Lecture 3: Carleson Measures via Harmonic Analysis

Much of the argument from this section is taken from the book by Garnett, [1]. The interested reader can also see variants of this argument in the book by Stein, [2]. The arguments in Stein are carried out more generally in $\mathbb{R}^n$.

Before we can determine the Carleson measures for the Hardy space, we need to make a excursion into maximal functions and non-tangential maximal functions. We work now in the case of $\mathbb{H}$ since many of the computations are easier. The results are true on the disc via appropriate modifications and changes. We indicate the necessary changes that need to be made in one of the exercises below.

Let I denote an interval in $\mathbb{R}$. Then, the *Hardy-Littlewood* maximal function of f is defined by

$$Mf(x) = \sup_{x\in I} \frac{1}{|I|}\int_I |f(t)|dt.$$

It is immediate to see that the maximal function is bounded on $L^\infty(\mathbb{R})$, i.e.,

$$\|Mf\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^\infty(\mathbb{R})}.$$

In fact, more is true.

Theorem 0.1 (Hardy-Littlewood Maximal Theorem)**.** *If $f \in L^p(\mathbb{R})$, $1 < p \leq \infty$, the $Mf(t)$ is finite almost everywhere.*

(a) *If $f \in L^1(\mathbb{R})$, then Mf is weak-type* $(1,1)$,

$$|\{t \in \mathbb{R} : Mf(t) > \lambda\}| \leq \frac{2}{\lambda}\|f\|_{L^1(\mathbb{R})};$$

(b) *If $f \in L^p(\mathbb{R})$, with $1 < p \leq \infty$, then $Mf \in L^p(\mathbb{R})$ and*

$$\|Mf\|_{L^p(\mathbb{R})} \leq C(p)\|f\|_{L^p(\mathbb{R})}$$

with $C(p)$ depending only on p.

The proof of this Theorem is standard in any harmonic analysis (or real analysis) textbook. First, one proves that (a) holds true. This is accomplished by the Calderón-Zygmund decomposition of a function $f \in L^1(\mathbb{R})$. Then one uses Marcinkiewicz Interpolation to conclude that the result is true for all $1 < p \leq \infty$.

Exercise 0.2. *If you are not familiar with this Theorem, look it up and write out the details of the Proof of Theorem 0.1.*

We now define a related maximal function on *harmonic* functions. First, recall that the Poisson kernel for the the upper half plane $\mathbb{H}$ is given by

$$P_z(t) = P_y(x-t) = \frac{1}{\pi}\text{Im}\left(\frac{1}{t-z}\right) = \frac{1}{\pi}\frac{y}{(x-t)^2+y^2}.$$

Theorem 0.3. *For $\alpha > 0$ and $t \in \mathbb{R}$, let $\Gamma_\alpha(t)$ be the cone in $\mathbb{H}$ with vertex t and angle $2\tan^{-1}\alpha$,*

$$\Gamma_\alpha(t) = \{(x,y) : |x-t| < \alpha y, 0 < y < \infty\}.$$

Suppose that $f \in L^1\left(\frac{dt}{1+t^2}\right)$ and let $u(x,y)$ denote the Poisson extension of $f(t)$,

$$u(x,y) = \int_{\mathbb{R}} f(x-s)P_y(s)ds$$

Then

$$\sup_{(x,y)\in\Gamma_\alpha(t)} |u(x,y)| \leq A_\alpha Mf(t), \quad t \in \mathbb{R}$$

The expression on the left-hand side is called the non-tangential maximal function associated to the function f, while the condition on the function f just makes sure that the integral defining u converges.

Proof. First, assume that $t = 0$. We focus now on points of the form $(0,y)$ that lie on the axis of the cone $\Gamma_\alpha(0)$. We have that

$$u(0,y) = \int_{\mathbb{R}} P_y(s)f(s)ds.$$

Now, note that the Poisson kernel is a positive even function which is decreasing for positive s, so it is a convex combination of box kernels

$$\frac{1}{2h}\chi_{(-h,h)}(s).$$

To see this, sketch the Poisson kernel and then it becomes obvious to see. Note that these kernels are what appears in the definition of the maximal function Mf.

