Lecture 6: $\overline{\partial}$-equations and The Corona Theorem

In the last lecture we studied the problem

$$\overline{\partial} u = v$$

and gave a solution operator of the form

$$u(z) = \frac{1}{2\pi i}\int_{\mathbb{D}} \frac{v(w)}{w-z} dA(w).$$

We then wanted to study the bounded of the solution u in terms of boundedness properties of the right hand side v. We finished the last lecture by showing that if the right hand side has some Carleson measure estimates, then we always have a bounded solution to this problem.

In this lecture we will return to studying the $\overline{\partial}$-problem when the right hand side is a Carleson measure. We will show that in this case, it is possible to give a constructive solution to this problem. This is a very clever construction due to Peter Jones. Finally, in this lecture we will use the results from last week to complete the proof of Carleson's Corona Theorem.

1. Peter Jones' Constructive Solution to $\overline{\partial}$

We now give another method to solve the $\overline{\partial}$ problem that is more constructive. The proof this time is based on the extremely clever solution operator constructed by Peter Jones, [3].

The starting point of the construction is the following observation. We have seen from above, that to solve the equation $\overline{\partial} F = G$ we can set

$$F = \frac{1}{2\pi i}\int_{\mathbb{D}} G(\xi)\frac{1}{z-\xi} d\xi\wedge d\overline{\xi}.$$

However, we can also use another kernel to accomplish this. Choose a function $K(z,\xi)$ that is analytic in z, $K(z,z)=1$ and that is smooth, then we will also have that

$$F = \frac{1}{2\pi i}\int_{\mathbb{D}} K(z,\xi) G(\xi)\frac{1}{z-\xi} d\xi\wedge d\overline{\xi}.$$

solves $\overline{\partial} F = G$.

Exercise 1.1. *Show that this is true. Hint: $K(z,\xi) = 1 + (z-\xi)\tilde{K}(z,\xi)$ and argue as before.*

This then gives us lots of freedom by which kernel we choose to solve the problem. The construction by Jones makes a very clever and judicious choice of the kernel so that one can obtain the solutions directly.

Theorem 1.2 (Jones, [3])**.** *Let μ be a complex $H^2(\mathbb{D})$ Carleson measure on $\mathbb{D}$. Then with $S(\mu)(z)$ given by*

$$S(\mu)(z) = \int_{\mathbb{D}} K(\sigma, z, \zeta)\, d\mu(\zeta) \tag{1.1}$$

where $\sigma = \frac{|\mu|}{\|\mu\|_{CM(H^2)}}$ *and*

$$K(\sigma, z, \zeta) \equiv \frac{2i}{\pi} \frac{1-|\zeta|^2}{(z-\zeta)(1-\overline{\zeta}z)} \exp\left\{\int_{|\omega|\geq|\zeta|} \left(-\frac{1+\overline{\omega}z}{1-\overline{\omega}z} + \frac{1+\overline{\omega}\zeta}{1-\overline{\omega}\zeta}\right) d\sigma(\omega)\right\},$$

we have that:

(1) $S(\mu) \in L^1_{loc}(\mathbb{D})$.
(2) $\overline{\partial} S(\mu) = \mu$ *in the sense of distributions.*
(3) $\int_{\mathbb{D}} \left|K\left(\frac{|\mu|}{\|\mu\|_{CM(H^2)}}, x, \zeta\right)\right| d|\mu|(\zeta) \lesssim \|\mu\|_{CM(H^2)}$ *for all* $x \in \mathbb{T} = \partial\mathbb{D}$,

so $\|S(\mu)\|_{L^\infty(\mathbb{T})} \lesssim \|\mu\|_{CM(H^2)}$.

