Lecture 7: Interpolation in $H^2(\mathbb{D})$ and $H^\infty(\mathbb{D})$

We are now interested in the following question: Given a collection of points $Z = \{z_j\}_{j=1}^\infty$ in $\mathbb{D}$ and a sequence of numbers $\{a_j\}_{j=1}^\infty$ does there exist a function f that interpolates the values a_j at the points z_j. Namely, can we find a function such that

$$f(z_j) = a_j \quad \forall j.$$

We now make this question more precise.

Definition 0.1. The sequence $Z = \{z_j\}$ is called an $H^\infty(\mathbb{D})$-*interpolating* sequence if for every $a = \{a_j\} \in \ell^\infty$ there exists a function $f \in H^\infty(\mathbb{D})$ such that

$$f(z_j) = a_j \quad \forall j$$

Note that, as we have seen, the functional $f \to f(z)$ is continuous on $H^2(\mathbb{D})$ with norm $(1-|z|^2)^{\frac{1}{2}}$. So, the map $f \to \{f(z_j)(1-|z_j|^2)^{\frac{1}{2}}\}$ is bounded from $H^2(\mathbb{D})$ to ℓ^∞. We now want to know if it is bounded as a map to a smaller space, ℓ^2. This is the content of the following definition.

Before the next definition, for the sequence Z let $\ell^2(\mu_Z)$ be the space of all sequences $a = \{a_j\}$ such that

$$\sum_{j=1}^\infty |a_j|\,(1-|z_j|^2) := \|a\|_{\ell^2(\mu_Z)}^2 < \infty.$$

Definition 0.2. The sequence $Z = \{z_j\}$ is called an $H^2(\mathbb{D})$-*interpolating* sequence if for every $a = \{a_j\} \in \ell^2(\mu_Z)$ there exists a function $f \in H^2(\mathbb{D})$ such that

$$f(z_j) = a_j \quad \forall j.$$

We first want to show that it if is possible to do the interpolation, then it is possible to do so with some norm control.

Proposition 0.3. *Suppose that Z is a H^∞-interpolating sequence. Given a sequence $a \in \ell^\infty$ there exists a function $f \in H^\infty(\mathbb{D})$ such that $f(z_j) = a_j$ and a constant C such that*

$$\|f\|_\infty \leq C\,\|a\|_{\ell^\infty}.$$

Exercise 0.4. *Prove this proposition. Hint Use the Open Mapping Theorem applied to the operator $T : H^\infty \to \ell^\infty$ given by $Tf(j) = f(z_j)$.*

We can then let

$$M_Z = \sup_{\|a_j\|_\infty \leq 1} \inf\{\|f\|_\infty : f \in H^\infty(\mathbb{D}) : f(z_j) = a_j \quad \forall j\}$$

denote the constant of interpolation for the sequence Z. The characterization of the interpolating sequences Z that we give will show that M_Z can be controlled in terms of information about the sequence Z.

Lemma 0.5. *Suppose that Z is an interpolating sequence for $H^2(\mathbb{D})$. Then there exists a constant C such that for every $a \in \ell^2(\mu_Z)$ there is an $f \in H^2(\mathbb{D})$ satisfying*

(i) $f(z_j) = a_j$;

(ii) $\|f\|_{H^2(\mathbb{D})} \leq C \|a\|_{\ell^2(\mu_Z)}$.

The proof is an application of the Closed Graph Theorem.

Proof. Set $M := \{f \in H^2(\mathbb{D}) : f(z_j) = 0 \quad \forall z_j \in Z\}$. Then since point evaluations are continuous M is closed. Thus, we have $H^2(\mathbb{D}) = M \oplus M^\perp$. Let $P^\perp$ denote the projection of $H^2(\mathbb{D})$ onto $M^\perp$. Then if $a \in \ell^2(\mu_Z)$ and $f_a \in H^2(\mathbb{D})$ satisfies $f_a(z_j) = a_j$ then we have that $P^\perp f_a(z_j) = a_j$ with the $P^\perp f_a$ the unique element in $M^\perp$. Then $\Lambda a := P^\perp f_a$ defines a linear map from $\ell^2(\mu_Z)$ to $M^\perp$.

