Lecture 5: The Maximal Ideal Space of $H^\infty(\mathbb{D})$, $\overline{\partial}$-equations and Wolff's Theorem

Now that we have introduced the multiplier algebra for $H^2(\mathbb{D})$ we will start looking at its maximal ideal space. In doing so, we will connect both analysis and algebra via Carleson's Corona Theorem.

1. The Maximal Ideal Space of $H^\infty(\mathbb{D})$

We now wish to study the maximal ideals space associated to the algebra $H^\infty(\mathbb{D})$. But, before we can do that, we need to review a little about the maximal ideal spaces associated to a Banach algebra.

Recall that a (commutative) Banach algebra $\mathcal{A}$ is a complex (commutative) algebra A that is also a Banach space under a norm that satisfies

$$\|fg\| \leq \|f\| \, \|g\| \quad f, g \in \mathcal{A}.$$

We will also assume that there is an identity element $1 \in \mathcal{A}$ and that our algebra is commutative. An element $f \in \mathcal{A}$ is *invertible* if there exists an element $g \in \mathcal{A}$ such that $fg = 1$. When this happens, we will simply write f^{-1} for g. We let

$$\mathcal{A}^{-1} = \{f \in \mathcal{A} : f^{-1} \text{ exists}\}.$$

Finally, we need to consider the multiplicative linear functionals on the algebra $\mathcal{A}$. These are simply the complex homomorphisms $m : \mathcal{A} \to \mathbb{C}$. Note that we trivially have that $m(1) = 1$.

The first observation is that the complex homomorphisms are continuous and bounded with norm at most 1.

Lemma 1.1. *Every complex homomorphism from $\mathcal{A}$ to $\mathbb{C}$ is a continuous linear functional with norm at most* 1. *Namely,*

$$\|m\| = \sup_{f \in \mathcal{A}, \|f\| \leq 1} |m(f)| \leq 1.$$

Proof. If m is unbounded or if $\|m\| > 1$ then we can find an element $f \in \mathcal{A}$ with $\|f\| < 1$ but $m(f) = 1$. Consider the element $\sum_{n=0}^{\infty} f^n \in \mathcal{A}$. This element exists since we have $\|f\| < 1$ and so

$$\left\| \sum_{n=0}^{\infty} f^n \right\| \leq \sum_{n=0}^{\infty} \|f^n\| \leq \sum_{n=0}^{\infty} \|f\|^n = \frac{1}{1 - \|f\|}.$$

Note that we have $(1 - f) \sum_{n=0}^{\infty} f^n = 1$. So $1 - f \in \mathcal{A}^{-1}$. However,

$$1 = m(1) = m((1-f)(1-f)^{-1}) = m((1-f)^{-1})(m(1) - m(f)) = 0$$

which is a contradiction. $\square$

Exercise 1.2. *Prove that $m_z(f) = f(z)$ is a multiplicative linear functional on $H^\infty(\mathbb{D})$.*

Next, we will connect the algebraic property of maximal ideals and the function analytic property of the multiplicative linear functionals.

Lemma 1.3. *Suppose that M is a maximal ideal of $\mathcal{A}$. Then M is the kernel of a multiplicative linear functional $m : \mathcal{A} \to \mathbb{C}$. Conversely, suppose that $m : \mathcal{A} \to \mathbb{C}$ is a multiplicative linear functional. Then* $\ker m$ *is a maximal ideal.*

Proof. We first show that the maximal ideal M is closed. Note that $M \subset \overline{M}$. If $\overline{M}$ is proper, i.e, $\overline{M} \neq \mathcal{A}$, then $\overline{M}$ is also an ideal. However, since M is maximal, if $\overline{M} \neq \mathcal{A}$ then we must have $\overline{M} = M$ and so M is closed. If $g \in M$ then $g \notin \mathcal{A}^{-1}$ (otherwise we would have $1 \in M$ and so $M = \mathcal{A}$). Consider the element $f = 1 - g$, then we have that $\|1 - g\| \geq 1$, and so $1 \notin \overline{M}$ and so M is closed.

