Lecture 11: The Corona Theorem for the Multiplier Algebra of $\mathcal{D}$

In 1962 Lennart Carleson demonstrated in [4] the absence of a corona in the maximal ideal space of $H^\infty(\mathbb{D})$ by showing that if $\{g_j\}_{j=1}^N$ is a finite set of functions in $H^\infty(\mathbb{D})$ satisfying

$$\sum_{j=1}^{N} |g_j(z)| \geq c > 0, \qquad z \in \mathbb{D}, \tag{0.1}$$

then there are functions $\{f_j\}_{j=1}^N$ in $H^\infty(\mathbb{D})$ with

$$\sum_{j=1}^{N} f_j(z)\, g_j(z) = 1, \qquad z \in \mathbb{D}, \tag{0.2}$$

While not immediately obvious, the result of Carleson is in fact equivalent to the following statement about the Hilbert space $H^2(\mathbb{D})$. If one is given a finite set of functions $\{g_j\}_{j=1}^N$ in $H^\infty(\mathbb{D})$ satisfying (0.1) and a function $h \in H^2(\mathbb{D})$, then there are functions $\{f_j\}_{j=1}^N$ in $H^2(\mathbb{D})$ with

$$\sum_{j=1}^{N} f_j(z)\, g_j(z) = h(z), \qquad z \in \mathbb{D}, \tag{0.3}$$

The key difference between (0.2) and (0.3) is that one is solving the problem in the Hilbert space setting as opposed to the multiplier algebra, which makes the problem somewhat easier.

In this chapter we discuss the Corona Theorem for the multiplier algebra of the Dirichlet space. The method of proof will be intimately connected with the resulting statements for $H^\infty(\mathbb{D})$ and $H^2(\mathbb{D})$. We also will connect this result to a related statement for the Hilbert space $\mathcal{D}$. One proof will be given by $\overline{\partial}$-methods and the connections between weak Carleson measures for the space $\mathcal{D}$. Another proof will be given by simply proving the Hilbert space version directly and then applying an abstract operator theory result. Implicit in both versions are certain solutions to $\overline{\partial}$-problems that arise.

This lecture is longer than previous lectures since the Corona problem is a question of particular interest to the author of the notes. For those readers more interested in function theory, they can focus more on Section 2 of the notes. While those with an interest should read both sections.

1. Corona Theorems and Complete Nevanlinna-Pick Kernels

Recall that we have already seen the Nevanlinna-Pick property when we studied the Pick interpolation question for the Hardy space $H^2(\mathbb{D})$. We will see a little later that the Dirichlet space also has this important property. It turns out that these spaces have a slightly stronger property of being able to solve the Nevanlinna-Pick interpolation problem, but with matrix targets or arbitrary size. When a reproducing kernel Hilbert space has this property, then we will say that it has the complete Nevanlinna-Pick property.

It turns out that for certain function spaces that have a complete Nevanlinna-Pick kernel, it is possible to solve the Corona problem for the multiplier algebra by solving a relatively easier Corona question for the space of analytic functions itself. It is an important fact that both the kernel for the Hardy space $H^2(\mathbb{D})$ and for the Dirichelet space $\mathcal{D}$ possess a complete Nevanlinna-Pick kernel. More generally, for any $0 \leq \sigma \leq \frac{1}{2}$ we have that the the space $B^2_\sigma(\mathbb{D})$ has a complete Nevanlinna-Pick kernel.

Let X be a Hilbert space of holomorphic functions in an open set Ω in $\mathbb{C}^n$ that is a reproducing kernel Hilbert space with a *complete irreducible Nevanlinna-Pick kernel* (see [1] for the definition). The following *Toeplitz corona theorem* is due to Ball, Trent and Vinnikov [3] (see also Ambrozie and Timotin [2] and Theorem 8.57 in [1]).

For $f = (f_\alpha)_{\alpha=1}^N \in \oplus^N X$ and $h \in X$, define $\mathbb{M}_f h = (f_\alpha h)_{\alpha=1}^N$ and

$$\|f\|_{Mult(X,\oplus^N X)} = \|\mathbb{M}_f\|_{X\to\oplus^N X} = \sup_{\|h\|_X\leq 1} \|\mathbb{M}_f h\|_{\oplus^N X}.$$

Note that $\max_{1\leq\alpha\leq N} \|\mathcal{M}_{f_\alpha}\|_{M_X} \leq \|f\|_{Mult(X,\oplus^N X)} \leq \sqrt{\sum_{\alpha=1}^N \|\mathcal{M}_{f_\alpha}\|^2_{M_X}}$.

