Lecture 12: Böe's Proof of Interpolation for Spaces with the Complete Nevanlinna-Pick Property

This will be the final lecture of this years Internet Analysis Seminar. The exposition in this lecture is adapted form the presentation given in the book [5].

The last result that we will need to explore for the Dirichlet space is the question of interpolation in the space $\mathcal{D}$. In this lecture we will present a proof of the result in the Hilbert space that works in a general reproducing kernel Hilbert space with the complete Nevanlinna-Pick property. This in fact covers the Dirichlet space $\mathcal{D}$ as well as the Besov-Sobolev spaces $B^2_\sigma(\mathbb{D})$. There are other proofs of this result that are more constructive, though we have selected to present an easier proof of this fact and will leave the more constructive proof as one of the projects for later in the seminar.

Background and Introduction. Suppose H is a Hilbert space of analytic functions with a complete Nevanlinna-Pick reproducing kernel $k(x,y)$, so that $H = \mathcal{H}_k$. There is a unique multiplier $\psi = \psi^{z_0}_{z_1} = \varphi_0 \in M_H$ of norm at most one satisfying the interpolation,

$$\psi(z_0) = d(z_0, z_1) = \sqrt{1 - \frac{|\langle k_{z_0}, k_{z_1}\rangle|^2}{\|k_{z_0}\|^2 \|k_{z_1}\|^2}} \text{ and } \psi(z_1) = 0,$$

and moreover, we have the explicit formula

$$\begin{aligned} \psi^{z_0}_{z_1}(z) &= \left(1 - \frac{|\langle k_{z_0}, k_{z_1}\rangle|^2}{\|k_{z_0}\|^2 \|k_{z_1}\|^2}\right)^{-\frac{1}{2}} \left(1 - \frac{\langle k_{z_0}, k_{z_1}\rangle k_{z_1}(z)}{\langle k_{z_1}, k_{z_1}\rangle k_{z_0}(z)}\right) \\ &= d(z_0, z_1)^{-1} \left(1 - \frac{\langle k_{z_0}, k_{z_1}\rangle k_{z_1}(z)}{\langle k_{z_1}, k_{z_1}\rangle k_{z_0}(z)}\right). \end{aligned} \tag{0.1}$$

We will refer to $\psi^{z_0}_{z_1}$ as the *generalized Blaschke function* associated to the pair of points (z_0, z_1). It vanishes at z_1 and is positive at z_0.

More generally, for $Z = \{z_n\}_{n=1}^\infty$, we will refer to the infinite product

$$B^{z_0}_Z(z) = \prod_{n=1}^\infty \psi^{z_0}_{z_n}(z), \tag{0.2}$$

as the *generalized Blaschke product* in M_H associated to the set $Z = \{z_n\}_{n=1}^\infty$ with pole at $z_0 \notin Z$. Suppose now that the measure $\mu_Z = \sum_{n=1}^\infty \|k_{z_n}\|^{-2} \delta_{z_n}$ associated to Z is a *finite* measure. Then we have

$$\sum_{n=1}^\infty \frac{|k_{z_0}(z_n)|^2}{\|k_{z_0}\|^2 \|k_{z_n}\|^2} = \int \left|\widetilde{k_{z_0}}(z)\right|^2 d\mu_Z(z) = C_{z_0}.$$

It now follows from the right hand inequality in

$$\exp\left(-\sum_{n=1}^\infty u_n\right) \ge \prod_{n=1}^\infty (1 - u_n) \ge \exp\left(-2\sum_{n=1}^\infty u_n\right), \tag{0.3}$$

that

$$B_Z^{z_0}(z_0)^2 = \prod_{n=1}^{\infty} \psi_{z_n}^{z_0}(z_0)^2 = \prod_{n=1}^{\infty} \left(1 - \frac{|k_{z_0}(z_n)|^2}{\|k_{z_0}\|^2 \|k_{z_n}\|^2}\right) = \prod_{n=1}^{\infty} d(z_0, z_n)^2 > 0.$$

Here we denote by $d(w,z)$ the metric associated to the kernel $k(w,z)$:

$$d(w,z) = \sqrt{1 - \frac{|k_w(z)|^2}{\|k_w\|^2 \|k_z\|^2}}.$$

Thus we see that when μ_Z is finite, the generalized Blaschke product $B_Z^{z_0}(z)$ is *not* identically zero, has norm at most one in the multiplier space M_H, vanishes on Z, and is positive at z_0. In fact, using the left hand inequality in (0.3), this argument can be reversed and yields the following characterization of nontrivial generalized Blaschke products.

Proposition 0.1. *Suppose H is a Hilbert space of analytic functions with a complete Nevanlinna-Pick reproducing kernel $k(x,y)$, so that $H = \mathcal{H}_k$. Fix a sequence $Z = \{z_j\}_{j=1}^{\infty}$ and $z_0 \notin Z$. Then $B_Z^{z_0}(z)$ is not identically zero* if and only if

$$B_Z^{z_0}(z_0)^2 \equiv \prod_{n=1}^{\infty} d(z_0, z_n)^2 > 0,$$

if and only if $\mu_Z = \sum_{j=1}^{\infty} \left\|k_{z_j}\right\|^{-2} \delta_{z_j}$ *is a* finite *measure.*

We can also consider separation and Carleson embedding for sequences Z using the bound $\int \left|\widetilde{k_{z_m}}(z)\right|^2 d\mu_Z(z) \leq \|\mu_Z\|_{H-Carleson}$ that is independent of m.