Note that the kernel $K(\sigma, z, \xi)$ is analytic in z,

$$K(\sigma, z, \xi) = \frac{2i}{\pi}\frac{1}{z-\xi}\tilde{K}(\sigma, z, \xi)$$

and $\tilde{K}(\sigma, z, \xi)$ is smooth with

$$\tilde{K}(\sigma, z, z) = \frac{1-|z|^2}{1-|z|^2} \exp\left\{\int_{|\omega|\geq|z|}\left(-\frac{1+\overline{\omega}z}{1-\overline{\omega}z} + \frac{1+\overline{\omega}z}{1-\overline{\omega}z}\right) d\sigma(\omega)\right\} = 1.$$

So (2) follows the argument above. While (1) follows from (3). We now turn to proving that (3) holds (though in the course of the proof we will address (1) and (3) at the same time).

Proof. Observe that if we prove $\int_{\mathbb{D}} \left|K\left(\frac{|\mu|}{\|\mu\|_{CM(H^2)}}, x, \zeta\right)\right| d|\mu|(\zeta) \lesssim \|\mu\|_{CM(H^2)}$ for all $x \in \mathbb{T} = \partial\mathbb{D}$, then we have

$$|S(\mu)(z)| \leq \int_{\mathbb{D}} \left|K\left(\frac{|\mu|}{\|\mu\|_{CM(H^2)}}, z, \zeta\right)\right| d|\mu|(\zeta) \lesssim \|\mu\|_{CM(H^2)}.$$

We turn now to the proof of this remaining fact. Note that for the measure $\sigma = \frac{\mu}{\|\mu\|_{CM(H^2)}}$ we have

$$\begin{aligned} \operatorname{Re}\left(\int_{|w|\geq|\zeta|}\left(\frac{1+\overline{w}\zeta}{1-\overline{w}\zeta}\right) d\sigma(w)\right) &= \int_{|w|\geq|\zeta|} \operatorname{Re}\left(\frac{1+\overline{w}\zeta}{1-\overline{w}\zeta}\right) d\sigma(w) \\ &\leq 2\int_{\mathbb{D}} \frac{1-|\zeta|^2}{|1-\overline{w}\zeta|^2} d\sigma(w) \\ &\leq 2\left\|\tilde{k}_\zeta\right\|^2_{H^2(\mathbb{D})} = 2, \end{aligned}$$

where $\tilde{k}_\zeta(z) = \frac{(1-|\zeta|^2)^{\frac{1}{2}}}{1-\overline{\zeta}z}$ is the normalized reproducing kernel and clearly has $H^2(\mathbb{D})$ norm equal to 1. Next, observe that it will suffice to control the boundary values of the function $S(\mu)$, and so we can take $z \in \mathbb{T}$. We then see, using the estimate from above and that $z \in \mathbb{T}$

that we have

$$\begin{aligned}
|K(\sigma,z,\xi)| &= \frac{2}{\pi}\frac{1-|\xi|^2}{\left|1-\overline{\xi}z\right|^2}\left|\exp\left\{\int_{|\omega|\geq|\xi|}\left(-\frac{1+\overline{\omega}z}{1-\overline{\omega}z}+\frac{1+\overline{\omega}\xi}{1-\overline{\omega}\xi}\right)d\sigma(\omega)\right\}\right| \\
&= \frac{2}{\pi}\frac{1-|\xi|^2}{\left|1-\overline{\xi}z\right|^2}\exp\left\{\mathrm{Re}\int_{|\omega|\geq|\xi|}-\frac{1+\overline{\omega}z}{1-\overline{\omega}z}d\sigma(\omega)\right\}\exp\left\{\mathrm{Re}\int_{|\omega|\geq|\xi|}\frac{1+\overline{\omega}z}{1-\overline{\omega}z}d\sigma(\omega)\right\} \\
&\leq \frac{2}{\pi}e^2\frac{1-|\xi|^2}{\left|1-\overline{\xi}z\right|^2}\exp\left\{-\int_{|\omega|\geq|\xi|}\frac{1-|\omega|^2}{|1-\overline{\omega}z|^2}d\sigma(\omega)\right\}
\end{aligned}$$