Now we next show that the quotient algebra $\mathcal{B} = \mathcal{A}/M$ satisfies

$$\mathcal{B} = \mathbb{C}\mathbf{1}$$

where $\mathbf{1} = 1 + M$ denotes the unit in the quotient algebra. It is obvious that $\mathbb{C}\mathbf{1} \subset \mathcal{B}$, and so we need to handle the other inclusion. Once we have shown this, then the quotient mapping will then define the multiplicative linear functional, and the kernel of this mapping with then be M.

Since we have M maximal, then we have that $\mathcal{B} = \mathcal{A}/M$ is a field. Moreover, since M is closed we have that $\mathcal{B}$ is complete in the quotient norm

$$\|f + M\| = \inf_{g \in M} \|f + g\|_{\mathcal{A}}.$$

It is also the case that the norm makes $\mathcal{B}$ into a Banach algebra (use the ideal property of M).

Suppose that $f \in \mathcal{B} \setminus \mathbb{C}\mathbf{1}$. Then, we have that $f - \lambda \in \mathcal{B}^{-1}$ for all $\lambda \in \mathbb{C}$ since $\mathcal{B}$ is a field. Choose λ_0 and note that on the disc centered at λ_0 of radius $\|(f - \lambda_0)^{-1}\|$ the series,

$$\sum_{n=0}^{\infty} (\lambda - \lambda_0)^n ((f - \lambda_0)^{-1})^{n+1}$$

converges in norm to the element $(f - \lambda)^{-1}$ because we have the following identity holding

$$\frac{1}{f - \lambda} = \frac{1}{f - \lambda_0} \frac{1}{1 - \frac{\lambda - \lambda_0}{(f - \lambda_0)}}.$$

Now $f^{-1} \neq 0$ and by the Hahn-Banach Theorem, there is a bounded linear functional L on $\mathcal{B}$ such that $\|L\| = 1$ and $L(f^{-1}) \neq 0$.

Define the function on the disc centered at λ_0 of radius $\|(f - \lambda_0)^{-1}\|$

$$F(\lambda) = L((f - \lambda)^{-1}) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n L(((f - \lambda_0)^{-1})^{n+1}).$$

Since the $\|L\| = 1$ and the series defining $(f - \lambda)^{-1}$ is norm convergent this function makes sense. Because λ_0 is arbitrary, we have that $F(\lambda)$ is an entire function. Suppose that $|\lambda|$ is large, then we have that

$$\left\|(f - \lambda)^{-1}\right\| = \frac{\left\|(1 - \frac{f}{\lambda})^{-1}\right\|}{|\lambda|} \leq \frac{1}{|\lambda|} \sum_{n=0}^{\infty} \frac{\|f\|^n}{|\lambda|^n}.$$

Note that this implies for $|\lambda|$ large that we have

$$|F(\lambda)| \leq \frac{C}{|\lambda|}$$

which by Louiville's Theorem implies that $F = 0$. But, this contradict the existence of the functional L and hence gives us that $\mathcal{B} = \mathbb{C}\mathbf{1}$.

For the converse, it is immediate that the $\ker m$ is an ideal. The maximality follows since the codimension of any linear functional is 1. Namely, $\dim(\mathcal{A} \setminus \ker m) = 1$. $\square$

Exercise 1.4. *Verify that if $\mathcal{A}$ is a Banach algebra, and if M is a closed proper maximal ideal, then $\mathcal{A}/M$ is a commutative Banach algebra.*

Exercise 1.5. *Show that for a linear functional that* $\dim(\mathcal{A} \setminus \ker m) = 1$.