One can then show that the graph of Λ is closed in $\ell^2(\mu_Z) \times M^\perp$. If we have $(a^k, \Lambda a^k) \to (a, h) \in \ell^2(\mu_Z) \times M^\perp$, the for each fixed j as $k \to \infty$ we have

$$a_j^k \to a_j \text{ and } \Lambda a^k(z_j) \to h(z_j).$$

However, $\Lambda a^k(z_j) = (RP^\perp f_{a^k})_j = a_j^k$, and this then shows that $h(z_j) = a_j = \Lambda a(z_j)$ for all j, and so by uniquness in $M^\perp$ that $\Lambda a = h$, namely that (a, h) is the the graph of Λ.

Since the graph is closed, and application of the closed graph theorem implies that Λ is continuous. So, we then take $f(z) := \Lambda a(z)$. This function then proves the Lemma. □

We have thus seen that solving these interpolation problems in either $H^\infty(\mathbb{D})$ and in $H^2(\mathbb{D})$ is equivalent to solving them with some norm control. The following Theorem of Carleson is important in our study of these questions.

Theorem 0.6 (Carleson, [1], Shapiro, Shields [3])**.** *The following are equivalent.*

(a) *The sequence Z is $H^2(\mathbb{D})$-interpolating;*
(b) *The sequence Z is $H^\infty(\mathbb{D})$-interpolating;*
(c) *The sequence Z is separated in the pseudo-hyperbolic metric and generates a H^2-Carleson measure. In particular, $\sum_{z_j \in Z}(1 - |z_j|^2)\delta_{z_j}$ is a $H^2(\mathbb{D})$ Carleson measure and*

$$\inf_{j \neq k} \left| \frac{z_j - z_k}{1 - \overline{z_k} z_j} \right| \geq \delta > 0$$

(d) *The sequence Z is strongly separated, namely there exists a constant $\delta > 0$ such that*

$$\inf_j \left| \prod_{j \neq k} \frac{z_j - z_k}{1 - \overline{z_k} z_j} \right| \geq \delta > 0$$

We remark that some of these conditions were not explicitly pointed out in the paper by Carleson. Additionally, the result about H^2-Interpolation was obtained by Shapiro and Shields, [3]

In the interest of seeing all the connections that hold, and the resulting situation when we change to the Dirichlet space, we will show as many of these connections as possible.

Equivalence between (d) *and* (c). First, suppose that (d) holds. Since each factor

$$\left| \frac{z_j - z_k}{1 - \overline{z_k} z_j} \right| \leq 1 \quad j \neq k$$

clearly we have that (d) implies

$$\inf_{j \neq k} \left| \frac{z_j - z_k}{1 - \overline{z_k} z_j} \right| \geq \delta > 0.$$

Next observe that we have

$$\begin{aligned} 2\log\delta &\leq \log\left|\prod_{j\neq k}\frac{z_j-z_k}{1-\overline{z_k}z_j}\right| \\ &= \sum_{z_j\neq z_k}\log\left|\frac{z_j-z_k}{1-\overline{z_k}z_j}\right|^2 \\ &= \sum_{z_j\neq z_k}\log\left(1-\frac{(1-|z_j|^2)(1-|z_k|^2)}{|1-\overline{z_k}z_j|^2}\right) \\ &\leq -\sum_{z_j\neq z_k}\frac{(1-|z_j|^2)(1-|z_k|^2)}{|1-\overline{z_k}z_j|^2}. \end{aligned}$$

This estimate gives that

$$\sum_{z_j\neq z_k}\frac{(1-|z_j|^2)(1-|z_k|^2)}{|1-\overline{z_k}z_j|^2}\leq\log\frac{1}{\delta^2}$$

and so

$$\sum_{j=1}^{\infty}\frac{(1-|z_j|^2)(1-|z_k|^2)}{|1-\overline{z_k}z_j|^2}\leq\left(1+\log\frac{1}{\delta^2}\right).$$

However, by our proof of the Carleson Embedding Theorem, we thus have that for all $f\in H^2(\mathbb{D})$ that

$$\sum_{j=1}^{\infty}|f(z_j)|^2\left(1-|z_j|^2\right)\lesssim\left(1+\log\frac{1}{\delta^2}\right)\|f\|_{H^2(\mathbb{D})}^2.$$

(Note that the proof of the Carleson embedding theorem that we actually gave showed that μ is a $H^2(\mathbb{D})$-Carleson measure if and only if

$$\int_{\mathbb{D}}|k_\lambda(z)|^2\,d\mu(z)\leq C\,\|k_\lambda\|_{H^2(\mathbb{D})}^2\quad\forall\lambda\in\operatorname{supp}\mu.$$

See Theorem 0.1 from Lecture 2.)