Now, take step functions $h_n(s)$ which are also nonnegative, even, and decreasing for $s > 0$ and such that $h_n(s)$ increases to $P_y(s)$. We have that

$$h_n(s) = \sum_{j=1}^{N} a_j \chi_{(-x_j,x_j)}(s)$$

with $a_j \geq 0$ and

$$\int_{\mathbb{R}} h_n(s)ds = \sum_{j=1}^{N} 2x_j a_j \leq 1 = \int_{\mathbb{R}} P_y(s)ds.$$

Using this we see that

$$\begin{aligned}\left|\int_{\mathbb{R}} h_n(s)f(s)ds\right| &\leq \int_{\mathbb{R}} h_n(s)\,|f(s)|\,ds \\ &\leq \sum_{j=1}^{N} 2x_j a_j \frac{1}{2x_j}\int_{-x_j}^{x_j} |f(s)|\,ds \\ &\leq Mf(0).\end{aligned}$$

Now fix $(x,y) \in \Gamma_\alpha(0)$. Then we have that $|x| < \alpha y$ and $P_y(x-s)$ is majorized by a positive even function $\psi(s)$ which is decreasing on $s > 0$, such that

$$\int_{\mathbb{R}} \psi(s)ds \leq A_\alpha.$$

Simply take $\psi(s) = \sup\{|P_y(x-t)| : |t| > s\}$. Then, approximating ψ by step functions h_n as before, we conclude in the same manner that

$$\int_{\mathbb{R}} \psi(s)\,|f(s)|\,ds \leq A_\alpha Mf(0).$$

These put together imply that

$$|u(x,y)| \leq \int_{\mathbb{R}} \psi(s)\,|f(s)|\,ds \leq A_\alpha Mf(0).$$

□

Exercise 0.4. *Show that the same theorem remains true for the case of the unit disc* $\mathbb{D}$*. Here, the cone* $\Gamma_\alpha(t)$ *must be replaced by the set*

$$\Gamma_\alpha(e^{i\theta}) = \left\{z \in \mathbb{D} : |1 - ze^{-i\theta}| < \alpha(1-|z|)\right\}$$

and one uses the Poisson kernel for the disc,

$$P_z(w) = \frac{1-|z|^2}{|1-\overline{z}w|^2}.$$

One need only repeat the proof in this context.

Since the non-tangential maximal function will play a role for the rest of this section, we will make a definition encapsulating what we have done. Fix $\alpha > 0$ and consider the cone

$$\Gamma_\alpha(t) = \{z \in \mathbb{H} : |x-t| < \alpha y\} \quad t \in \mathbb{R}.$$

For a harmonic function u on $\mathbb{H}$, we define the non-tangential maximal function of u at $t \in \mathbb{R}$ as

$$u^*(t) = \sup_{z \in \Gamma_\alpha(t)} |u(z)|.$$

We then have the following theorem

Theorem 0.5. *Let* $u(z)$ *be harmonic in* $\mathbb{H}$ *and suppose that* $1 \leq p < \infty$*. Assume that*

$$\sup_y \int_{\mathbb{R}} |u(x+iy)|^p dx < \infty.$$

If $p > 1$ *then* $u^*(t) \in L^p(\mathbb{R})$ *and*

$$\|u^*\|_{L^p(\mathbb{R})}^p \leq C(p,\alpha) \sup_y \int_{\mathbb{R}} |u(x+iy)|^p dx.$$

If $p = 1$ *then* u^* *is weak-type* $(1,1)$ *and*

$$|\{t \in \mathbb{R} : u^*(t) > \lambda\}| \leq \frac{C(p,\alpha)}{\lambda} \sup_y \int_{\mathbb{R}} |u(x+iy)|dx.$$

