Lecture 2: Carleson Measures for $H^2(\mathbb{D})$

Our next goal is to prove the following important theorem about certain measures for $H^2(\mathbb{D})$.

Theorem 0.1 (Carleson Embedding Theorem)**.** *Let μ be a non-negative Borel measure in $\mathbb{D}$. Then the following are equivalent:*

(i) *The embedding operator $\mathcal{J} : L^2(\mathbb{T}) \to L^2(\mathbb{D}, \mu)$, with $\mathcal{J}(f)(z) = f(z)$, is bounded.*

(ii) $C(\mu)^2 := \sup_{z\in\mathbb{D}} \left\| \mathcal{J}\tilde{k}_z \right\|_{L^2(\mu)}^2 = \sup_{z\in\mathbb{D}} \|P_z\|_{L^1(\mu)} < \infty$*, where $\tilde{k}_z(\xi) = \frac{(1-|z|^2)^{1/2}}{(1-\xi\overline{z})}$, the reproducing kernel for the Hardy space $H^2(\mathbb{D})$.*

(iii) $I(\mu) = \sup\left\{\frac{1}{r}\mu(\mathbb{D}\bigcap Q(\xi,r)) : r > 0, \xi \in \mathbb{T}\right\} < \infty$*, where $Q(\xi, r)$ is a ball measured with respect to the non-isotropic metric associated to $\mathbb{D}$.*

Moreover, the following inequalities hold

$$C(\mu) \leq \|\mathcal{J}\| \leq 4C(\mu)$$

and

$$32^{-1}I(\mu) \leq C(\mu)^2 \leq 32I(\mu)$$

Here the embedding operator $\mathcal{J}$ is given by

$$\mathcal{J}(f)(z) = \int_{\mathbb{T}} P_z(\xi)f(\xi)dm(\xi)$$

where

$$P_z(\xi) = \frac{1-|z|^2}{|1-\overline{\xi}z|^2}$$

is the Poisson kernel.

We will give two proofs of this fact. One will be based on the tools of harmonic analysis (maximal functions). The other will take advantage of the special structure of the reproducing kernel of the space $H^2(\mathbb{D})$. We will see that the proof via harmonic analysis will allow us to study the Dirichlet space, while the proof that exploits the structure of the kernel unfortunately doesn't carry over.

First, we say that a non-negative Borel measure μ on $\mathbb{D}$ is a $H^2(\mathbb{D})$-Carleson measure if

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq C(\mu)^2 \|f\|_{H^2(\mathbb{D})}^2 \quad \forall f \in H^2(\mathbb{D}).$$

Observe that this is saying that $H^2(\mathbb{D})$ continuously embeds into $L^2(\mathbb{D};\mu)$. This is nothing other than condition (i) in Theorem 0.1 when applied to the boundary values of $f \in H^2(\mathbb{D})$ and then carried back to the unit disc via harmonic extension.

Note that since we have a natural set of functions by which to test this inequality, namely the reproducing kernels, $\{k_z\}_{z\in\mathbb{D}}$, we see that a *necessary* condition is that

$$\int_{\mathbb{D}} |k_z(w)|^2 \, d\mu(w) \leq C(\mu)^2 \|k_z\|_{H^2(\mathbb{D})}^2 .$$

Upon rearrangement, this gives condition (ii) from Theorem 0.1.

Finally, observe that if we restrict the integration to a natural set associated to the kernel k_z then we see that condition (iii) arises naturally. Namely, note that to each point $z \in \mathbb{D}$ we can associate an interval in $\mathbb{T}$ with center $\frac{z}{|z|}$ and total length $(1-|z|)$. We let $T(I)$ denote the "tent" over an interval I,

$$T(I) = \{z = re^{i\theta} \in \mathbb{D} : 0 < 1 - r \leq |I|, \theta \in I\}.$$

Then it is possible to phrase condition (iii) as

$$\mu\left(T(I)\right) \leq T(\mu)|I| \quad \forall I \subset \mathbb{T}.$$

0.1. **Carleson Measures via Reproducing Kernels.** We first give a proof of Theorem 0.1 that uses the reproducing kernel directly. This proof can be found as an exercise in the book [1].