Proposition 0.2. *Suppose H is a Hilbert space of analytic functions with a complete Nevanlinna-Pick reproducing kernel $k(x,y)$, so that $H = \mathcal{H}_k$. Then a sequence $Z = \{z_j\}_{j=1}^{\infty}$ is separated, i.e. there is $\varepsilon > 0$ such that*

$$\frac{|\langle k_n, k_{z_m}\rangle|}{\|k_{z_n}\| \|k_{z_m}\|} \leq 1 - \varepsilon,$$

and $\mu_Z = \sum_{j=1}^{\infty} \left\|k_{z_j}\right\|^{-2} \delta_{z_j}$ is a Carleson measure for H only if

$$\inf_{m \geq 1} B_{Z\setminus\{z_m\}}^{z_m}(z_m)^2 \equiv \inf_{m \geq 1} \prod_{n \neq m} d(z_m, z_n)^2 > 0.$$

We recall a theorem of B. Böe [3] (see Theorem 0.4 below) which says that for certain Hilbert spaces with reproducing kernel, in the presence of the separation condition (which is necessary for an interpolating sequence, see Ch. 9 of [1]) a necessary and sufficient condition for a sequence to be interpolating is that the Grammian matrix associated with Z is bounded. That matrix is built from normalized reproducing kernels; it is

$$\left[\left\langle \frac{k_{z_i}}{\|k_{z_i}\|}, \frac{k_{z_j}}{\|k_{z_j}\|} \right\rangle\right]_{i,j=1}^{\infty}. \tag{0.4}$$

The spaces to which Böe's Theorem applies are those where the kernel has the Nevanlinna-Pick property, and which have the following additional technical property. Whenever we have a sequence for which the matrix (0.4) is bounded on ℓ^2 then the matrix with absolute values

$$\left[\left|\left\langle \frac{k_{z_i}}{\|k_{z_i}\|}, \frac{k_{z_j}}{\|k_{z_j}\|} \right\rangle\right|\right]_{i,j=1}^{\infty}$$

is also bounded on ℓ^2. This property holds in our case because, for σ in the range of interest, $\mathrm{Re}\left(\frac{1}{1-\overline{z_j}\cdot z_i}\right)^{2\sigma} \approx \left|\frac{1}{1-\overline{z_j}\cdot z_i}\right|^{2\sigma}$ which, as noted in [3], insures that the Gramm matrix has the desired property. Finally, as also pointed out in [3], the boundedness on ℓ^2 of the Grammian matrix is equivalent to $\mu_Z = \sum_{j=1}^{\infty} \|k_{z_j}\|^{-2} \delta_{z_j} = \sum_{j=1}^{\infty}(1-|z_j|^2)^{2\sigma}\delta_{z_j}$ being a Carleson measure.

Remark 0.3. Böe presents his work for the dimension $n = 1$, but, as he notes, it extends directly to general n. However, in that context only for values of $\sigma < \frac{1}{2}$. It is an important open question to characterize the interpolating sequences for the Drury-Arveson space. The interested reader should consult [2] for the generalization to higher dimensions of Theorem 0.4.

In order to state Böe's Theorem, we briefly recall the theory of Hilbert spaces with a Nevanlinna-Pick kernel $k(x,y)$ in Agler and M^{c}Carthy [1], keeping in mind the classical model of the Szego kernel $k(x,y) = \frac{1}{1-\overline{x}y}$ on the unit disk $\mathbb{D}$. Let Ω be an infinite set and $k(x,y)$ be a positive definite kernel function on Ω, i.e. for all finite subsets $\{x_i\}_{i=1}^{m}$of Ω,

$$\sum_{i,j=1}^{m} a_i\overline{a_j}k(x_i,x_j) \geq 0 \text{ with equality } \Leftrightarrow \text{ all } a_i = 0.$$

Denote by $H = \mathcal{H}_k$ the Hilbert space obtained by completing the space of finite linear combinations of k_{x_i}'s, where $k_x(y) = k(x,y)$, with respect to the inner product

$$\left\langle \sum_{i=1}^{m} a_i k_{x_i}, \sum_{j=1}^{m} b_j k_{y_j} \right\rangle = \sum_{i,j=1}^{m} a_i\overline{b_j}k(x_i,y_j).$$

Let $Z = \{z_j\}_{j=1}^{J}$ be a finite set of points in Ω and consider the Nevanlinna-Pick interpolation problem: For which sequences of data $\{\xi_j\}_{j=1}^{J} \subset \mathbb{C}$ is there $\varphi \in M_H$ with muliplier norm at most one satisfying

$$\varphi(z_j) = \xi_j, \qquad 1 \leq j \leq J? \tag{0.5}$$

There is an easy necessary condition for the data in terms of a certain matrix being positive semidefinite. If $\|\mathcal{M}_\varphi\| \equiv \|\varphi\|_{M_H} \leq 1$ then $\|\mathcal{M}_\varphi^*\| \leq 1$ and for every choice of scalars $\{\lambda_j\}_{j=1}^{J} \subset \mathbb{C}$ we have

$$0 \leq \left\|\sum_{j=1}^{J} \lambda_j k_{z_j}\right\|^2 - \left\|\mathcal{M}_\varphi^*\left(\sum_{j=1}^{J} \lambda_j k_{z_j}\right)\right\|^2 = \sum_{j,m=1}^{J} \left(1-\xi_j\overline{\xi_m}\right) k_{z_j}(z_m)\,\lambda_j\overline{\lambda_m},$$

which is

$$\left[\left(1-\xi_j\overline{\xi_m}\right) k_{z_j}(z_m)\right]_{j,m=1}^{J} \succeq 0. \tag{0.6}$$

We say that the Hilbert space H (more precisely the *reproducing kernel* of H) has the Nevanlinna-Pick property if the implication above can be reversed.