Given a Banach algebra $\mathcal{A}$, we let $\mathfrak{M}_{\mathcal{A}}$ denote the set of complex homorphisms of $\mathcal{A}$. This is called the *maximal ideal space* of the Banach algebra. By the above, we have that $\mathfrak{M}_{\mathcal{A}}$ is contained in the unit ball of the dual Banach algebra $\mathcal{A}^*$. We now endow $\mathfrak{M}_{\mathcal{A}}$ with the weak-* topology of $\mathcal{A}^*$. Namely, the basic neighborhood of a $m_0 \in \mathfrak{M}_{\mathcal{A}}$ is determined by $\epsilon > 0$ and by elements $f_1, \ldots, f_n \in \mathcal{A}$ such that

$$V = \{m \in \mathcal{A}^* : \|m\| \leq 1, |m(f_j) - m_0(f_j)| < \epsilon, \quad 1 \leq j \leq n\}.$$

Also, note that we have that $\mathfrak{M}_{\mathcal{A}}$ is a weak-* closed subset of the unit ball of $\mathcal{A}^*$. This is because

$$\mathfrak{M}_{\mathcal{A}} = \{m \in \mathcal{A}^* : \|m\| \leq 1, \quad m(fg) = m(f)m(g), f, g \in \mathcal{A}\}.$$

This topology on $\mathfrak{M}_{\mathcal{A}}$ is called the *Gelfand topology*. In this topology we have that $\mathfrak{M}_{\mathcal{A}}$ is a weak-* closed subset of the unit ball of $\mathcal{A}^*$. Now by the Banach-Alaoglu Theorem, we have that the ball of $\mathcal{A}^*$ is weak-* compact and so we can have that $\mathfrak{M}_{\mathcal{A}}$ is compact Hausdorff space.

We now turn from these abstractions and focus on a particular case of interest $\mathcal{A} = H^\infty(\mathbb{D})$. By the exercise above we see that $\mathbb{D} \subset \mathfrak{M}_{H^\infty}$. Now, we know that the disc $\mathbb{D}$ is open, and that the space $\mathfrak{M}_H^\infty$ is compact, so we can not have them being equal. However, it is conceivable that by taking the closure of the disc $\mathbb{D}$ is the Gelfand topology we could have

$$\mathfrak{M}_{H^\infty} = \overline{\mathbb{D}}^{\text{Gelfand}}.$$

We then define the "Corona" of the algebra $H^\infty(\mathbb{D})$ to be $\mathfrak{M}_{H^\infty} \setminus \overline{\mathbb{D}}^{\text{Gelfand}}$. We now translate this question of density to a question about analytic functions.

Theorem 1.6. *The open disc $\mathbb{D}$ is dense in $\mathfrak{M}_{H^\infty}$ if and only if the following condition holds: If $f_1, \ldots, f_n \in H^\infty(\mathbb{D})$ and if*

$$\max_{1 \leq j \leq n} |f_j(z)| \geq \delta > 0$$

then there exists $g_1, \cdots, g_n \in H^\infty(\mathbb{D})$ such that

$$f_1 g_1 + \cdots + f_n g_n = 1.$$

Proof. Suppose that $\mathbb{D}$ is dense in $\mathfrak{M}_{H^\infty}$. Then, by continuity we have that

$$\max_{1 \leq j \leq n} |m(f_j)| \geq \delta$$

for all $m \in \mathfrak{M}_{H^\infty}$. This implies that $\{f_1, \ldots, f_n\}$ is in no proper ideal of $H^\infty(\mathbb{D})$. Hence the ideal generated by $\{f_1, \ldots, f_n\}$ must contain the constant function 1 and so there exists $g_1, \ldots, g_n \in H^\infty(\mathbb{D})$ such that

$$1 = f_1 g_1 + \cdots + f_n g_n.$$

Conversely, suppose that $\mathbb{D}$ is not dense in $\mathfrak{M}_{\mathcal{A}}$. Then for some $m_0 \in \mathfrak{M}_{\mathcal{A}}$ has a neighborhood disjoint from $\mathbb{D}$ and this neighborhood has the form