0.1.1. *(i)* $\iff$ *(ii)*. One direction of this equivalence is immediate, namely (i)⇒(ii). For if we know that the embedding operator $\mathcal{J}$ is bounded then we have

$$\left\|\mathcal{J}\tilde{k}_z\right\|_{L^2(\mu)} \leq \|\mathcal{J}\| \left\|\tilde{k}_z\right\|_{L^2(\mathbb{T})} = \|\mathcal{J}\|$$

since the reproducing kernel $\tilde{k}_z$ is normalized to have $L^2(\mathbb{T})$ norm one. This also proves that $C(\mu) \leq \|\mathcal{J}\|$. Finally, we should indicate why the equality

$$\sup_{z\in\mathbb{D}} \left\|\mathcal{J}\tilde{k}_z\right\|^2_{L^2(\mu)} = \sup_{z\in\mathbb{D}} \|P_z\|_{L^1(\mu)}$$

holds. Since $\tilde{k}_z \in H^2(\mathbb{D})$ then the Poisson kernel P_w reproduces the function value at w, namely

$$\tilde{k}_z(w) = \int_{\mathbb{T}} \tilde{k}_z(\xi) P_w(\xi) dm(\xi).$$

This then implies that $\mathcal{J}(\tilde{k}_z)(w) = k_z(w)$, and using this we have

$$\begin{aligned}
\left\|\mathcal{J}\tilde{k}_z\right\|^2_{L^2(\mu)} &= \int_{\mathbb{D}} \left|\mathcal{J}(\tilde{k}_z)(w)\right|^2 d\mu(w) \\
&= \int_{\mathbb{D}} \left|\tilde{k}_z(w)\right|^2 d\mu(w) \\
&= \int_{\mathbb{D}} \left|\frac{(1-|z|^2)^{1/2}}{(1-\overline{z}w)}\right|^2 d\mu(w) \\
&= \int_{\mathbb{D}} \frac{(1-|z|^2)}{|1-z\overline{w}|^2} d\mu(w) \\
&= \int_{\mathbb{D}} P_z(w) d\mu(w).
\end{aligned}$$

This then shows that $\left\|\mathcal{J}\tilde{k}_z\right\|^2_{L^2(\mu)} = \|P_z\|_{L^1(\mu)}$, and then these suprema over $z \in \mathbb{D}$ are of course equal.

It only remains to prove that (ii)⇒(i). To prove this direction it is enough to show that the formal adjoint $\mathcal{J}^* : L^2(\mathbb{D},\mu) \to L^2(\mathbb{T})$ with

$$\mathcal{J}^*(f)(\xi) = \int_{\mathbb{D}} P_z(\xi) f(z) d\mu(z),$$

is bounded. With this in mind we have,

$$\begin{aligned}
\|\mathcal{J}^*(f)\|^2_{L^2(\mathbb{T})} &= \int_{\mathbb{T}} |\mathcal{J}^*(f)(\xi)|^2 \, dm(\xi) \\
&= \int_{\mathbb{T}} \mathcal{J}^*(f)(\xi)\overline{\mathcal{J}^*(f)(\xi)} dm(\xi) \\
&= \int_{\mathbb{T}} \left(\int_{\mathbb{D}} P_z(\xi) f(z) d\mu(z) \right) \left(\int_{\mathbb{D}} P_{z'}(\xi) \overline{f(z')} d\mu(z') \right) dm(\xi) \\
&= \int_{\mathbb{D}} \int_{\mathbb{D}} \left(\int_{\mathbb{T}} P_z(\xi) P_{z'}(\xi) dm(\xi) \right) f(z) \overline{f(z')} d\mu(z) d\mu(z').
\end{aligned}$$

The following lemma will be used to prove that $\mathcal{J}^*$ is bounded.

Lemma 0.2 (Vinogradov-Senichkin Test)**.** *Let $\mathcal{Z}$ be a measurable space and k a non-negative measurable function on $\mathcal{Z} \times \mathcal{Z}$. If*

$$\int_{\mathcal{Z}} k(s,t)k(s,x)ds \leq C(k(t,x) + k(x,t))$$

for a.e. $(t,x) \in \mathcal{Z} \times \mathcal{Z}$, then

$$Q := \iint_{\mathcal{Z}\times\mathcal{Z}} k(s,t)g(s)g(t)dsdt \leq 2C$$

for any non-negative function g with $\|g\|_{L^2(\mathcal{Z})} \leq 1$ and $Q < \infty$.