There is a surprising consequence of the Nevanlinna-Pick property for certain extremal problems. Let $Z = \{z_j\}_{j=1}^{\infty}$ and $z_0 \notin Z$. To make the following argument rigorous, we may take Z finite and then pass to a limit. Let f_0 be the unique solution to the extremal problem

$$\text{Re} f_0(z_0) = \{\text{Re} f(z_0) : f(z_j) = 0 \text{ for } 1 \leq j < \infty \text{ and } \|f\| \leq 1\}. \tag{0.7}$$

Note that the solution exists by a normal families argument, and is unique because for each real t, the element of minimal norm in the closed convex set

$$E_t = \{f \in H : \text{Re} f(z_0) = t, f(z_j) = 0 \text{ for } 1 \leq j < \infty \text{ and } \|f\| \leq 1\}$$

is unique. From the definition of f_0 we have

$$|\lambda_0 f_0(z_0)| = \left|\left\langle \sum_{j=0}^{\infty} \lambda_j k_{z_j}, f_0 \right\rangle\right| \leq \left\|\sum_{j=0}^{\infty} \lambda_j k_{z_j}\right\|,$$

which in terms of the data $\xi_0 = \frac{|f_0(z_0)|}{\|k_{z_0}\|}$ and $\xi_j = 0$ for $1 \leq j < \infty$ can be rewritten as

$$0 \leq \left\|\sum_{j=0}^{\infty} \lambda_j k_{z_j}\right\|^2 - |\lambda_0 f_0(z_0)|^2 = \sum_{j,m=0}^{\infty} \left(1 - \xi_j \overline{\xi_m}\right) k_{z_j}(z_m) \lambda_j \overline{\lambda_m}.$$

Since H has the Nevanlinna-Pick property, there is $\varphi_0 \in M_H$ with norm at most one satisfying

$$\varphi_0(z_0) = \xi_0 = \frac{|f_0(z_0)|}{\|k_{z_0}\|} \text{ and } \varphi_0(z_j) = 0 \text{ for } 1 \leq j < \infty.$$

Thus the function $\rho(z) \equiv \varphi_0(z) \frac{k_{z_0}(z)}{\|k_{z_0}\|}$ satisfies

$$\|\rho\| = \left\|\varphi_0 \frac{k_{z_0}}{\|k_{z_0}\|}\right\| \leq \|\mathcal{M}_{\varphi}\| \left\|\frac{k_{z_0}}{\|k_{z_0}\|}\right\| \leq 1,$$

and

$$\text{Re}\rho(z_0) = \text{Re}\left(\varphi_0(z_0) \frac{k_{z_0}(z_0)}{\|k_{z_0}\|}\right) = \frac{|f_0(z_0)|}{\|k_{z_0}\|} \frac{\|k_{z_0}\|^2}{\|k_{z_0}\|} = |f_0(z_0)|$$

and $\rho(z_j) = 0$ for $1 \leq j < \infty$. By the uniqueness of the solution to the extremal problem (0.7), we obtain the remarkable formula,

$$f_0(z) = \varphi_0(z) \frac{k_{z_0}(z)}{\|k_{z_0}\|}. \tag{0.8}$$

Theorem 0.4 (Böe, [3])**.** *Suppose H is a Hilbert space of analytic functions with a Nevanlinna - Pick reproducing kernel $k(x,y)$, so that $H = \mathcal{H}_k$. Suppose also that the Grammian property mentioned above holds: whenever $\{z_j\}_{j=1}^{\infty}$ is a sequence for which the matrix (0.4) is bounded on ℓ^2 then the matrix with absolute values is also bounded on ℓ^2. Then a sequence $Z = \{z_j\}_{j=1}^{\infty}$ is interpolating for H if and only if Z is separated and $\mu_Z = \sum_{j=1}^{\infty} \left\|k_{z_j}\right\|^{-2} \delta_{z_j}$ is a Carleson measure for H.*

Remark 0.5. The Grammian matrix (0.4) is bounded on ℓ^2 if and only if μ_Z is a Carleson measure for H. To see this let $T : H \to \ell^{\infty}$ be the normalized restriction map $Tf = \left\{\frac{f(z_j)}{\|k_{z_j}\|}\right\}_{j=1}^{\infty}$. Then μ_Z is a Carleson measure for H if and only if T is bounded into ℓ^2. But

$T^{*}\left\{\xi_j\right\}_{j=1}^{\infty}=\sum_{j=1}^{\infty}\xi_j\frac{k_{z_j}}{\left\|k_{z_j}\right\|}$ and so the matrix representation of TT^{*} relative to the standard basis $\left\{\mathbf{e}_j\right\}_{j=1}^{\infty}$ of ℓ^2 is the Grammian:

$$\begin{aligned}\left[\left\langle TT^{*}\mathbf{e}_i,\mathbf{e}_j\right\rangle\right]_{i,j=1}^{\infty} &= \left[\left\langle T\left(\frac{k_{z_i}}{\left\|k_{z_i}\right\|}\right),\mathbf{e}_j\right\rangle\right]_{i,j=1}^{\infty} \\ &= \left[\left\langle\frac{k_{z_i}\left(z_j\right)}{\left\|k_{z_i}\right\|\left\|k_{z_j}\right\|}\right\rangle\right]_{i,j=1}^{\infty}=\left[\left\langle\frac{k_{z_i}}{\left\|k_{z_i}\right\|},\frac{k_{z_j}}{\left\|k_{z_j}\right\|}\right\rangle\right]_{i,j=1}^{\infty}.\end{aligned}$$

Now use that T is bounded if and only if TT^{*} is bounded.