$$V = \cap_{j=1}^{n} \{m : |m(j_j)| < \delta\}$$

where $\delta > 0$ and $f_1, \ldots, f_n \in H^\infty(\mathbb{D})$ with $m_0(f_j) = 0$. Since $\mathbb{D} \cap V = \emptyset$ we have that

$$\max_{1 \leq j \leq n} |f_j(z)| \geq \delta > 0.$$

But, it is not possible that we have

$$1 = f_1 g_1 + \cdots + f_n g_n$$

since we have that $m_0(f_j) = 0$ for all $1 \leq j \leq n$. □

Exercise 1.7. *Let $A(\mathbb{D}) = \{f \in \mathrm{Hol}(\mathbb{D}) \cap C(\overline{\mathbb{D}}) : \sup_{z \in \mathbb{D}} |f(z)| < \infty\}$ denote the disc algebra. Show that*

$$\mathfrak{M}_{A(\mathbb{D})} = \overline{\mathbb{D}}.$$

2. The Corona Theorem for $H^\infty(\mathbb{D})$

Our goal is to prove that that a collection of functions $f_1, \ldots, f_n \in H^\infty(\mathbb{D})$ that don't simultaneously vanish as in Theorem 1.6 generate all of $H^\infty(\mathbb{D})$ as an ideal. This was originally shown to be true by Carleson, but we will give a proof due to Wolff's. It turns out to be easy to solve this problem is we want smooth functions (not necessarily analytic), and then our task is to modify the smooth solutions to be analytic. This is accomplished by solving certain inhomogeneous $\overline{\partial}$-equations.

2.1. Solving $\overline{\partial}$-equations.

Recall that a function h is analytic if

$$\overline{\partial} h = 0$$

where $\overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$. We first begin showing how to solve equations of the form

$$\overline{\partial} F = G.$$

Equations of this type will arise naturally in the proof of the Corona Theorem. Suppose that G is smooth with compact support. Then we have the following Theorem

Theorem 2.1. *Suppose that G is a smooth compactly supported function in $\mathbb{D}$. Then*

$$F(z) = \frac{1}{2\pi i} \int_{\mathbb{D}} G(\xi) \frac{1}{z - \xi} d\xi \wedge d\overline{\xi}$$

solves

$$\overline{\partial} F = G.$$

Before we prove this theorem, since we will apply Stokes Theorem, we briefly recall the theory of differential forms in this context. Recall that Stokes Theorem can (roughly) be stated as

$$\int_{\partial \Omega} \omega = \int_{\Omega} d\omega.$$

Here Ω is a nice domain, $\partial\Omega$ is the boundary of Ω, ω is a differential form, and d is the exterior differential.

In two variables, for a smooth function we have that

$$df = \partial_x dx + \partial_y dy$$

Recall also that $dx \wedge dy = -dy \wedge dx$ and that $dx \wedge dx = dy \wedge dy = 0$. Since we are working with complex variables it is more conducive to write this in the variables z and $\overline{z}$. In this notation we have

$$df = \overline{\partial} f d\overline{z} + \partial f dz$$

where

$$dz = dx + idy \quad d\overline{z} = dx - idy \quad \overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y) \quad \partial = \frac{1}{2}(\partial_x - i\partial_y) \quad dz \wedge d\overline{z} = -2idx \wedge dy.$$

To apply Stokes Theorem, we will have to integrate 2-forms, i.e. expressions like $\omega(\xi)d\xi \wedge d\overline{\xi}$ over the domain Ω. And we will have to integrate 1-forms, i.e., expressions of the form $\omega(z)dz + \sigma(z)d\overline{z}$. With these notions out of the way, we can turn to the proof