Theorem 1.1 (Toeplitz Corona Theorem)**.** *Let X be a Hilbert function space in an open set Ω in $\mathbb{C}^n$ with an irreducible complete Nevanlinna-Pick kernel. Let $\delta > 0$ and $N \in \mathbb{N}$. Then $g_1, \ldots, g_N \in M_X$ satisfy the following "baby corona property"; for every $h \in X$, there are $f_1, \ldots, f_N \in X$ such that*

$$\begin{aligned} \|f_1\|^2_X + \cdots + \|f_N\|^2_X &\leq \frac{1}{\delta}\|h\|^2_X, \\ g_1(z) f_1(z) + \cdots + g_N(z) f_N(z) &= h(z), \qquad z \in \Omega, \end{aligned} \tag{1.1}$$

if and only if *$g_1, \ldots, g_N \in M_X$ satisfy the following "multiplier corona property"; there are $\varphi_1, \ldots, \varphi_N \in M_X$ such that*

$$\begin{aligned} \|\varphi\|_{Mult(X,\oplus^N X)} &\leq 1, \\ g_1(z)\varphi_1(z) + \cdots + g_N(z)\varphi_N(z) &= \sqrt{\delta}, \qquad z \in \Omega. \end{aligned} \tag{1.2}$$

The *baby corona theorem* is said to hold for X if whenever $g_1, \cdots, g_N \in M_X$ satisfy

$$|g_1(z)|^2 + \cdots + |g_N(z)|^2 \geq c > 0, \qquad z \in \Omega, \tag{1.3}$$

then $g_1, \ldots, g_N$ satisfy the baby corona property (1.1).

More succinctly, (1.1) is equivalent to the operator lower bound

$$\mathcal{M}_g \mathcal{M}_g^* - \delta I_X \geq 0, \tag{1.4}$$

where $g \equiv (g_1, \ldots g_N)$, $\mathcal{M}_g : \oplus^N X \to X$ by $\mathcal{M}_g f = \sum_{\alpha=1}^N g_\alpha f_\alpha$, and $\mathcal{M}_g^* h = \left(\mathcal{M}_{g_\alpha}^* f\right)_{\alpha=1}^N$. We note that (1.3) with $c = \delta$ is necessary for (1.4) as can be seen by testing on reproducing kernels k_z.

Remark 1.2. A standard abstract argument applies to show that the absence of a corona for the multiplier algebra M_X, i.e. the density of the linear span of point evaluations in the maximal ideal space of M_X, is equivalent to the following assertion: for each finite set $\{g_j\}_{j=1}^N \subset M_X$ such that (1.3) holds for some $c > 0$, there are $\{\varphi_j\}_{j=1}^N \subset M_X$ and $\delta > 0$ such that condition (1.2) holds. See for example Lemma 9.2.6 in [6] or the proof of Criterion 3.5 on page 39 of [8].

Here we recall the proof of the Toeplitz Corona Theorem 1.1 for holomorphic Hilbert function spaces with a complete Nevanlinna-Pick kernel. First we note the equivalence of (1.1) and (1.4). To see this note that (1.4) is equivalent to

$$\delta \langle h,h\rangle_X \leq \left\langle h,\mathcal{M}_\varphi\mathcal{M}_\varphi^* h\right\rangle_X = \left\langle \mathcal{M}_\varphi^* h,\mathcal{M}_\varphi^* h\right\rangle_{\oplus^N X}. \tag{1.5}$$

From functional analysis, we obtain that the bounded map $\mathcal{M}_\varphi : \oplus^N X \to X$ is onto. If $\mathcal{N} = \ker \mathcal{M}_\varphi$, then $\widehat{\mathcal{M}_\varphi} : \mathcal{N}^\perp \to X$ is invertible. Now (1.5) implies that $\widehat{\mathcal{M}_\varphi}^* : X \to \mathcal{N}^\perp$ is invertible and that $\left\|\left(\widehat{\mathcal{M}_\varphi}^*\right)^{-1}\right\| \leq \frac{1}{\sqrt{\delta}}$. By duality we then have $\left\|\left(\widehat{\mathcal{M}_\varphi}\right)^{-1}\right\| \leq \frac{1}{\sqrt{\delta}}$. Thus given $h \in X$, there is $f \in \mathcal{N}^\perp$ satisfying $\mathcal{M}_\varphi f = h$ and