Proof of Theorem 0.4. If Z is interpolating for H, standard arguments show that Z is separated and that μ_Z is a Carleson measure for H.

Conversely, Remark 0.5 shows that the Grammian matrix (0.4) is bounded on ℓ^2. To show that Z is interpolating for H it suffices by Bari's Theorem 0.7 to show that $\left\{\widetilde{k_{z_i}}\right\}_{j=1}^{\infty}$ is a Riesz basis, where $\widetilde{k_{z_i}}=\frac{k_{z_i}}{\left\|k_{z_i}\right\|}$ is the normalized reproducing kernel for H. Let $\left\{f_j\right\}_{j=1}^{\infty}$ be the biorthogonal functions defined as the unique minimal norm solutions of

$$\frac{f_n\left(z_m\right)}{\left\|k_{z_m}\right\|}=\left\langle f_n,\widetilde{k_{z_m}}\right\rangle=\delta_m^n.$$

If P denotes projection onto the closed linear span $\vee_{j=1}^{\infty}k_{z_j}$ of the k_{z_j}, then $\left\langle Pf_n,\widetilde{k_{z_m}}\right\rangle=\left\langle f_n,\widetilde{k_{z_m}}\right\rangle=\delta_m^n$ and so $f_n=Pf_n\in\vee_{j=1}^{\infty}k_{z_j}$. By Bari's Theorem 0.7 again, $\left\{\widetilde{k_{z_i}}\right\}_{j=1}^{\infty}$ is a Riesz basis if and only if both $\left[\left\langle\widetilde{k_{z_n}},\widetilde{k_{z_m}}\right\rangle\right]_{m,n=1}^{\infty}$ and $\left[\left\langle f_n,f_m\right\rangle\right]_{m,n=1}^{\infty}$ are bounded matrices on ℓ^2. We already know that $\left[\left\langle\widetilde{k_{z_n}},\widetilde{k_{z_m}}\right\rangle\right]_{m,n=1}^{\infty}$ is bounded, so it remains to show that $\left[\left\langle f_n,f_m\right\rangle\right]_{m,n=1}^{\infty}$ is also.

0.0.1. *Calculations that hold in an arbitrary Hilbert function space.* For $A\subset Z=\left\{z_j\right\}_{j=1}^{\infty}$ let $H_A=\left\{f\in H:f\left(a\right)=0\text{ for }a\in A\right\}$. If $k_w^A\left(z\right)$ is the reproducing kernel for H_A, then $\left\|k_w^A\right\|^2=k_w^A\left(w\right)$ and

$$k_w^A\left(w\right)=\sup\left\{\left|f\left(w\right)\right|:f\in H_A\text{ with }\left\|f\right\|=\left\|k_w^A\right\|\right\}.$$

It follows that with $Z_n=Z\setminus\left\{z_n\right\}$, we have

$$f_n\left(z\right)=\frac{\left\|k_{z_n}\right\|}{\left\|k_{z_n}^{Z_n}\right\|^2}k_{z_n}^{Z_n}\left(z\right),\qquad n\geq 1.$$

Note in particular that

$$\left\|f_n\right\|=\frac{\left\|k_{z_n}\right\|}{\left\|k_{z_n}^{Z_n}\right\|}\text{ and }\frac{k_{z_n}^{Z_n}\left(z_m\right)}{\left\|k_{z_n}^{Z_n}\right\|\left\|k_{z_m}\right\|}=\frac{f_n\left(z_m\right)}{\left\|k_{z_m}\right\|\left\|f_n\right\|}=\frac{\delta_m^n}{\left\|f_n\right\|}.$$

We now compute the entries $\langle f_n, f_m\rangle$ in the biorthogonal Grammian $[\langle f_n, f_m\rangle]_{m,n=1}^{\infty}$ in terms of the corresponding entries $\left\langle \widetilde{k_{z_n}}, \widetilde{k_{z_m}} \right\rangle$ in the Grammian $\left[\left\langle \widetilde{k_{z_n}}, \widetilde{k_{z_m}} \right\rangle\right]_{m,n=1}^{\infty}$. We have

$$\langle f_n, f_m\rangle = \frac{\|k_{z_n}\| \, \|k_{z_m}\|}{\left\|k_{z_n}^{Z_n}\right\|^2 \left\|k_{z_m}^{Z_m}\right\|^2} \left\langle k_{z_n}^{Z_n}, k_{z_m}^{Z_m} \right\rangle . \tag{0.9}$$

Now we use that the reproducing kernels $k_w^{A\cup\{a\}}$ for $H_{A\cup\{a\}}$ are given in terms of those k_w^A for H_A by the formula

$$k_w^{A\cup\{a\}}(z) = k_w^A(z) - \frac{k_a^A(z)\, k_w^A(a)}{k_a^A(a)}.$$

Indeed, the right hand side is in $H_{A\cup\{a\}}$ and its inner product with $f \in H_{A\cup\{a\}}$ is $f(w)$.