Thus we have shown that (d) implies (c). We now turn to showing that (c) implies (d). Note that by the Carleson measure hypothesis of (c) we have that

$$C\geq\sum_{j=1}^{\infty}\frac{(1-|z_j|^2)(1-|z_k|^2)}{|1-\overline{z_j}z_k|^2}\geq\sum_{z_j\neq z_k}\frac{(1-|z_j|^2)(1-|z_k|^2)}{|1-\overline{z_j}z_k|^2}.$$

Now, we apply the following trivial inequality for $a^2<t<1$ that

$$-\log t\leq-\frac{2\log a}{1-a^2}(1-t)\leq\left(1+2\log\frac{1}{a}\right)(1-t).$$

Since we have that the sequence Z is separated we have that

$$\inf_{j\neq k}\left|\frac{z_j-z_k}{1-\overline{z_k}z_j}\right|\geq\delta$$

we apply the trivial inequality with $t = \left|\frac{z_j - z_k}{1-\overline{z_k}z_j}\right|^2$ and $a = \delta$ to obtain

$$\begin{aligned} C\left(1+2\log\frac{1}{\delta}\right) &\geq \left(1+2\log\frac{1}{\delta}\right)\sum_{z_j\neq z_k}\frac{(1-|z_j|^2)(1-|z_k|^2)}{|1-\overline{z_j}z_k|^2} \\ &= \left(1+2\log\frac{1}{\delta}\right)\sum_{z_j\neq z_k}\left(1-\left|\frac{z_j-z_k}{1-\overline{z_k}z_j}\right|^2\right) \\ &\geq \sum_{z_j\neq z_k}\log\left|\frac{z_j-z_k}{1-\overline{z_k}z_j}\right|^2 \\ &= -\log\left|\prod_{j\neq k}\frac{z_j-z_k}{1-\overline{z_k}z_j}\right| \end{aligned}$$

This then can be rearranged to give

$$\inf_j\left|\prod_{j\neq k}\frac{z_j-z_k}{1-\overline{z_k}z_j}\right| \geq C(\delta) > 0.$$

□

(a) *implies* (d) *(implies* (c)*).* First, lets see that we can easily obtain one half of the condition appearing in (c). If Z is an interpolating sequence for $H^2(\mathbb{D})$, then by Lemma 0.5 we know that there is a solution to the interpolation problem $f(z_j) = a_j$ satisfying

$$\|f\|^2_{H^2(\mathbb{D})} \leq C\sum_{j=1}^{\infty}|a_j|^2(1-|z_j|^2)$$

with C independent of the sequence $\{a_j\}$. Fix j and let $a_j := \delta_{j,l}$. Then we can find $f \in H^2(\mathbb{D})$ such that $f(z_j) = \delta_{l,j}$ and

$$\|f\|^2_{H^2(\mathbb{D})} \leq \frac{C}{k_{z_j}(z_j)}.$$

Next observe that for any constant α we have

$$\left\langle f, k_{z_j} - \alpha k_{z_l}\right\rangle_{H^2(\mathbb{D})} = 1.$$

An application of Cauchy–Schwarz and the reproducing kernel property implies that

$$1 \leq \frac{C}{k_{z_j}(z_j)}\left(k_{z_j}(z_j) - 2\mathrm{Re}(\alpha k_{z_l}(z_j)) + |\alpha|^2 k_{z_l}(z_l)\right).$$