We can then use this estimate on the kernel $K(\sigma,z,\zeta)$ to give

$$\begin{aligned}
|S(\mu)(z)| &= \left|\int_{\mathbb{D}}K(\sigma,z,\zeta)\,d\mu(\zeta)\right| \\
&\leq \|\mu\|_{CM(H^2)}\int_{\mathbb{D}}|K(\sigma,z,\zeta)|\,d\sigma(\zeta) \\
&\leq e^2\|\mu\|_{CM(H^2)}\frac{2}{\pi}\int_{\mathbb{D}}\frac{1-|\zeta|^2}{\left|1-\overline{\zeta}z\right|^2}\exp\left\{-\int_{|\omega|\geq|\zeta|}\frac{1-|\omega|^2}{|1-\overline{\omega}z|^2}d\sigma(\omega)\right\}d\sigma(\zeta).
\end{aligned}$$

It remains to show that

$$\int_{\mathbb{D}}\frac{1-|\zeta|^2}{\left|1-\overline{\zeta}z\right|^2}\exp\left\{-\int_{|\omega|\geq|\zeta|}\frac{1-|\omega|^2}{|1-\overline{\omega}z|^2}d\sigma(\omega)\right\}d\sigma(\zeta)\leq 1.$$

First, suppose that we have $d\sigma=\sum_{j=1}^N a_j\delta_{\zeta_j}$ with $|\zeta_j|\leq|\zeta_{j+1}|$. Further, set $\beta_j=a_j\frac{1-|\zeta_j|^2}{\left|1-\overline{\zeta_j}z\right|^2}$ and

$$t_j=\sum_{k=j}^N\beta_k,\ \text{and so}\ \beta_j=t_j-t_{j-1}.$$

If we evaluate the above integral for this measure and use the resulting notation, we see that the integral becomes

$$\sum_{j=1}^N(t_j-t_{j-1})e^{-t_j}\leq\int_0^\infty e^{-t}dt=1.$$

A standard measure theory argument then finishes the proof that

$$\int_{\mathbb{D}}\frac{1-|\zeta|^2}{\left|1-\overline{\zeta}z\right|^2}\exp\left\{-\int_{|\omega|\geq|\zeta|}\frac{1-|\omega|^2}{|1-\overline{\omega}z|^2}d\sigma(\omega)\right\}d\sigma(\zeta)\leq 1.$$

This then completes the proof of the Theorem. □

The original proof that Jones gave was for the upper half-plane $\mathbb{H}$. The idea is identical, but requires certain modifications.

Exercise 1.3. *State and give the proof of Jones' Theorem in the case of the upper half plane* $\mathbb{H}$.

2. Wolff's Proof of the Corona Theorem

Recall that in the last lecture we finished by proving the following Theorem of Wolff.

Theorem 2.1 (Wolff, [2]). *Suppose that $G(z)$ is bounded and smooth on the disc $\mathbb{D}$. Further, assume that the measures*

$$|G|^2 \log \frac{1}{|z|} dA(z) \text{ and } |\partial G| \log \frac{1}{|z|} dA(z)$$

are $H^2(\mathbb{D})$-Carleson measures. Then there exists a continuous function $b(z)$ on $\overline{\mathbb{D}}$, smooth on $\mathbb{D}$ such that

$$\overline{\partial} b = G$$

and there exists constants C_1 and C_2 such that

$$\|b\|_{L^\infty(\mathbb{D})} \leq C_1 \left\| |G|^2 \log \frac{1}{|z|} dA(z) \right\|_{H^2(\mathbb{D})-Carl} + C_2 \left\| |\partial G| \log \frac{1}{|z|} dA(z) \right\|^2_{H^2(\mathbb{D})-Carl}$$

With this tool, we are now is a place to prove the following important Theorem due to Carleson. The proof we give, will exploit the Theorem above due to Wolff.

Theorem 2.2 (Carleson, [1]). *Suppose that $f_1, \ldots, f_n \in H^\infty(\mathbb{D})$ and there exists a $\delta > 0$ such that*

$$1 \geq \max_{1 \leq j \leq n} \{|f_j(z)|\} \geq \delta > 0.$$

Then there exists $g_1, \ldots, g_n \in H^\infty(\mathbb{D})$ such that

$$1 = f_1(z) g_1(z) + \cdots + f_n(z) g_n(z) \quad \forall z \in \mathbb{D}$$

and

$$\|g_j\|_{H^\infty(\mathbb{D})} \leq C(\delta, n) \quad \forall j = 1, \ldots, n.$$

An obvious remark is that the condition on the functions f is clearly necessary. We can't have all the functions simultaneously vanish if they can generate the function 1.