If we set

$$Z_{m,n} = Z \setminus \{m,n\} = Z_n \setminus \{m\} = Z_m \setminus \{n\},$$

we thus obtain

$$k_{z_n}^{Z_n}(z) = k_{z_n}^{Z_{m,n}}(z) - \frac{k_{z_m}^{Z_{m,n}}(z)\, k_{z_n}^{Z_{m,n}}(z_m)}{k_{z_m}^{Z_{m,n}}(z_m)}, \tag{0.10}$$

and the same formula with m and n interchanged. Then we have from the interchanged formula,

$$\begin{aligned}
\left\langle k_{z_n}^{Z_n}, k_{z_m}^{Z_m} \right\rangle &= \left\langle k_{z_n}^{Z_n}, k_{z_m}^{Z_{m,n}} - \frac{k_{z_n}^{Z_{m,n}} k_{z_m}^{Z_{m,n}}(z_n)}{k_{z_n}^{Z_{m,n}}(z_n)} \right\rangle \\
&= \left\langle k_{z_n}^{Z_n}, k_{z_m}^{Z_{m,n}} \right\rangle - \frac{\overline{k_{z_m}^{Z_{m,n}}(z_n)}}{k_{z_n}^{Z_{m,n}}(z_n)} \left\langle k_{z_n}^{Z_n}, k_{z_n}^{Z_{m,n}} \right\rangle \\
&= k_{z_n}^{Z_n}(z_m) - \frac{\overline{k_{z_m}^{Z_{m,n}}(z_n)}}{k_{z_n}^{Z_{m,n}}(z_n)} k_{z_n}^{Z_n}(z_n).
\end{aligned}$$

Now from (0.10) we have

$$k_{z_n}^{Z_n}(z_n) = k_{z_n}^{Z_{m,n}}(z_n) - \frac{k_{z_m}^{Z_{m,n}}(z_n)\, k_{z_n}^{Z_{m,n}}(z_m)}{k_{z_m}^{Z_{m,n}}(z_m)} = \sigma_m^n k_{z_n}^{Z_{m,n}}(z_n),$$

where σ_m^n satisfies two equalities:

$$\sigma_m^n = \frac{k_{z_n}^{Z_n}(z_n)}{k_{z_n}^{Z_{m,n}}(z_n)} = \frac{\left\|k_{z_n}^{Z_n}\right\|^2}{\left\|k_{z_n}^{Z_{m,n}}\right\|^2} = 1 - \frac{k_{z_m}^{Z_{m,n}}(z_n)\, k_{z_n}^{Z_{m,n}}(z_m)}{k_{z_n}^{Z_{m,n}}(z_n)\, k_{z_m}^{Z_{m,n}}(z_m)}, \tag{0.11}$$

and is at most 1 since $\left\|k_{z_n}^{Z_n}\right\| \le \left\|k_{z_n}^{Z_{m,n}}\right\|$ or

$$\left|k_{z_m}^{Z_{m,n}}(z_n)\right| = \left|\left\langle k_{z_m}^{Z_{m,n}}, k_{z_n}^{Z_{m,n}} \right\rangle\right| \le \left\|k_{z_m}^{Z_{m,n}}\right\| \left\|k_{z_n}^{Z_{m,n}}\right\| = \sqrt{k_{z_m}^{Z_{m,n}}(z_m)\, k_{z_n}^{Z_{m,n}}(z_n)}$$

by Cauchy-Schwarz. Note that $\left\|k_{z_n}^{Z_n}\right\|^2=\sigma_m^n\left\|k_{z_n}^{Z_{m,n}}\right\|^2$. Combining equalities yields

$$\begin{aligned}
(0.12)\qquad \left\langle k_{z_n}^{Z_n},k_{z_m}^{Z_m}\right\rangle &= k_{z_n}^{Z_n}(z_m)-\frac{\overline{k_{z_m}^{Z_{m,n}}(z_n)}}{k_{z_n}^{Z_{m,n}}(z_n)}k_{z_n}^{Z_n}(z_n)\\
&= k_{z_n}^{Z_n}(z_m)-\frac{\overline{k_{z_m}^{Z_{m,n}}(z_n)}}{k_{z_n}^{Z_{m,n}}(z_n)}\sigma_m^n k_{z_n}^{Z_{m,n}}(z_n)\\
&= k_{z_n}^{Z_n}(z_m)-\sigma_m^n\overline{k_{z_m}^{Z_{m,n}}(z_n)},
\end{aligned}$$

and

$$(0.13)\qquad \|f_n\|=\frac{\|k_{z_n}\|}{\left\|k_{z_n}^{Z_n}\right\|}\text{ and }\sigma_m^n=\frac{\left\|k_{z_n}^{Z_n}\right\|^2}{\left\|k_{z_n}^{Z_{m,n}}\right\|^2}.$$

Note that $\overline{k_{z_m}^{Z_{m,n}}(z_n)}=k_{z_n}^{Z_{m,n}}(z_m)$ and that $k_{z_n}^{Z_n}(z_m)=0$ for $m\neq n$.