Now choose $\alpha = \frac{|k_{z_l}(z_j)|\sqrt{k_{z_j}(z_j)}}{k_{z_l}(z_j)\sqrt{k_{z_l}(z_l)}}$. Simple algebra then produces that

$$\frac{|k_{z_l}(z_j)|^2}{k_{z_l}(z_l)k_{z_j}(z_j)} = \frac{(1-|z_j|^2)(1-|z_k|^2)}{|1-\overline{z_j}z_k|^2} \leq \left(1-\frac{1}{2C}\right)^2.$$

This then gives that

$$\left|\frac{z_j-z_k}{1-\overline{z_k}z_j}\right|^2 = 1 - \frac{(1-|z_j|^2)(1-|z_k|^2)}{|1-\overline{z_j}z_k|^2} \geq C$$

and so the sequence Z is separated as claimed.

We now will modify this slightly so that we give the proof that (a) implies (d). Let $a_j = \delta_{j,k}$ and choose a function f that interpolates this sequence. We also know that we have norm control on the solution

$$\|f\|_{H^2(\mathbb{D})} \leq C(1-|z_k|^2)^{\frac{1}{2}}.$$

Again, let B_k denote the Blaschke product with zeros at $\{z_j\}_{j\neq k}$ and consider the function $g(z) = \frac{f(z)}{B_k(z)}$. Then it is a fact that $\|f\|_{H^2} = \|g\|_{H^2}$ and so

$$\begin{aligned} \frac{1}{|B_k(z_k)|} &= \left|\frac{f(z_k)}{B_k(z_k)}\right| = |g(z_k)| = \left|\langle g, k_{z_k}\rangle_{H^2(\mathbb{D})}\right| \\ &\leq \|g\|_{H^2}\|k_{z_k}\|_{H^2} = \|f\|_{H^2}\|k_{z_k}\|_{H^2} \leq C. \end{aligned}$$

□

(b) *implies* (d) *(implies* (c)*).* Fix the point z_k and let $f \in H^\infty(\mathbb{D})$ interpolate the values $\delta_{k,j}$. Let B_k be the Blaschke product with zeros at $\{z_j\}_{j\neq k}$. Then we have that $f = B_k g$ with $g \in H^\infty(\mathbb{D})$ and $\|g\|_\infty \leq M$. But,

$$1 = |f(z_k)| = |B_k(z_k)|\,|g(z_k)| \leq M\,|B_k(z_k)|\,.$$

This then gives

$$|B_k(z_k)| \geq \frac{1}{M}$$

and since z_k was arbitrary that

$$\inf_j \left|\prod_{j\neq k} \frac{z_j - z_k}{1-\overline{z_k}z_j}\right| \geq \delta > 0.$$

Since we now have that (a) implies (d), then by the computations above, we have shown that (a) also implies (c).

□

(d) *implies* (b)*.* We will prove the result in the upper half plane $\mathbb{H}$ since the ideas are easier there. As a reduction to the problem, we are allowed to suppose that the sequence Z is finite as long as any estimates we obtain are independent on the number of elements in the sequence Z. A normal families argument then will complete the proof.

First, we need to translate the condition (d) into the appropriate language. In this setting, a Blaschke product takes the form

$$B(z) = \prod_{j=1}^{n} \frac{z - z_j}{z-\overline{z_j}}.$$

So

$$B_j(z) = \frac{z-\overline{z_j}}{z-z_j} B(z)$$

and a simple computation gives that

$$B'(z_j) = \frac{B_j(z_j)}{z_j - \overline{z_j}}.$$

In our case then the hypothesis is that

$$\inf_j |B_j(z_j)| \geq \delta > 0.$$

Since the sequence Z is finite, it is very easy to construct a function which solves $f(z_j) = a_j$, though this frequently will not have good norm estimates. Here is a very simple way to construct such a function

$$f_n(z) = \sum_{j=1}^n a_j \frac{B_j(z)}{B_j(z_j)} = \sum_{j=1}^n a_j \frac{\frac{z-\overline{z_j}}{z-z_j} B(z)}{B'(z_j)(z_j - \overline{z_j})} = \sum_{j=1}^n a_j \frac{B(z)}{B'(z_j)(z-z_j)} \frac{z-\overline{z_j}}{z_j - \overline{z_j}}.$$