Here, the constant $C(p,\alpha)$ *depends only on* p *and* α*.*

Proof. Suppose that $p > 1$. Then, note that $u(z)$ is the Poisson integral of a function $f \in L^p(\mathbb{R})$ and

$$\|f\|_{L^p(\mathbb{R})} \leq \sup_y \left(\int_{\mathbb{R}} |u(x+iy)|^p dx\right)^{\frac{1}{p}}$$

By Theorem 0.3 we have that $u^* \leq C(\alpha)Mf$. If we use this, along with Theorem 0.1 then we see

$$\begin{aligned}
\|u^*\|_{L^p(\mathbb{R})}^p &\leq C(\alpha)\,\|Mf\|_{L^p(\mathbb{R})}^p \\
&\leq C(p,\alpha)\,\|f\|_{L^p(\mathbb{R})}^p \\
&\leq \sup_y \left(\int_{\mathbb{R}} |u(x+iy)|^p dx\right)^{\frac{1}{p}}.
\end{aligned}$$

□

Exercise 0.6. *Complete the details of the proof when $p = 1$.*

We now, for the remainder of this section, take as the *definition* of a Carleson measure μ on $\mathbb{H}$ as a measure for which,

$$\mu(T(I)) \leq C(\mu)|I|$$

where $I = (x_0, x_0 + h)$ and

$$T(I) = \{z = x + iy \in \mathbb{H} : x_0 < x < x_0 + h, 0 < y < h\}$$

We then begin with a Lemma that will allow us to give the necessary characterization of the Carleson measures for $H^2(\mathbb{H})$. One can compare this with the related conditions for the case of the disc $\mathbb{D}$.

Exercise 0.7. *Show that in the definition of a Carleson measure, one can take "any" sort of reasonable geometric object to test over that uses one-parameter to describe its geometry. For example, take any $x \in \mathbb{R}$ and $r > 0$ and consider the $B_r(x)$, ball of radius r centered at x. Show that μ is a Carleson measure if and only if for any $x \in \mathbb{R}$ and $r > 0$ that*

$$\mu\left(B_r(x) \cap \mathbb{H}\right) \leq Cr.$$

Lemma 0.8. *Let μ be a positive measure on $\mathbb{H}$ and let $\alpha > 0$. Then μ is a Carleson measure if and only if there exists $C(\alpha)$ such that for every harmonic function u on $\mathbb{H}$*

$$\mu\left(\{z \in \mathbb{H} : |u(z)| > \lambda\}\right) \leq C(\alpha)\left|\{t \in \mathbb{R} : u^*(t) > \lambda\}\right| \quad \lambda > 0.$$

Moreover, we have that $C_1 C(\alpha) \leq C(\mu) \leq C_2 C(\alpha)$.

Proof. Take $\alpha = 1$, the case of different α is left as an exercise. First, assume that μ is a Carleson measure. Note that the set

$$\{t : u^*(t) > \lambda\}$$

is an open set, and so is a disjoint union of open intervals $\{I_j\}$ with centers c_j. Let $T_j = T(I_j)$ denote the corresponding tent

$$T_j = \left\{z : |x - c_j| + y < \frac{|I_j|}{2}\right\}$$

If it is the case that $|u(z)| > \lambda$, then on the interval $\{t \in \mathbb{R} : |t - x| < y\}$ we have that $u^*(t) > \lambda$ and that this interval is contained in one of the intervals I_j.