Calculations that use the Nevanlinna-Pick property. From the solution (0.8) to the extremal problem (0.7) with $Z_{m,n}$ in place of Z, and z_m in place of z_0, we obtain after renormalizing φ_0,

$$(0.14)\qquad \frac{k_{z_m}^{Z_{m,n}}(z)}{\left\|k_{z_m}^{Z_{m,n}}\right\|^2}=\varphi_n^m(z)\frac{k_{z_m}(z)}{\|k_{z_m}\|^2},$$

where $\varphi_n^m\in M_H$ is the unique extremal solution to

$$C_{M_H}(m,n)=\inf\left\{\|\varphi\|_{M_H}:\varphi(z_m)=1\text{ and }\varphi(z_j)=0\text{ for }j\in Z_{m,n}\right\}.$$

Indeed, (0.8) with φ_0 denoted by $\varphi_0^{m,n}$ yields

$$\frac{k_{z_m}^{Z_{m,n}}(z)}{\left\|k_{z_m}^{Z_{m,n}}\right\|}=\varphi_0^{m,n}(z)\frac{k_{z_m}(z)}{\|k_{z_m}\|},$$

and with

$$\varphi_n^m(z)=\frac{\varphi_0^{m,n}(z)}{\varphi_0^{m,n}(z_m)},$$

we obtain (0.14). The connection with the multiplier $\varphi_0=\varphi_0^{m,n}$ in (0.8) is

$$\frac{\varphi_n^m(z)}{C_{M_H}(m,n)}=\varphi_0^{m,n}(z).$$

Calculations that use separation and the Carleson condition. Moreover, we have the inequality

$$(0.15)\qquad C_{M_H}(m,n)\leq C,\qquad m,n\geq 1.$$

Indeed, from Proposition 23 of [4] (see [6] for the initial theorem on the Dirichlet space) we have

$$\begin{aligned} C_{M_H}(m,n) &\leq \prod_{j\notin\{m,n\}}\left(1-\frac{\left|\left\langle k_{z_j},k_{z_m}\right\rangle\right|^2}{\left\|k_{z_j}\right\|^2\left\|k_{z_m}\right\|^2}\right)^{-1} \\ &\leq \sup_{m\geq 1}\prod_{j\neq m}\left(1-\frac{\left|\left\langle k_{z_j},k_{z_m}\right\rangle\right|^2}{\left\|k_{z_j}\right\|^2\left\|k_{z_m}\right\|^2}\right)^{-1}. \end{aligned}$$

By the Carleson condition applied to $\widetilde{k_{z_m}}=\frac{k_{z_m}}{\|k_{z_m}\|}$, we obtain

$$C=C\left\|\widetilde{k_{z_m}}\right\|^2\geq\int\left|\widetilde{k_{z_m}}(z)\right|^2 d\mu_Z(z)=\sum_{j=1}^{\infty}\frac{\left|k_{z_m}(z_j)\right|^2}{\left\|k_{z_m}\right\|^2\left\|k_{z_j}\right\|^2}.$$

This together with separation, $\frac{|k_{z_m}(z_j)|^2}{\|k_{z_m}\|^2\|k_{z_j}\|^2}\leq 1-\varepsilon$ for some $\varepsilon>0$, and (0.3) yields

$$\prod_{j\neq m}\left(1-\frac{\left|\left\langle k_{z_j},k_{z_m}\right\rangle\right|^2}{\left\|k_{z_j}\right\|^2\left\|k_{z_m}\right\|^2}\right)\geq c>0,\qquad m\geq 1,$$

and hence (0.15).

Calculation of the off-diagonal terms in the biorthogonal Grammian. For $m\neq n$ we have $k_{z_n}^{Z_n}(z_m)=0$, and hence from (0.9), (0.12), (0.14) and (0.11) we obtain

$$\begin{aligned} \langle f_n,f_m\rangle &= \frac{\|k_{z_n}\|\,\|k_{z_m}\|}{\left\|k_{z_n}^{Z_n}\right\|^2\left\|k_{z_m}^{Z_m}\right\|^2}\left\{-\sigma_m^n\overline{k_{z_m}^{Z_{m,n}}(z_n)}\right\} \\ &= -\frac{\|k_{z_n}\|\,\|k_{z_m}\|}{\left\|k_{z_n}^{Z_n}\right\|^2\left\|k_{z_m}^{Z_m}\right\|^2}\sigma_m^n\left\|k_{z_m}^{Z_{m,n}}\right\|^2\overline{\varphi_n^m(z_n)}\frac{\overline{k_{z_m}(z_n)}}{\|k_{z_m}\|^2} \\ &= -\|f_n\|^2\frac{\sigma_m^n}{\sigma_n^m}\overline{\varphi_n^m(z_n)}\frac{\overline{k_{z_m}(z_n)}}{\|k_{z_m}\|\,\|k_{z_n}\|} \\ &= -\|f_n\|^2\,\overline{\varphi_n^m(z_n)}\,\overline{\left\langle\widetilde{k_{z_m}},\widetilde{k_{z_n}}\right\rangle}, \end{aligned}$$

since $\|f_n\|^2=\frac{\|k_{z_n}\|^2}{\left\|k_{z_n}^{Z_n}\right\|^2}$ and $\sigma_m^n=\frac{\left\|k_{z_n}^{Z_n}\right\|^2}{\left\|k_{z_n}^{Z_{m,n}}\right\|^2}=\sigma_n^m$ by (0.11). Taking complex conjugates we can write this as

$$\langle f_m,f_n\rangle=-\|f_n\|^2\varphi_n^m(z_n)\left\langle\widetilde{k_{z_m}},\widetilde{k_{z_n}}\right\rangle.$$

At this point we use (0.15) to conclude that $|\langle f_m,f_n\rangle|\leq C\left|\left\langle\widetilde{k_{z_m}},\widetilde{k_{z_n}}\right\rangle\right|$ for all m,n. Our hypothesis on the Grammian $\left[\left\langle\widetilde{k_{z_m}},\widetilde{k_{z_n}}\right\rangle\right]_{m,n=1}^{\infty}$ shows that $\left[\left|\left\langle\widetilde{k_{z_m}},\widetilde{k_{z_n}}\right\rangle\right|\right]_{m,n=1}^{\infty}$ is bounded on ℓ^2, and thus so is $[|\langle f_m,f_n\rangle|]_{m,n=1}^{\infty}$, hence $[\langle f_m,f_n\rangle]_{m,n=1}^{\infty}$. This completes the proof of Theorem 0.4.