So we also have that

$$g(z) = \sum_{j=1}^n a_j \frac{B(z)}{B'(z_j)(z-z_j)}$$

also solves $g(z_j) = a_j$. Now note that any other solution to the problem can be obtained by looking at

$$g + Bh$$

where $h \in H^\infty(\mathbb{H})$ is arbitrary. So the function which solves the problem and has minimal norm is given by

$$\begin{aligned}
\inf_{h \in H^\infty} \|g + Bh\|_\infty &= \inf_{h \in H^\infty} \left\| B(z) \left(\sum_{j=1}^n a_j \frac{1}{B'(z_j)(z-z_j)} + h(z) \right) \right\|_{L^\infty(\mathbb{R})} \\
&= \inf_{h \in H^\infty} \left\| \sum_{j=1}^n a_j \frac{1}{B'(z_j)(z-z_j)} + h(z) \right\|_{L^\infty(\mathbb{R})} \\
&= \sup_{f \in H^1(\mathbb{H})} \left| \sum_{j=1}^n \int_{\mathbb{R}} \frac{a_j f(x)}{B'(z_j)(x-z_j)} dx \right|.
\end{aligned}$$

Here the last line follows since for andy Banach space X with closed subspace Y we have that $Y^* \simeq \frac{X^*}{Y^\perp}$, and

$$\inf\{\|x^* + k\| : k \in Y^\perp\} = \sup\{|\langle x^*, y\rangle| : y \in Y, \|y\| \leq 1\}.$$

and applying this with $X = L^1$, and $Y = H^1$, and so $Y^\perp = H^\infty$. An application of Cauchy's Theorem then gives that

$$\begin{aligned}
\inf_{h \in H^\infty} \|g + Bh\|_\infty &= 2\pi \sup_{f \in H^1(\mathbb{H})} \left| \sum_{j=1}^n \frac{a_j f(z_j)}{B'(z_j)} \right| \\
&= 4\pi \sup_{f \in H^1(\mathbb{H})} \left| \sum_{j=1}^n \frac{a_j y_j f(z_j)}{B_j(z_j)} \right|
\end{aligned}$$

Now recalling how we defined the constant of interpolation M_Z we have

$$\begin{aligned} M_Z &= \sup_{a\in\ell^\infty} 4\pi \sup_{f\in H^1(\mathbb{H})}\left|\sum_{j=1}^n \frac{a_j y_j f(z_j)}{B_j(z_j)}\right| \\ &= 4\pi \sup_{f\in H^1(\mathbb{H})} \sum_{j=1}^n \frac{y_j\,|f(z_j)|}{|B_j(z_j)|} \end{aligned}$$

Now set ν_Z to be the measure

$$\nu_Z = \sum_{j=1}^n \frac{y_j}{|B_j(z_j)|}\delta_{z_j}$$

then we see that M_Z is the norm of the embedding operator $H^1(\mathbb{H}) \subset L^1(\mathbb{H}; d\nu_Z)$. We then use the fact that $H^1(\mathbb{H}) = H^2(\mathbb{H})\cdot H^2(\mathbb{H})$ with equality of norms. Namely, given $f \in H^1(\mathbb{H})$ there exists a $g, h \in H^2(\mathbb{H})$ such that $f = gh$ and $\|f\|_1 = \|g\|_2\,\|h\|_2$. We compute the norm of this embedding operator

$$\begin{aligned} \int_{\mathbb{H}} |f(z)|\,d\nu_Z &\leq \left(\int_{\mathbb{H}} |g(z)|^2\,d\nu_Z(z)\right)^{\frac{1}{2}}\left(\int_{\mathbb{H}} |h(z)|^2\,d\nu_Z(z)\right)^{\frac{1}{2}} \\ &\leq \|\nu_Z\|^2_{H^2-Carl}\,\|g\|_2\,\|h\|_2 = \|f\|_1\,\|\nu_Z\|^2_{H^2-Carl}. \end{aligned}$$

So we have that $\|\nu_Z\|_{H^1-Carl} \leq \|\nu_Z\|_{H^2-Carl}$ and we are left computing this norm. However, (at least in the case of the disc) we have already done this.