Before we prove this Theorem, we first consider the case of two functions so that we can see the connections between this problem and the $\overline{\partial}$-problem we initially studied. Suppose we have two functions $f_1, f_2 \in H^\infty(\mathbb{D})$ such that $\max(|f_1(z)|, |f_2(z)|) \geq \delta$. Define the following functions

$$\varphi_1(z) = \frac{\overline{f_1(z)}}{|f_1(z)|^2 + |f_2(z)|^2} \quad \varphi_2(z) = \frac{\overline{f_2(z)}}{|f_1(z)|^2 + |f_2(z)|^2}.$$

The hypotheses on f_1 and f_2 imply that the functions φ_1 and φ_2 are in fact bounded and smooth on $\mathbb{D}$. Note that we have that

$$1 = f_1(z) \varphi_1(z) + f_2(z) \varphi_2(z) \quad \forall z \in \mathbb{D}$$

but the functions φ_1 and φ_2 are in general *not* analytic. Now, observe for any function r we have that the functions

$$g_1 = \varphi_1 + r f_2 \quad g_2 = \varphi_2 - r f_1$$

also solve the problem

$$f_1 g_1 + f_2 g_2 = 1.$$

Our goal is to select a good choice of function r so that the resulting choice will make g_1 and g_2 be analytic and bounded. Now, we have that g_1 is analytic if and only if

$$0 = \overline{\partial} g_1 = \overline{\partial} \varphi_1 + f_2 \overline{\partial} r.$$

Similarly, g_2 is analytic if and only if

$$0 = \overline{\partial} g_2 = \overline{\partial} \varphi_2 - f_1 \overline{\partial} r.$$

Using these two equations and the condition that $f_1\varphi_1 + f_2\varphi_2 = 1$ gives that the function r must satisfy the equation

$$\overline{\partial} r = \varphi_1 \overline{\partial} \varphi_2 - \varphi_2 \overline{\partial} \varphi_1.$$

Thus, we need to solve for the choice of r that will give a bounded solution. Here we will use the fact that when we express what appears on the right hand side of the above equation, we have certain Carleson measures appearing.

Proof. With out lose of generality, we may assume that the functions are analytic in a neighborhood of the closed disc $\overline{\mathbb{D}}$. Define the functions

$$\varphi_j(z) = \frac{\overline{f_j(z)}}{\sum_{j=1}^n |f_j(z)|^2} \quad \forall z \in \mathbb{D}.$$

Then by the hypotheses on the functions f_j we clearly have that $|\varphi_j(z)| \leq C(n, \delta)$. These functions are in general not analytic, and so we must correct them to be so. We thus, will set

$$g_j(z) = \varphi_j(z) + \sum_{k=1}^n a_{j,k}(z) f_k(z)$$

where the functions $a_{j,k}(z)$ are to be determined. However, we will require that $a_{j,k}(z) = -a_{k,j}(z)$. Note that this alternating condition implies that

$$\sum_{j=1}^n f_j(z) g_j(z) = \sum_{j=1}^n f_j(z)\varphi_j(z) + \sum_{j=1}^n \sum_{k=1}^n a_{j,k}(z) f_j(z) f_k(z) = 1.$$

To have the alternating characterisitic of $a_{j,k}$ we set $a_{j,k} = b_{j,k}(z) - b_{k,j}(z)$ for some yet to be determined functions. We will chose the functions $b_{j,k}$ to be solutions to the follow $\overline{\partial}$ problem:

$$\overline{\partial} b_{j,k} = \varphi_j \overline{\partial} \varphi_k := G_{j,k}.$$