Proof of Theorem 2.1. Fix $\epsilon > 0$ and $z \in \mathbb{D}$ let $\mathbb{D}_\epsilon(z) = \{\xi \in \mathbb{D} : |z - \xi| \geq \epsilon\}$. Note that $\partial\mathbb{D}_\epsilon = \mathbb{T} \cup \{\xi : |\xi - z| = \epsilon\}$. Suppose that φ is a smooth compactly supported function in $\mathbb{D}$. Then, we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\mathbb{D}_\epsilon} \overline{\partial}\varphi(\xi)\frac{1}{\xi - z} d\xi \wedge d\overline{\xi} &= -\frac{1}{2\pi i}\int_{\mathbb{D}_\epsilon} \overline{\partial}\left(\frac{\varphi(\xi)}{\xi - z}\right) d\overline{\xi} \wedge d\xi \\
&= \frac{1}{2\pi i}\int_{|\xi - z| = \epsilon} \frac{\varphi(\xi)}{\xi - z} d\xi + \int_{\mathbb{T}} \frac{\varphi(\xi)}{\xi - z} d\xi \\
&= \frac{1}{2\pi i}\int_{|\xi - z| = \epsilon} \frac{\varphi(\xi)}{\xi - z} d\xi.
\end{aligned}$$

Here we have used the fact that the support of $\varphi \subset \mathbb{D}$ to conclude that the last integral is 0. We have also used the following computation

$$d\left(\frac{\varphi(\xi)}{\xi - z}\right) d\xi = \overline{\partial}\left(\frac{\varphi(\xi)}{\xi - z}\right) d\overline{\xi} \wedge d\xi + \partial\left(\frac{\varphi(\xi)}{\xi - z}\right) d\xi \wedge d\xi$$

Now note that as $\epsilon \to 0$ we have that

$$\frac{1}{2\pi i}\int_{|\xi - z| = \epsilon} \frac{\varphi(\xi)}{\xi - z} d\xi \to \varphi(z).$$

This says that

$$\frac{1}{2\pi i}\int_{\mathbb{D}} \overline{\partial}\varphi(\xi)\frac{1}{\xi - z} d\xi \wedge d\overline{\xi} = \varphi(z).$$

So, if we have a solution to the problem $\overline{\partial} F = G$ then one solution should be given by

$$F(z) = \frac{1}{2\pi i}\int_{\mathbb{D}} G(\xi)\frac{1}{\xi - z} d\xi \wedge d\overline{\xi}.$$

Note that this solution is continuous in the complex plane and smooth in the disc since it is the convolution of a continuous function and a bounded function. We now show that we do indeed have $\overline{\partial} F = G$. First, note that

$$\begin{aligned}
\int_{\mathbb{D}} F\overline{\partial}\varphi dz \wedge d\overline{z} + \int_{\mathbb{D}} \overline{\partial} F\varphi dz \wedge d\overline{z} &= \int_{\mathbb{D}} \overline{\partial}(F\varphi) dz \wedge d\overline{z} \\
&= \int_{\mathbb{T}} F\varphi dz = 0.
\end{aligned}$$

Here again we have used the support of φ and the analogous computations from above. This then implies

$$\int_{\mathbb{D}} F\overline{\partial}\varphi dz \wedge d\overline{z} = -\int_{\mathbb{D}} \overline{\partial}F\varphi dz \wedge d\overline{z}.$$

Using these computations we see that

$$\begin{aligned}\int_{\mathbb{D}} \overline{\partial}F\varphi dz \wedge d\overline{z} &= -\int_{\mathbb{D}} F\overline{\partial}\varphi dz \wedge d\overline{z} \\ &= -\int_{\mathbb{D}} \left(\frac{1}{2\pi i}\int_{\mathbb{D}} G(\xi)\frac{1}{\xi - z}d\xi \wedge d\overline{\xi}\right)\overline{\partial}\varphi dz \wedge d\overline{z} \\ &= -\int_{\mathbb{D}} G\left(\frac{1}{2\pi i}\int_{\mathbb{D}} \overline{\partial}\varphi\frac{1}{\xi - z}dz \wedge d\overline{z}\right) d\xi \wedge d\overline{\xi} \\ &= \int_{\mathbb{D}} G\varphi d\xi \wedge d\overline{\xi}.\end{aligned}$$