Consider the measure

$$d\nu_Z = \sum_{j=1}^n \frac{y_j}{|B_j(z_j)|}\delta_{z_j}.$$

If we were to translate this measure to an equivalent measure on the disc, it would take the form

$$d\tilde{\nu}_Z = \sum_{j=1}^n \frac{(1-|z_j|^2)}{|B_j(z_j)|}\delta_{z_j}.$$

Note that by the strong separation condition, the denominator of the terms above is bounded below by δ. This observation, and our previous argument shows that this is a Carleson measure with

$$\|d\tilde{\nu}_Z\|_{H^2-Carl} \lesssim \sqrt{\frac{1}{\delta}\left(1+\log\frac{1}{\delta^2}\right)}.$$

The proof in the case of the upper half plane is identical (modulo writing things appropriately). □

Exercise 0.7. *Show that the Carleson norm of the measure*

$$d\nu_Z = \sum_{j=1}^n \frac{y_j}{|B_j(z_j)|}\delta_{z_j}$$

on $\mathbb{H}$ *is controlled by*

$$\frac{1}{\delta}\left(1+\log\frac{1}{\delta^2}\right).$$

We briefly illustrate why we are saying that the sequence Z is *separated* if the condition

$$\inf_{j\neq k}\left|\frac{z_j-z_k}{1-\overline{z_k}z_j}\right|\geq\delta>0$$

holds. This is a geometric condition on the sequence of points, and it can be interpreted as the the points are far away from each other in the hyperbolic metric.

Recall that the hyperbolic metric between two points $z,w\in\mathbb{D}$ is given by

$$\rho(z,w)=\log\frac{1+|b_w(z)|}{1-|b_w(z)|}$$

where $b_z(w)=\frac{z-w}{1-\overline{w}z}$.

Exercise 0.8. *Show that $\rho(z,w)$ is a metric on $\mathbb{D}$. Hint: To check the triangle inequality, first show that the metric $\rho(z,w)$ is conformally invariant, i.e.,*

$$\rho(b_\lambda(z),b_\lambda(w))=\rho(z,w).$$

Then the condition

$$\inf_{j\neq k}\left|\frac{z_j-z_k}{1-\overline{z_k}z_j}\right|\geq\delta>0$$

is equivalent to the condition that the hyperbolic discs

$$\left\{w:\rho(z_j,w)<\log\frac{1+\delta}{1-\delta}\right\}$$

are disjoint. The proof of this fact is just a simple computation.

Exercise 0.9. *Show that the condition*

$$\inf_{j\neq k}\left|\frac{z_j-z_k}{1-\overline{z_k}z_j}\right|\geq\delta>0$$

can also be interpreted as saying that Euclidean discs centered about the points z_j with radius a multiple (depending on δ) of $(1-|z_j|)$ must also be disjoint.

0.1. **Peter Jones' Constructive Proof of Interpolation.** We now give another proof of this Carleson's Interpolation Theorem, but will give an explicit formula that solves the interpolation problem. This construction is very similar to the constructive solution of the $\overline{\partial}$-problem.

First, some notation. We are again given a sequence $Z=\{z_j\}\subset\mathbb{D}$ and we let

$$B_j(z)=\prod_{k\neq j}b_{z_k}(z)=\prod_{k\neq j}\frac{z-z_k}{1-\overline{z_k}z}$$

denote the infinite Blaschke factor that vanishes on the set of points $Z\setminus\{z_j\}$. We also set

$$\delta_j:=|B_j(z_j)|=\prod_{k\neq j}\left|\frac{z_j-z_k}{1-\overline{z_k}z_j}\right|.$$

Finally, ℓ^∞ will denote the collection of sequences $\{a_k\}$ such that

$$\|a\|_{\ell^\infty}:=\sup_n|a_n|<\infty.$$

Theorem 0.10 (Jones, [2]). *Suppose that the sequence Z satisfies*

$$\inf_{j}\left|\prod_{j\neq k}\frac{z_j-z_k}{1-\overline{z_k}z_j}\right|\geq\delta>0.$$