Using this, we see that

$$\begin{aligned}
\overline{\partial} g_j &= \overline{\partial}\varphi_j + \sum_{k=1}^n f_k \overline{\partial} a_{j,k} \\
&= \overline{\partial}\varphi_j + \sum_{k=1}^n f_k \left(\overline{\partial} b_{j,k} - \overline{\partial} b_{k,j} \right) \\
&= \overline{\partial}\varphi_j + \sum_{k=1}^n f_k \left(\varphi_j \overline{\partial}\varphi_k - \varphi_k \overline{\partial}\varphi_j \right) \\
&= \overline{\partial}\varphi_j + \varphi_j \overline{\partial} \left(\sum_{k=1}^n f_k \varphi_k \right) - \overline{\partial}\varphi_j \sum_{k=1}^n f_k \varphi_k \\
&= \overline{\partial}\varphi_j + \varphi_j \overline{\partial} 1 - \overline{\partial}\varphi_j 1 = 0.
\end{aligned}$$

So the functions g_j are analytic. Suppose that we prove the functions $b_{j,k}$ are bounded by $C(n,\delta)$, then similarly we have that $a_{j,k}$ are bounded, and so

$$|g_j(z)| \leq C(n,\delta).$$

We are thus left with proving that the functions $b_{j,k}$ are bounded. With this in mind, and having the result of Wolff at our disposal, we must show that the measures

$$|G_{j,k}|^2 \log\frac{1}{|z|} dA(z) \text{ and } |\partial G_{j,k}| \log\frac{1}{|z|} dA(z)$$

are $H^2(\mathbb{D})$-Carleson measures. We claim that each of these measures can be dominated (up to a constant $C(n,\delta)$) by the following measure

$$\sum_{j=1}^n \left|f_j'(z)\right|^2 \log\frac{1}{|z|} dA(z).$$

To see that this is a $H^2(\mathbb{D})$-Carleson measure, it suffices to show that for $f\in H^\infty(\mathbb{D})$ that we have $|f'(z)|^2\log\frac{1}{|z|}dA(z)$ is a $H^2(\mathbb{D})$-Carleson measure, which follows easily from the alternate norm on $H^2(\mathbb{D})$ we previous defined (see the exercise below).

Consider the expression $|G_{j,k}|^2$. Note that by the hypotheses on f_j we have that $|\varphi_j|\leq C(n,\delta)$ and so,

$$|G_{j,k}|^2 \leq C(n,\delta)\left|\overline{\partial}\varphi_k\right|^2.$$

Now, if we compute we see that

$$\begin{aligned}\overline{\partial}\varphi_k &= \frac{\overline{f_k'}}{\sum_{j=1}^n |f_j|^2} - \frac{\overline{f_k}\sum_{j=1}^n f_j\overline{f_j'}}{\left(\sum_{j=1}^n |f_j|^2\right)^2}\\ &= \frac{\sum_{j=1}^n f_j(\overline{f_j f_k'}-\overline{f_k f_j'})}{\left(\sum_{j=1}^n |f_j|^2\right)^2}.\end{aligned}$$

Using this, we see that

$$\left|\overline{\partial}\varphi_k\right|^2 \leq C\frac{\sum_{j=1}^n |f_j|^2\sum_{j=1}^n\left|f_j'\right|^2}{\left(\sum_{j=1}^n |f_j|^2\right)^2} \leq C(n,\delta)\sum_{j=1}^n\left|f_j'\right|^2$$

which proves that

$$|G_{j,k}|^2\log\frac{1}{|z|} \leq C(n,\delta)\sum_{j=1}^n\left|f_j'\right|^2\log\frac{1}{|z|}dA(z).$$

We now turn to showing that $|\partial G_{j,k}|\log\frac{1}{|z|}dA(z)$ is dominated appropriately. First, observe that

$$\partial G_{j,k} = \partial\varphi_j\overline{\partial}\varphi_k + \varphi_j\partial\overline{\partial}\varphi_k$$

By the computations above, we have that

$$\overline{\partial}\varphi_k = \frac{\sum_{j=1}^n f_j(\overline{f_j f_k'}-\overline{f_k f_j'})}{\left(\sum_{j=1}^n |f_j|^2\right)^2}.$$