Since this is true for all smooth compactly supported φ in $\mathbb{D}$ we have that

$$\overline{\partial}F = G$$

as claimed. □

This Theorem demonstrates that it is possible to solve equations for the form $\overline{\partial}F = G$. However, we will want to solve the equation with some norm control, in particular we want to solve the equation and obtain estimates on $\|F\|_{\infty}$ in terms of information from G. To accomplish this, we will assume the the function G "generates" Carleson measures for $H^2(\mathbb{D})$. We now prove a result of Wolff that gives the desired estimates.

Theorem 2.2 (Wolff, [1]). *Suppose that $G(z)$ is bounded and smooth on the disc $\mathbb{D}$. Further, assume that the measures*

$$|G|^2 \log\frac{1}{|z|}dA(z) \text{ and } |\partial G| \log\frac{1}{|z|}dA(z)$$

are $H^2(\mathbb{D})$-Carleson measures. Then there exists a continuous function $b(z)$ on $\overline{\mathbb{D}}$, smooth on $\mathbb{D}$ such that

$$\overline{\partial}b = G$$

and there exists constants C_1 and C_2 such that

$$\|b\|_{L^\infty(\mathbb{D})} \leq C_1 \left\| |G|^2 \log\frac{1}{|z|}dA(z)\right\|_{H^2(\mathbb{D})-Carl} + C_2 \left\| |\partial G| \log\frac{1}{|z|}dA(z)\right\|^2_{H^2(\mathbb{D})-Carl}$$

The proof of this Theorem is a clever application of Green's Theorem and using the conditions on the measures appropriately.

Proof. By Theorem 2.1 above, we clearly have one solution to the problem

$$\overline{\partial}b = G.$$

Note now that we can obtain lots of solutions by adding functions that are in the kernel of the operator $\overline{\partial}$ and any function h in the disc algebra $A(\mathbb{D})$ allows us to have that $b + h$ also satisfies that $\overline{\partial}(b + h) = G$. The goal is to select a good choice of the function h that allows us to obtain the estimates we seek.

A duality argument show that

$$\inf\left\{\|b\|_\infty : \overline{\partial} b = G\right\} = \sup\left\{\left|\int_{\mathbb{T}} F\overline{k_1 k_2}dm\right| : k_1 \in \overline{H^2(\mathbb{D})}, k_2 \in \overline{H^2_0(\mathbb{D})}\right\}.$$

Here we have that the function F is defined as in the Theorem 2.1. Now since we are supposing that G is bounded and smooth, we have that F is smooth on the $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. A density argument lets us further assume that the functions k_1 and k_2 are smooth across the boundary of $\mathbb{D}$ (just consider dilates of the functions $f_r(z) = f(rz)$ and apply a normal family argument). Without loss of generality, we can also assume that $\|k_1\|_2 \leq 1$ and $\|k_1\|_2 \leq 1$.

Now, we apply Green's Theorem to the function $F\overline{k_1 k_2}$. First, we compute the Laplacian of the function $F\overline{k_1 k_2}$ since it will appear in Green's Theorem. Doing so, we find

$$\begin{aligned}\Delta(F\overline{k_1 k_2}) &= 4\partial\left(\overline{\partial} F\overline{k_1 k_2} + F\overline{\partial}(\overline{k_1 k_2})\right)\\ &= 4\left(\partial G\overline{k_1 k_2} + G\left(\overline{k_1' k_2 + k_1 k_2'}\right)\right).\end{aligned}$$

Here we have used that $\overline{\partial} F = G$ and that $k_1 k_2$ is anti-holomorphic.