Let $C(\delta)$ denote the best constant for the Carleson measure

$$\mu_Z=\sum_{j=1}^{\infty}(1-|z_j|^2)\delta_{z_j}.$$

Define

$$f_j(z):=\frac{B_j(z)}{B_j(z_j)}\left(\frac{1-|z_j|^2}{1-\overline{z_j}z}\right)^2\exp\left(-\frac{1}{2C(\delta)}\sum_{|z_m|\geq|z_j|}\left(\frac{1+\overline{z_m}z}{1-\overline{z_m}z}-\frac{1+\overline{z_m}z_j}{1-\overline{z_m}z_j}\right)(1-|z_m|^2)\right).$$

Then for any $a\in\ell^\infty$ we have

$$f(z)=\sum_{j=1}^{\infty}a_jf_j(z)\in H^\infty(\mathbb{D})$$

with $f(z_j)=a_j$ and

$$|f(z)|\leq\|a\|_{\ell^\infty}\sum_{j=1}^{\infty}|f_j(z)|\leq C(\delta)\|a\|_{\ell^\infty}.$$

The proof of this Theorem is related to the constructive solution of the the $\overline{\partial}$ problem, but in a discrete setting. The construction of the function was done by Jones in [2].

Proof. First, some simple observations. We have that $\delta_j\geq\delta$ for all j. We have also seen that under the hypothesis that

$$\inf_j|B_j(z_j)|\geq\delta$$

that the following embedding condition holds

$$\sum_{j=1}^{\infty}|f(z_j)|^2(1-|z_j|^2)\leq C(\delta)\|f\|_{H^2(\mathbb{D})}^2.$$

Assuming that we have proved the estimate claimed in the Theorem, it is then clear that this function is holomoprhic and moreover that $f_j(z_k)=\delta_{jk}$ and so $f(z_j)=a_j$. The proof of analyticity is a normal families argument. Then, again assuming the claimed estimate, given $\{a_j\}\in\ell^\infty$ and having

$$f(z)=\sum_{j=1}^{\infty}a_jf_j(z)$$

gives

$$|f(z)|\leq\|a\|_{\ell^\infty}\sum_{j=1}^{\infty}|f_j(z)|\leq C(\delta)\|a\|_{\ell^\infty}.$$

Since we want to show that the function $f \in H^\infty(\mathbb{D})$ we can suppose that $z \in \mathbb{T}$, and then if we can show that the above sum converges, by the maximum principle we will have the same result in $\mathbb{D}$. Now, letting $\alpha_j = \frac{(1-|z_j|^2)^2}{|1-\overline{z_j}z|^2}$ it is easy to see that we have

$$
\begin{aligned}
|f_j(z)| &\leq \alpha_j \exp\left(-\frac{1}{2C(\delta)}\mathrm{Re}\sum_{|z_m|\geq|z_j|}\left(\frac{1+\overline{z_m}z}{1-\overline{z_m}z}-\frac{1+\overline{z_m}z_j}{1-\overline{z_m}z_j}\right)(1-|z_m|^2)\right)\\
&\leq \alpha_j \exp\left(-\frac{1}{2C(\delta)}\sum_{|z_m|\geq|z_j|}\alpha_m\right)\exp\left(\frac{1}{C(\delta)}\sum_{|z_m|\geq|z_j|}\frac{1-|z_j|^2}{|1-\overline{z_m}z_j|^2}(1-|z_m|^2)\right).
\end{aligned}
$$

Here in the computations we used the fact that $|z|=1$, the simple fact that

$$\mathrm{Re}\left(\frac{1+u}{1-u}\right) = \frac{1-|u|^2}{|1-u|^2}$$

and obvious estimates. Now, note that we have by the hypothesis on the sequence Z that

$$\exp\left(\frac{1}{C(\delta)}\sum_{|z_m|\geq|z_j|}\frac{1-|z_j|^2}{|1-\overline{z_m}z_j|^2}(1-|z_m|^2)\right) \leq e.$$

This follows from using the Carleson measure condition on the reproducing kernel. Using this estimate, and setting $\beta_j = \frac{\alpha_j}{2C(\delta)}$ we see that

$$\sum_{j=1}^\infty |f_j(z)| \leq 2eC'(\delta)\sum_{j=1}^\infty \beta_j \exp\left(-\sum_{|z_m|\geq|z_j|}\beta_m\right).$$