The Hardy Space $H^2(\mathbb{D})$. The above proof just fails to capture the classical Hardy space $H^2(\mathbb{D})$ on the disk since we no longer have $|k_w(\zeta)| = \left|\frac{1}{1-\overline{w}\zeta}\right| \approx \mathrm{Re}\frac{1}{1-\overline{w}\zeta} = \mathrm{Re}k_w(\zeta)$, despite the nonnegativity of $\mathrm{Re}k_w$. However, if $B_W(z) = \prod_{w\in W} \frac{w-z}{1-\overline{w}z}\frac{|w|}{w}$ denotes the Blaschke product with zeroes $W \subset \mathbb{D}$, then it is known that the extremal functions f_n and φ_n^m are given by Blaschke products, namely

$$f_n(z) = \frac{B_{Z_n}(z)}{B_{Z_n}(z_n)}\frac{k_{z_n}(z)}{\|k_{z_n}\|} \text{ and } \varphi_n^m(z) = \frac{B_{Z_{m,n}}(z)}{B_{Z_{m,n}}(z_m)},$$

so that

$$\|f_n\|^2 = \left\|\frac{B_{Z_n}(z)}{B_{Z_n}(z_n)}\frac{k_{z_n}(z)}{\|k_{z_n}\|}\right\|^2 = \frac{1}{|B_{Z_n}(z_n)|^2}$$

and

$$\begin{aligned}
\varphi_n^m(z_n) &= \frac{B_{Z_{m,n}}(z_n)}{B_{Z_{m,n}}(z_m)} = \frac{\prod_{j\notin\{m,n\}}\left(\frac{z_j-z_n}{1-\overline{z_j}z_n}\frac{|z_j|}{z_j}\right)}{\prod_{j\notin\{m,n\}}\left(\frac{z_j-z_m}{1-\overline{z_j}z_m}\frac{|z_j|}{z_j}\right)} \\
&= \frac{\left(\frac{z_n-z_m}{1-\overline{z_n}z_m}\frac{|z_n|}{z_n}\right)\prod_{j\notin\{n\}}\left(\frac{z_j-z_n}{1-\overline{z_j}z_n}\frac{|z_j|}{z_j}\right)}{\left(\frac{z_m-z_n}{1-\overline{z_m}z_n}\frac{|z_m|}{z_m}\right)\prod_{j\notin\{m\}}\left(\frac{z_j-z_m}{1-\overline{z_j}z_m}\frac{|z_j|}{z_j}\right)} \\
&= -\frac{\left(\frac{1}{1-\overline{z_n}z_m}\frac{|z_n|}{z_n}\right)}{\left(\frac{1}{1-\overline{z_m}z_n}\frac{|z_m|}{z_m}\right)}\frac{B_{Z_n}(z_n)}{B_{Z_m}(z_m)}.
\end{aligned}$$

Altogether we then have

$$-\|f_n\|^2\varphi_n^m(z_n) = \frac{1-\overline{z_m}z_n}{1-\overline{z_n}z_m}\frac{|z_n|}{z_n}\frac{z_m}{|z_m|}\frac{1}{B_{Z_m}(z_m)\overline{B_{Z_n}(z_n)}}. \tag{0.16}$$

Now we use

$$\begin{aligned}
\frac{1-\overline{z_m}z_n}{1-\overline{z_n}z_m}\left\langle\widetilde{k_{z_m}},\widetilde{k_{z_n}}\right\rangle &= \frac{1-\overline{z_m}z_n}{1-\overline{z_n}z_m}\frac{k_{z_m}(z_n)}{\|k_{z_n}\|\|k_{z_m}\|} \\
&= \frac{k_{z_n}(z_m)}{\|k_{z_n}\|\|k_{z_m}\|} = \left\langle\widetilde{k_{z_n}},\widetilde{k_{z_m}}\right\rangle,
\end{aligned}$$

to obtain that

$$\begin{aligned}
\langle f_m, f_n\rangle &= -\|f_n\|^2\varphi_n^m(z_n)\left\langle\widetilde{k_{z_m}},\widetilde{k_{z_n}}\right\rangle \\
&= \overline{\left(\frac{z_n}{|z_n|B_{Z_n}(z_n)}\right)}\left(\frac{z_m}{|z_m|B_{Z_m}(z_m)}\right)\left\langle\widetilde{k_{z_n}},\widetilde{k_{z_m}}\right\rangle.
\end{aligned}$$

It is now clear that $[\langle f_m, f_n\rangle]_{m,n=1}^\infty$ is bounded on ℓ^2 if $\left[\left\langle\widetilde{k_{z_n}},\widetilde{k_{z_m}}\right\rangle\right]_{m,n=1}^\infty$ is bounded on ℓ^2 since the factors $\frac{z_n}{|z_n|B_{Z_n}(z_n)}$ satisfy

$$1 \le \left|\frac{z_n}{|z_n|B_{Z_n}(z_n)}\right| \le C, \qquad n \ge 1.$$

Note that this argument provides a purely Hilbert space proof of Carleson's Theorem.

Bari's characterization of Riesz bases.