$$\|f\|^2_{\oplus^N X} = \left\|\left(\widehat{\mathcal{M}_\varphi}\right)^{-1} h\right\|^2_{\oplus^N X} \leq \frac{1}{\delta}\|h\|^2_X,$$

which is (1.1). Conversely, using (1.1) we compute that

$$\begin{aligned} \left\|\mathcal{M}_\varphi^* h\right\|_{\oplus^N X} &= \sup_{\|g\|_{\oplus^N X}\leq 1}\left|\left\langle g,\mathcal{M}_\varphi^* h\right\rangle_{\oplus^N X}\right| = \sup_{\|g\|_{\oplus^N X}\leq 1}\left|\left\langle \mathcal{M}_\varphi g,h\right\rangle_X\right| \\ &\geq \left|\left\langle \mathcal{M}_\varphi \frac{f}{\|f\|_{\oplus^N X}},h\right\rangle_X\right| = \frac{\|h\|^2_X}{\|f\|_{\oplus^N X}} \geq \sqrt{\delta}\,\|h\|_X, \end{aligned} \tag{1.6}$$

which is (1.5), and hence (1.4).

Next we note that (1.3) with $c = \delta$ is necessary for (1.4) as can be seen by testing (1.5) on reproducing kernels k_z:

$$\delta\langle k_z,k_z\rangle \leq \left\langle \mathcal{M}_\varphi^* k_z,\mathcal{M}_\varphi^* k_z\right\rangle_{\oplus^N X} = |\varphi(z)|^2\langle k_z,k_z\rangle$$

since $\mathcal{M}_\varphi^* k_z = \left(\overline{\varphi_\alpha(z)}k_z\right)_{\alpha=1}^N$.

1.1. **Calculus of kernel functions and proof of the Toeplitz Corona Theorem.** A crucial theme for the proof of the Toeplitz Corona Theorem is that operator bounds for Hilbert function spaces, such as $\mathcal{M}_\varphi\mathcal{M}_\varphi^* - \delta I_X \geq 0$ for X in (1.4), can be recast in terms of kernel functions, namely

$$\left\{\langle\varphi(\zeta),\varphi(\lambda)\rangle_{\mathbb{C}^N} - \delta\right\}k(\zeta,\lambda) \succeq 0. \tag{1.7}$$

Indeed, if we let $h = \sum_{i=1}^J \xi_i k_{x_i}$ in (1.5) we obtain

$$\begin{aligned} \delta\sum_{i,j=1}^J \xi_i\overline{\xi_j}k(x_j,x_i) &= \delta\sum_{i,j=1}^J \xi_i\overline{\xi_j}\left\langle k_{x_i},k_{x_j}\right\rangle \\ &\leq \sum_{\alpha=1}^N\left\langle \sum_{i=1}^J \xi_i\overline{\varphi_\alpha(x_i)}k_{x_i},\sum_{j=1}^J \xi_j\overline{\varphi_\alpha(x_j)}k_{x_j}\right\rangle_X \\ &= \sum_{i,j=1}^J \xi_i\overline{\xi_j}\left\{\sum_{\alpha=1}^N \overline{\varphi_\alpha(x_i)}\varphi_\alpha(x_j)\right\}k(x_j,x_i), \end{aligned}$$

which is (1.7). A similar calculation shows that the operator upper bound $I_X - \mathcal{M}_f\mathcal{M}_f^* \geq 0$, which is equivalent to $\|\mathcal{M}_f\|_{\oplus^N X \to X} \leq 1$, is recast in terms of kernel functions as

$$\{1 - \langle f(\zeta), f(\lambda)\rangle_{\mathbb{C}^N}\} k(\zeta,\lambda) \succeq 0. \tag{1.8}$$