Exercise 0.3. *Prove the Vinogradov-Senichkin Test.*

We want to apply the Vinogradov-Senichkin Test, so we define the integral operator $T_k : L^2(\mathbb{D},\mu) \to L^2(\mathbb{D},\mu)$ with kernel $k(z',z) = P_z(z')$ by

$$T_k(g)(z) := \int_{\mathbb{D}} k(z',z) g(z') d\mu(z') \quad \forall g \in L^2(\mathbb{D},\mu).$$

Then we have that

$$\begin{aligned}
\|\mathcal{J}^*(f)\|^2_{L^2(\mathbb{T})} &\leq \int_{\mathbb{D}} \int_{\mathbb{D}} P_z(z') \, |f(z)| \, |f(z')| \, d\mu(z) d\mu(z') \\
&= (T_k \, |f| \, , |f|).
\end{aligned}$$

Now, we make the following claim:

Proposition 0.4. *Let $z, z', w \in \mathbb{D}$ then*

$$P_z(z')P_w(z') \leq 8 \left(P_z(w)P_w(z') + P_w(z)P_z(z') \right)$$

Proof. Begin by noting that the following inequality holds

$$a^{-1} := |1 - z\overline{w}|^{1/2} \leq \left|1 - z\overline{z'}\right|^{1/2} + |1 - z'\overline{w}|^{1/2} := b^{-1} + c^{-1}.$$

Exercise 0.5. *Prove this. Hint: This is the triangle inequality for a certain metric.*

Using this inequality we have that

$$bc \leq a(b + c),$$

which in turn implies that

$$b^4c^4 \leq a^4(b + c)^4 \leq 2^3a^4(b^4 + c^4).$$

Using this inequality, but with the appropriate substitutions for a,b, and c and using the numerators for $P_z(z')$ and $P_w(z')$ we find that

$$\begin{aligned} P_z(z')P_w(z') &\leq \frac{2^3}{|1 - z\overline{w}|^2}(1 - |z|^2)(1 - |w|^2)\left(\frac{1}{\left|1 - z\overline{z'}\right|^2} + \frac{1}{|1 - z'\overline{w}|^2}\right) \\ &= 2^3\left[\frac{(1 - |z|^2)}{|1 - z\overline{w}|^2}\frac{(1 - |w|^2)}{\left|1 - z\overline{z'}\right|^2} + \frac{(1 - |z|^2)}{|1 - z'\overline{w}|^2}\frac{(1 - |w|^2)}{|1 - z\overline{w}|^2}\right] \\ &= 2^3(P_w(z)P_z(z') + P_z(w)P_w(z')). \end{aligned}$$

□

Since we want to apply the Vinogradov-Senichkin Test we need to know that the kernel $k(z', z) = P_z(z')$ satisfies the hypothesis of the lemma. To this end we need to estimate

$$\int_{\mathbb{D}} k(z', z)k(z', w)d\mu(z').$$

Using the estimate we found for the product of two Poisson kernels that we illustrated above, we have

$$\begin{aligned} \int_{\mathbb{D}} k(z', z)k(z', w)d\mu(z') &= \int_{\mathbb{D}} P_z(z')P_w(z')d\mu(z') \\ &\leq 2^3\int_{\mathbb{D}} P_z(w)P_w(z') + P_w(z)P_z(z')d\mu(z') \\ &= 2^3\left[P_z(w)\int_{\mathbb{D}} P_w(z')d\mu(z') + P_w(z)\int_{\mathbb{D}} P_z(z')d\mu(z')\right] \\ &\leq 2^3C(\mu)^2(P_z(w) + P_w(z)). \end{aligned}$$

The last inequality holds because we are trying to prove that (ii)⇒(i). So the kernel satisfies the Vinogradov-Senichkin Test, thus we have

$$\begin{aligned} \|\mathcal{J}^*(f)\|^2_{L^2(\mathbb{T})} &\leq (T_k\,|f|\,,|f|) \\ &\leq 2\cdot(2^3C(\mu)^2)\,\|f\|^2_{L^2(\mu)} \\ &= 2^4C(\mu)^2\,\|f\|^2_{L^2(\mu)}\,. \end{aligned}$$

A duality argument then gives that $\|\mathcal{J}\| \leq 4C(\mu)$, proving that $\mathcal{J}$ is bounded and giving the relationship between $\|\mathcal{J}\|$ and $C(\mu)$.