Substituting in this information we find:

$$\begin{aligned}\int_{\mathbb{T}} F\overline{k_1 k_2}dm &= F(0)\overline{k_1(0)k_2(0)} + \int_{\mathbb{D}} \Delta(F\overline{k_1 k_2})\log\frac{1}{|z|}dA\\ &= 2\int_{\mathbb{D}} \partial G\overline{k_1 k_2}\log\frac{1}{|z|}dA(z) + 2\int_{\mathbb{D}} G\left(\overline{k_1' k_2 + k_1 k_2'}\right)\log\frac{1}{|z|}dA(z)\\ &= I + II.\end{aligned}$$

We estimate each of these integrals separately. First, consider the integral corresponding to I. Making obvious estimates, we have

$$\begin{aligned}\left|\int_{\mathbb{D}} \partial G\overline{k_1 k_2}\log\frac{1}{|z|}dA(z)\right| &\leq \int_{\mathbb{D}} |k_1|\,|k_2|\,|\partial G|\log\frac{1}{|z|}dA(z)\\ &\leq \left(\int_{\mathbb{D}} |k_1|^2\,|\partial G|\log\frac{1}{|z|}dA(z)\right)^{\frac{1}{2}}\left(\int_{\mathbb{D}} |k_2|^2\,|\partial G|\log\frac{1}{|z|}dA(z)\right)^{\frac{1}{2}}\\ &\leq \left\||\partial G|\log\frac{1}{|z|}dA(z)\right\|^2_{H^2(\mathbb{D})-Carl}\|k_1\|_{H^2(\mathbb{D})}\|k_2\|_{H^2(\mathbb{D})}.\end{aligned}$$

Next, turning to term II, one easily sees that it suffices to handle the term $k_1' k_2$ since the other will follow by symmetry. So, consider the following,

$$\begin{aligned}\left|\int_{\mathbb{D}} G\overline{k_1' k_2}\log\frac{1}{|z|}dA(z)\right| &\leq \int_{\mathbb{D}} \left|G\overline{k_1' k_2}\right|\log\frac{1}{|z|}dA(z)\\ &\leq \left(\int_{\mathbb{D}} |k_1'|^2\log\frac{1}{|z|}dA(z)\right)^{\frac{1}{2}}\left(\int_{\mathbb{D}} |k_2|^2\,|G|^2\log\frac{1}{|z|}dA(z)\right)^{\frac{1}{2}}\\ &\leq \left\||G|^2\log\frac{1}{|z|}\right\|_{H^2(\mathbb{D})-Carl}\|k_1\|_{H^2(\mathbb{D})}\|k_2\|_{H^2(\mathbb{D})}\end{aligned}$$

Thus, we see that we have the following estimate for term II

$$\int_{\mathbb{D}} G\left(\overline{k_1' k_2 + k_1 k_2'}\right)\log\frac{1}{|z|}dA(z) \leq C\left\||G|^2\log\frac{1}{|z|}\right\|_{H^2(\mathbb{D})-Carl}\|k_1\|_{H^2(\mathbb{D})}\|k_2\|_{H^2(\mathbb{D})}.$$

Putting the estimates for terms I and II together gives us that

$$\begin{aligned}\left|\int_{\mathbb{T}} F\overline{k_1 k_2} dm\right| \quad \leq \quad & C_1 \left\| |G|^2 \log\frac{1}{|z|}\right\|_{H^2(\mathbb{D})-Carl} \|k_1\|_{H^2(\mathbb{D})} \|k_2\|_{H^2(\mathbb{D})} \\ & + C_2 \left\| |\partial G| \log\frac{1}{|z|} dA(z)\right\|_{H^2(\mathbb{D})-Carl}^2 \|k_1\|_{H^2(\mathbb{D})} \|k_2\|_{H^2(\mathbb{D})},\end{aligned}$$

which then proves the claim. □

References

[1] John B. Garnett, *Bounded analytic functions*, 1st ed., Graduate Texts in Mathematics, vol. 236, Springer, New York, 2007. ↑6