However, it is easy to see that this last sum is a lower Riemann sum for the integral $\int_0^\infty e^{-t}dt = 1$. So, we have proven that

$$|f(z)| \leq 2eC'(\delta)\left\|a\right\|_{\ell^\infty}.$$

□

We remark that being careful and recalling what the constants $C(\delta)$ looks like, the proof of this theorem shows that the norm of the operator of interpolation is controlled by an absolute constant times

$$\frac{1}{\delta}\left(1+\log\frac{1}{\delta^2}\right)$$

which also appeared in the other proof we gave characterizing the interpolating sequences.

We also can use this result to prove the following result. We will say that a sequence Z is interpolating for $H^p(\mathbb{D})$ if given any $a \in \ell^p(\mathbb{N})$ there exists a $f \in H^p(\mathbb{D})$ such that

$$f(z_j) = a_j(1-|z_j|^2)^{-\frac{1}{p}}.$$

Theorem 0.11. *Let $1 \leq p \leq \infty$. Suppose that the sequence Z is strongly separated, i.e.,*

$$\inf_j\left|\prod_{j\neq k}\frac{z_j-z_k}{1-\overline{z_k}z_j}\right| \geq \delta > 0.$$

Then Z is $H^p(\mathbb{D})$ interpolating, and we have norm control on the solution function.

Proof. The proof can be given via an interpolation argument. By Theorem 0.10 we have the result being true when $p = \infty$. We then focus on the case of $p = 1$. Given a sequence $a \in \ell^1$ we need a function $f \in H^1(\mathbb{D})$ such that

$$f(z_j) = a_j(1 - |z_j|^2)^{-1}.$$

Based on Theorem 0.10, we set

$$f(z) = \sum_{j=1}^{\infty} a_j(1 - |z_j|^2)^{-1} f_j(z)$$

where f_j are the functions defined in the statement of Theorem 0.10. This function clearly has that

$$f(z_j) = a_j(1 - |z_j|^2)^{-1}$$

Then, we compute that

$$\|f\|_{H^1(\mathbb{D})} \leq \sum_{j=1}^{\infty} |a_j| (1 - |z_j|^2)^{-1} \|f_j\|_{H^1(\mathbb{D})}.$$

Now since $f_j \in H^\infty(\mathbb{D})$ we definitely have that $f_j \in H^1(\mathbb{D})$, but if we use the coarse estimate that

$$\|f\|_{H^1(\mathbb{D})} \leq \|f_j\|_\infty$$

then we unfortunately won't prove the result. Instead, we need to show that

$$\|f_j\|_{H^1(\mathbb{D})} \leq C(\delta)(1 - |z_j|^2).$$

Using this estimate, then we immediately have that

$$\|f\|_{H^1(\mathbb{D})} \leq \sum_{j=1}^{\infty} |a_j|.$$

The estimate on the $H^1(\mathbb{D})$ norm is an easy computation we leave as an exercise.

The case of general $1 < p < \infty$ then follows by a standard interpolation argument. □

This proof gives another proof of the result of Shapiro and Shields [4].

Exercise 0.12. *Prove that*

$$\|f_j\|_{H^1(\mathbb{D})} \leq C(\delta)(1 - |z_j|^2).$$

Hint: Compute the norm of $H^1(\mathbb{D})$ by integrating $f_j(z)$ over the boundary $\mathbb{T}$. Then think about harmonic functions.

Exercise 0.13. *Complete the proof by supplying the details of the interpolation argument.*

References

[1] Lennart Carleson, *An interpolation problem for bounded analytic functions*, Amer. J. Math. **80** (1958), 921–930. ↑2

[2] Peter W. Jones, L^∞ *estimates for the $\bar{\partial}$ problem in a half-plane*, Acta Math. **150** (1983), no. 1-2, 137–152. ↑9

[3] H. S. Shapiro and A. L. Shields, *On the zeros of functions with finite Dirichlet integral and some related function spaces*, Math. Z. **80** (1962), 217–229. ↑2

[4] ———, *On some interpolation problems for analytic functions*, Amer. J. Math. **83** (1961), 513–532. ↑11