Definition 0.6. A set $\mathcal{X} = \{x_\alpha\}_{\alpha\in A}$ in a Hilbert space is a *Riesz basis for* H if there is a linear isomorphism $U : H \to K$ onto another Hilbert space K (i.e. U and its inverse U^{-1} are linear and bounded, but do not necessarily preserve the inner products) such that $U\mathcal{X} = \{Ux_\alpha\}_{\alpha\in A}$ is an orthonormal set in K. The operator U is called an orthogonalizer of $\mathcal{X}$. More generally, we say that $\mathcal{X}$ is a *Riesz basis* if it is a Riesz basis for its closed linear span $\vee\mathcal{X}$.

If $\mathcal{X} = \{x_\alpha\}_{\alpha\in A}$ is a Riesz basis, then every $x \in \vee\mathcal{X}$ can be expanded in a Fourier series relative to $\mathcal{X}$:

$$x = U^{-1}Ux = U^{-1}\sum_{\alpha\in A}\langle Ux, Ux_\alpha\rangle Ux_\alpha = \sum_{\alpha\in A}\langle x, U^*Ux_\alpha\rangle x_\alpha.$$

Now let H be a Hilbert space and let $\mathcal{X} = \{x_\alpha\}_{\alpha\in A}$ be a Riesz basis for H. Denote by $\mathcal{X}' = \{x'_\alpha\}_{\alpha\in A}$ the biorthogonal system defined by the relations $\langle x_\alpha, x'_\beta\rangle = \delta^\alpha_\beta$. Define $J_\mathcal{X}x$ to be the sequence $\{\langle x, x'_\alpha\rangle\}_{\alpha\in A}$ on A. Bari's Theorem characterizes Riesz bases in a number of ways.

Theorem 0.7. *(Bari's Theorem) Let H be a Hilbert space and suppose that $\mathcal{X} = \{x_\alpha\}_{\alpha\in A}$ is a subset of H that satisfies $x_\alpha \notin \vee(\mathcal{X}\setminus\{\alpha\})$ for each $\alpha \in A$. Then the following statements are equivalent:*

1. $\mathcal{X} = \{x_\alpha\}_{\alpha\in A}$ *is a* Riesz basis for H.
2. $\vee\mathcal{X} = H$ *and* $J_\mathcal{X}H = \ell^2(A)$.
3. $\vee\mathcal{X} = H$ *and* $J_\mathcal{X}H \subset \ell^2(A)$ *and* $J_{\mathcal{X}'}H \subset \ell^2(A)$.
4. $\vee\mathcal{X} = H$ *and the Gram matrices* $\Gamma_\mathcal{X} = [\langle x_\alpha, x_\beta\rangle]_{\alpha,\beta\in A}$ *and* $\Gamma_{\mathcal{X}'} = [\langle x'_\alpha, x'_\beta\rangle]_{\alpha,\beta\in A}$ *are bounded on* $\ell^2(A)$.
5. $\vee\mathcal{X} = H$ *and the Gram matrix* $\Gamma_\mathcal{X} = [\langle x_\alpha, x_\beta\rangle]_{\alpha,\beta\in A}$ *defines a continuous and invertible linear map in* $\ell^2(A)$.

Proof: **1** implies **2** is obvious. If **2** holds, then $J_\mathcal{X}$ is continuous by the closed graph theorem, and invertible by the open mapping theorem. Thus **1** holds with orthogonalizer $J_\mathcal{X}$ since $J_\mathcal{X}\mathcal{X}$ is the standard basis $\{\mathbf{e}_\alpha\}_{\alpha\in A}$ in $\ell^2(A)$. **1** implies **3** is obvious. To prove the remaining assertions **3** $\Longrightarrow$ **4** $\Longrightarrow$ **5** $\Longrightarrow$ 1, we use the following two identities for the operator $V_\mathcal{X}$ defined by $V_\mathcal{X}a = \sum_{\alpha\in A}a_\alpha x_\alpha$ for $a = \{a_\alpha\}_{\alpha\in A} \in \ell^2(A)$:

$$\begin{aligned}(0.17)\qquad \|V_\mathcal{X}a\|^2 &= \left\langle \sum_{\alpha\in A}a_\alpha x_\alpha, \sum_{\alpha\in A}a_\alpha x_\alpha\right\rangle = \sum_{\alpha\in A}a_\alpha\overline{a_\beta}\langle x_\alpha, x_\beta\rangle = \langle\Gamma_\mathcal{X}a, a\rangle,\\ \langle V_\mathcal{X}a, b\rangle &= \left\langle\sum_{\alpha\in A}a_\alpha x_\alpha, b\right\rangle = \sum_{\alpha\in A}a_\alpha\overline{\langle b, x_\alpha\rangle} = \langle a, J_{\mathcal{X}'}b\rangle.\end{aligned}$$

Now **3** implies **4** because $J_{\mathcal{X}'}$ is continuous by the closed graph theorem, then $V_\mathcal{X}$ is continuous by the second identity in (0.17), and finally $\Gamma_\mathcal{X}$ is continuous by the first identity in (0.17). Similarly for $\Gamma_{\mathcal{X}'}$. We see that **4** implies **5** because by the first identity in (0.17), $\Gamma_\mathcal{X}$ is invertible if and only if $V_\mathcal{X}$ is invertible; then $J_\mathcal{X}V_\mathcal{X} = I$ and $V_\mathcal{X}J_\mathcal{X} = I$ where once more by the identities in (0.17), continuity of $\Gamma_{\mathcal{X}'}$ implies that of $V_{\mathcal{X}'}$ implies that of $J_\mathcal{X}$.

Finally, **5** implies **1** because $V_{\mathcal{X}}$ is an isomorphism, which follows from the first identity in (0.17) and the relation $V_{\mathcal{X}}\ell^2(A) = \vee\mathcal{X} = H$.

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