0.1.2. *(ii)* $\Longleftrightarrow$ *(iii).* To finish the proof of this theorem we only need to dispose of the final equivalence. Again, we have an easy implication and a harder implication. We begin by showing that (ii)⇒(iii). Under the hypothesis of (ii) we have

$$C(\mu)^2 \geq \int_{\mathbb{D}} P_z(z')d\mu(z') = \int_{\mathbb{D}} \frac{(1-|z|^2)}{\left|1-z\overline{z'}\right|^2} d\mu(z').$$

Now take $\xi \in \mathbb{T}$ and $0 < r < 2$, and set $z = (1-\frac{r}{2})\xi$. Now consider the non-isotropic ball $Q(\xi, r)$. A simple calculation shows that $z \in Q(\xi, r)$. Then we have

$$\begin{aligned}
C(\mu)^2 &\geq \int_{\mathbb{D}} \frac{(1-|z|^2)}{\left|1-z\overline{z'}\right|^2} d\mu(z') \\
&\geq \int_{\mathbb{D}\bigcap Q(\xi,r)} \frac{(1-|z|^2)}{\left|1-z\overline{z'}\right|^2} d\mu(z') \\
&\geq 16^{-1}r^{-2}(1-|z|)\int_{\mathbb{D}\bigcap Q(\xi,r)} d\mu(z') \\
&= 16^{-1}r^{-2}(1-|z|)\mu\left(\mathbb{D}\bigcap Q(\xi,r)\right).
\end{aligned}$$

The last inequality follows since $z \in \mathbb{D}$ and we are considering $z, z' \in Q(\xi, r)$. Because if we have two points $z, z' \in Q(\xi, r)$ then by the triangle inequality for the non-isotropic metric we have

$$\left|1-z\overline{z'}\right|^{1/2} \leq \left|1-z\overline{\xi}\right|^{1/2} + \left|1-z'\overline{\xi}\right|^{1/2}.$$

Squaring this last inequality gives

$$\left|1-z\overline{z'}\right| \leq 2\left(\left|1-z\overline{\xi}\right| + \left|1-\xi\overline{z'}\right|\right) \leq 4r.$$

This then gives $\left|1-z\overline{z'}\right|^{-2} \geq 2^{-4}r^{-2}$. We also have that

$$1-|z| = \frac{r}{2}.$$

Combining these estimates gives,

$$\begin{aligned}
C(\mu)^2 &\geq 16^{-1}r^{-2}(1-|z|)\mu\left(\mathbb{D}\bigcap Q(\xi,r)\right) \\
&\geq 16^{-1}2^{-1}rr^{-2}\mu\left(\mathbb{D}\bigcap Q(\xi,r)\right).
\end{aligned}$$

Taking the supremum over $0 < r < 2$ then gives

$$32^{-1}I(\mu) \leq C(\mu)^2.$$

It only remains to prove that (iii)⇒(ii). We will break this part up into two different cases. First, consider the case where $|z| \leq \frac{3}{4}$. Now we have the following inequality holding for the Poisson kernel.

$$\begin{aligned} P_z(\xi) &= \frac{(1-|z|^2)}{\left|1-z\overline{\xi}\right|^2} \\ &\leq \frac{(1-|z|^2)}{(1-|z|)^2} \\ &= \frac{(1+|z|)^2}{(1-|z|^2)} \\ &\leq \frac{2^2}{(1-|z|^2)}. \end{aligned}$$

Then if $|z| \leq \frac{3}{4}$ we have that

$$\begin{aligned} \int_{\mathbb{D}} P_z(w)d\mu(w) &\leq 4\frac{16}{7}\mu(\mathbb{D}) \\ &= \frac{2^6}{7}2\mu(\mathbb{D}\bigcap Q(\xi,2)) \\ &\leq \frac{2^7}{7}I(\mu) \\ &\leq 20I(\mu). \end{aligned}$$

So we only need to deal with the case when $|z| > \frac{3}{4}$. Let $\tilde{z} = \frac{z}{|z|}$ and define the following sets

$$Q_k := \mathbb{D}\bigcap Q\left(\tilde{z}, 2^{k+1}(1-|z|^2)\right) \qquad \forall k \in \mathbb{N}.$$