Direct computation also gives, that

$$\partial\varphi_j = -\frac{\overline{f_j}\sum_{l=1}^{n} f_l'\overline{f_l}}{\left(\sum_{j=1}^{n}|f_j|^2\right)^2}.$$

Finally, we have that

$$\partial\overline{\partial}\varphi_k = \frac{\sum_{j=1}^{n} f_j'(\overline{f_j f_k'} - \overline{f_k f_j'})}{\left(\sum_{j=1}^{n}|f_j|^2\right)^2} - 2\frac{\left(\sum_{l=1}^{n} f_l'\overline{f_l}\right)\left(\sum_{l=1}^{n}\left(\overline{f_l f_k'} - \overline{f_k f_l'}\right)\right)}{\left(\sum_{j=1}^{n}|f_j|^2\right)^3}.$$

Now consider the term $\left|\partial\varphi_j\overline{\partial}\varphi_k\right|$. It is obvious that we can dominate this expression by

$$C(n,\delta)\sum_{j,k}\left|f_j'\right|\left|f_k'\right| \leq C(n,\delta)\sum_{k=1}^{n}\left|f_k'\right|^2.$$

Similarly, we have that $\left|\varphi_j\partial\overline{\partial}\varphi_k\right|$ can be dominated by an identical expression. Altogether this then gives that

$$\left|\partial G_{j,k}\right|\log\frac{1}{|z|}dA(z) \leq C(n,\delta)\sum_{k=1}^{n}\left|f_k'\right|^2\log\frac{1}{|z|}dA(z).$$

□

Exercise 2.3. *Show that for $f \in H^\infty(\mathbb{D})$ that we have $|f'(z)|^2\log\frac{1}{|z|}dA(z)$ is a $H^2(\mathbb{D})$-Carleson measure. Hint: Use the alternate norm for $H^2(\mathbb{D})$ and think about the product rule for derivatives.*

Exercise 2.4. *Suppose that $f_1,\dots,f_n,g \in H^\infty(\mathbb{D})$ such that*

$$|g(z)| \leq \sum_{j=1}^{n}|f_j(z)|.$$

Show that there exists $g_1,\dots,g_n \in H^\infty(\mathbb{D})$ such that

$$g^3 = \sum_{j=1}^{n} f_j g_j.$$

Hint: Mimic Wolff's proof of the Corona Theorem but start with $\psi_j = g\varphi_j$, with φ_j as defined above.

We state one more theorem, closely related to Carleson's Corona Theorem.

Theorem 2.5. *Suppose that $f_1,\dots,f_n \in H^\infty(\mathbb{D})$ and there exists a $\delta > 0$ such that*

$$1 \geq \max_{1\leq j\leq n}\{|f_j(z)|\} \geq \delta > 0.$$

Let $h \in H^2(\mathbb{D})$. Then there exists $g_1,\dots,g_n \in H^2(\mathbb{D})$ such that

$$h(z) = f_1(z)g_1(z) + \cdots + f_n(z)g_n(z) \quad \forall z \in \mathbb{D}$$

and

$$\|g_j\|_{H^2(\mathbb{D})} \leq C(\delta,n)\|h\|_{H^2(\mathbb{D})} \quad \forall j = 1,\dots,n.$$

It is clear that Carleson's Corona Theorem implies this result. What isn't immediately obvious though, is that knowing this result, one can deduce Carleson's Corona Theorem. This is connected to deeper facts about operator theory and will play a role in later lectures.

Exercise 2.6. *Give a proof of the above Theorem without appealing to Carleson's Theorem directly, but instead appealing to the proof of the Theorem.*

References

[1] Lennart Carleson, *Interpolations by bounded analytic functions and the corona problem*, Ann. of Math. (2) **76** (1962), 547–559. ↑4

[2] John B. Garnett, *Bounded analytic functions*, 1st ed., Graduate Texts in Mathematics, vol. 236, Springer, New York, 2007. ↑4

[3] Peter W. Jones, L^∞ *estimates for the* $\bar\partial$ *problem in a half-plane*, Acta Math. **150** (1983), no. 1-2, 137–152. ↑1