w_k}}{\overline{z_k}}}{1-z_j\overline{z_k}}z_j\overline{z_k}$$
and so
$$Q_n(Z,W)(a)=\tilde{Q}_n(Z,W)(z_1a_1,\ldots,z_{n-1}a_{n-1})+\left|\sum_{j=1}^{n}a_j\right|^2$$
Thus $\tilde{Q}_n(Z,W)\geq 0$ implies that $Q_n(Z,W)\geq 0$. Similarly, setting $a_n=-\sum_{j=1}^{n-1}a_j$ we see that $Q_n(Z,W)\geq 0$ implies $\tilde{Q}_n(Z,W)\geq 0$ as well. □

Exercise 0.4. *Complete the proof of the sufficiency when $|w_j|=1$.*