So by the computations above, it is the case that

$$\{z \in \mathbb{H} : |u(z)| > \lambda\} \subset \bigcup_j T_j$$

Thus, we have

$$\begin{aligned}\mu\left(\{z\in\mathbb{H}:|u(z)|>\lambda\}\right) &\leq \sum_j \mu\left(T_j\right)\\ &\leq C(\mu)\sum_j |I_j|\\ &= C(\mu)\left|\{t\in\mathbb{R}:u^*(t)>\lambda\}\right|.\end{aligned}$$

Conversely, suppose that $I\subset\mathbb{R}$ is an open interval, $I=(x_0,x_0+h)$. Let $u(z)=P_y*f(x)$ where $f(t)=4\lambda\chi_I(t)$. Then, by the hypotheses of the lemma we have

$$\begin{aligned}\mu\left(T(I)\right) &\leq C(\alpha)\left|\{t\in\mathbb{R}:u^*(t)>\lambda\}\right|\\ &\leq \frac{C(\alpha)}{\lambda}\left\|f\right\|_{L^1(\mathbb{R})}\\ &\leq C(\alpha)\left|I\right|.\end{aligned}$$

□

Exercise 0.9. *Complete the details for the case of different α.*

Exercise 0.10. *Show that if $z\in\Gamma_\alpha(t)$ then $z\in\Gamma_\alpha(t_0)$ for all $\{t_0:|x-t_0|<\alpha y\}$. Use this to conclude that if $|u(z)|>\lambda$ then $u^*(t)>\lambda$ for $\{t:|x-t|<\alpha y\}$*

Exercise 0.11. *Show that for $u(z)=P_y*f(x)$ where $f(t)=4\lambda\chi_I(t)$ that $|u(z)|>\lambda$ on $T(I)$. Hint: Draw a picture.*

Using this we can conclude the proof of Carleson's Embedding Theorem.

Theorem 0.12. *Let $f\in L^p(\mathbb{R})$ and let $u(z)$ denote the Poisson integral of f. If μ is a positive measure on the upper half plane, then the following are equivalent*

(a) *μ is a (geometric) Carleson measure, i.e., $\mu\left(T(I)\right)\leq C(\mu)\left|I\right|$ for any $I\subset\mathbb{R}$;*
(b) *For $1<p<\infty$ and for all $f\in L^p(\mathbb{R})$, $u(z)\in L^p(\mathbb{H};\mu)$;*
(c) *For $1<p<\infty$*

$$\int_{\mathbb{H}}|u(z)|^p\,d\mu(z)\leq C(p)\int_{\mathbb{R}}|f(t)|^p\,dt\quad\forall f\in L^p(\mathbb{R}).$$

Proof. It is clear the (b) and (c) are equivalent. Also, we have already seen that if (c) holds, then by testing on the box-type kernel then we have (a) holding as well.

It only remains to handle the case of (a) implies (c). To accomplish this, we use Lemma 0.8 and compute the integral via the distribution inequality.

$$\begin{aligned}\int_{\mathbb{H}}|u(z)|^p\,d\mu(z) &= \int_0^\lambda p\lambda^{p-1}\{z\in\mathbb{H}:|u(z)|>\lambda\}d\lambda\\ &\leq C(\alpha)\int_0^\lambda p\lambda^{p-1}\{t\in\mathbb{R}:u^*(t)>\lambda\}d\lambda\\ &= C(\alpha)\int_{\mathbb{R}}|u^*(t)|^p\,dt\\ &\leq C(\alpha)\int_{\mathbb{R}}Mf(t)^p dt\\ &\leq C(\alpha,p)\int_{\mathbb{R}}|f(t)|^p\,dt.\end{aligned}$$

□

The careful reader will notice that this proof actually only gives the Carleson measures for the harmonic embedding. However since any function in $H^2(\mathbb{D})$ is harmonic, this gives another proof of the Carleson Embedding Theorem we proved in the last lecture.

We gave two different proofs characterizing the Carleson measures for $H^2(\mathbb{D})$ for a reason. When we study the Besov-Sobolev spaces $B^2_\sigma(\mathbb{D})$, then it will turn out that the proof via reproducing kernels unfortunately fails. However, the proof via harmonic analysis will carry through with almost the same proof as what appears in this Lecture.

References

[1] John B. Garnett, *Bounded analytic functions*, 1st ed., Graduate Texts in Mathematics, vol. 236, Springer, New York, 2007. ↑1

[2] Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy; Monographs in Harmonic Analysis, III. ↑1