Then for $w \in Q_{k+1} \setminus Q_k$ we have

$$\left|1-w\overline{\tilde{z}}\right| \geq 2^{k+1}(1-|z|^2).$$

By the triangle inequality for the non-isotropic metric we have

$$\begin{aligned} \left|1-w\overline{\tilde{z}}\right|^{1/2} &\leq |1-w\overline{z}|^{1/2} + \left|1-z\overline{\tilde{z}}\right|^{1/2} \\ &= |1-w\overline{z}|^{1/2} + (1-|z|)^{1/2} \\ &\leq |1-w\overline{z}|^{1/2} + (1-|z|^2)^{1/2}, \end{aligned}$$

with the last inequality following since $z \in \mathbb{D}$. Squaring this last inequality gives,

$$\left|1-w\overline{\tilde{z}}\right| \leq 2\left(|1-w\overline{z}| + (1-|z|^2)\right).$$

Now using this we can conclude that

$$\begin{aligned} |1-w\overline{z}| &\geq 2^{-1}\left|1-w\overline{\tilde{z}}\right| - (1-|z|^2) \\ &\geq 2^k(1-|z|^2) - (1-|z|^2) \\ &\geq 2^{k-1}(1-|z|^2). \end{aligned}$$

But this last inequality implies that

$$|1-w\overline{z}|^{-2} \leq 2^{-2(k-1)}(1-|z|^2)^{-2}$$

when $w \in Q_{k+1} \setminus Q_k$. Now using this we have

$$\begin{aligned}\int_{\mathbb{D}} P_z(w)d\mu(w) &= \int_{Q_1} P_z(w)d\mu(w) + \sum_{k=1}^{\infty} \int_{Q_{k+1}\setminus Q_k} P_z(w)d\mu(w) \\ &\leq \int_{Q_1} \frac{2^2}{(1-|z|^2)} d\mu(w) + \sum_{k=1}^{\infty} \int_{Q_{k+1}\setminus Q_k} \frac{(1-|z|^2)}{4^{k-1}(1-|z|^2)^2} d\mu(w) \\ &\leq 4\frac{\mu(Q_1)}{(1-|z|^2)} + \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^{k-1} \frac{\mu(Q_{k+1})}{(1-|z|^2)}.\end{aligned}$$

Now we need to recall how $I(\mu)$ was defined and how each of the Q_k was defined. Doing this we have

$$\begin{aligned}\int_{\mathbb{D}} P_z(w)d\mu(w) &\leq 16\frac{\mu(Q_1)}{(2^2(1-|z|^2))} + \sum_{k=1}^{\infty} (4^{-1})^{k-1}(2^{k+2})\frac{\mu(Q_{k+1})}{(2^{k+2}(1-|z|^2))} \\ &\leq 16I(\mu) + 2^4 I(\mu)\sum_{k=1}^{\infty}(2^{-1})^k \\ &\leq 2^5 I(\mu).\end{aligned}$$

Combining the estimates if $|z| \leq \frac{3}{4}$ and if $|z| > \frac{3}{4}$, we have that

$$\int_{\mathbb{D}} P_z(w)d\mu(w) \leq 2^5 I(\mu),$$

then taking the supremum over $z \in \mathbb{D}$ proves the theorem.

The careful reader will have noticed that this proof can actually be used to show the following theorem.

Theorem 0.6 (Carleson Embedding Theorem)**.** *Let μ be a non-negative Borel measure in $\mathbb{D}$. Then the following are equivalent:*

(i) *The embedding operator $\mathcal{J} : L^2(\mathbb{T}) \to L^2(\mathbb{D}, \mu)$, with $\mathcal{J}(f)(z) = f(z)$, is bounded.*

(ii) $C(\mu)^2 := \sup_{z\in\operatorname{supp}\mu} \left\| \mathcal{J}\tilde{k}_z \right\|^2_{L^2(\mu)} = \sup_{z\in\operatorname{supp}\mu} \|P_z\|_{L^1(\mu)} < \infty$*, where* $\tilde{k}_z(\xi) = \frac{(1-|z|^2)^{1/2}}{(1-\xi\bar{z})}$*, the reproducing kernel for the Hardy space $H^2(\mathbb{D})$.*

References

[1] Nikolai K. Nikolski, *Operators, functions, and systems: an easy reading. Vol. 1*, Mathematical Surveys and Monographs, vol. 92, American Mathematical Society, Providence, RI, 2002. Hardy, Hankel, and Toeplitz; Translated from the French by Andreas Hartmann. ↑2