In order to recast the multiplier bound in the first line of (1.2) in terms of kernel functions, we must consider $N \times N$ matrix-valued kernel functions. Recall that $\mathbb{M}_f : X \to \oplus^N X$ by $\mathbb{M}_f h = (f_\alpha h)_{\alpha=1}^N$ (compare with $\mathcal{M}_f : \oplus^N X \to X$ by $\mathcal{M}_f g = \sum_{\alpha=1}^N f_\alpha g_\alpha$). Then for $g \in \oplus^N M_X$,

$$\langle \mathbb{M}_f h, g\rangle_{\oplus^N X} = \sum_{\alpha=1}^N \langle f_\alpha h, g_\alpha\rangle_X = \left\langle h, \sum_{\alpha=1}^N \mathcal{M}_{f_\alpha}^* g_\alpha \right\rangle_X,$$

and so $\mathbb{M}_f^* g = \sum_{\alpha=1}^N \mathcal{M}_{f_\alpha}^* g_\alpha$. Thus the first line in (1.2) is equivalent to $\left\|\mathbb{M}_f^*\right\|_{\oplus^N X \to X}^2 \leq 1$, hence to

$$\begin{aligned} 0 &\leq \sum_{\alpha=1}^N \|g_\alpha\|_X^2 - \left\|\sum_{\alpha=1}^N \mathcal{M}_{f_\alpha}^* g_\alpha\right\|_X^2 = \langle g,g\rangle_{\oplus^N X} - \left\langle \mathbb{M}_f^* g, \mathbb{M}_f^* g\right\rangle_X \\ &= \left\langle \left(I_{\oplus^N X} - \mathbb{M}_f\mathbb{M}_f^*\right) g, g\right\rangle_{\oplus^N X}, \end{aligned} \tag{1.9}$$

which is the operator bound

$$I_{\oplus^N X} - \mathbb{M}_f\mathbb{M}_f^* \geq 0. \tag{1.10}$$

To obtain an equivalent kernel estimate, let

$$g = (g_\alpha)_{\alpha=1}^N = \left(\sum_{i=1}^J \xi_i^\alpha k_{x_i^\alpha}\right)_{\alpha=1}^N,$$

so that

$$\mathbb{M}_f^* g = \sum_{\alpha=1}^N \mathcal{M}_{f_\alpha}^* g_\alpha = \sum_{\alpha=1}^N \sum_{i=1}^J \xi_i^\alpha \overline{f_\alpha(x_i^\alpha)} k_{x_i^\alpha}.$$

If we substitute this in (1.9) we obtain

$$\begin{aligned} \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^J \xi_i^\alpha \overline{\xi_j^\beta}\, \overline{f_\alpha(x_i^\alpha)} f_\beta\left(x_j^\beta\right) k\left(x_j^\beta, x_i^\alpha\right) &= \left\langle \mathbb{M}_f^* g, \mathbb{M}_f^* g\right\rangle_X \\ &\leq \langle g,g\rangle_{\oplus^N X} = \sum_{\alpha,\beta=1}^N \delta_\beta^\alpha \sum_{i,j=1}^J \xi_i^\alpha \overline{\xi_j^\beta} k\left(x_j^\beta, x_i^\alpha\right), \end{aligned}$$

or

$$\sum_{\alpha,\beta=1}^N \sum_{i,j=1}^J \xi_i^\alpha \overline{\xi_j^\beta} \left[\left\{\delta_\beta^\alpha - \overline{f_\alpha(x_i^\alpha)} f_\beta\left(x_j^\beta\right)\right\} k\left(x_j^\beta, x_i^\alpha\right)\right] \geq 0.$$

If we view $f(\zeta) \in \mathcal{B}\left(\mathbb{C}, \mathbb{C}^N\right)$ we can rewrite this last expression as

$$\{I_{\mathbb{C}^N} - f(\zeta) f(\lambda)^*\} k(\zeta,\lambda) \succeq 0, \tag{1.11}$$

which is the required matrix-valued kernel equivalence of the multiplier bound in (1.2).

Now we turn to the proof of Theorem 1.1. To see that (1.2) implies (1.1) just multiply $\mathcal{M}_\varphi f = \sqrt{\delta}$ by $\frac{h}{\sqrt{\delta}}$ to get $\mathcal{M}_\varphi \frac{fh}{\sqrt{\delta}} = h$ where

$$\begin{aligned}\left\|\frac{fh}{\sqrt{\delta}}\right\|_{\oplus^N X}^2 &= \frac{1}{\delta}\left(\|f_1 h\|_X^2 + \ldots + \|f_N h\|_X^2\right)\\ &\leq \frac{1}{\delta}\left(\|f_1\|_{M_X}^2 + \ldots + \|f_N\|_{M_X}^2\right)\|h\|_X^2 \leq \frac{1}{\delta}\|h\|_X^2 .\end{aligned}$$

The computation (1.6) above then shows that (1.4) holds for the same $\delta > 0$.

However, we can give another short proof, but using the language of positive semidefinite kernel functions to characterize operator boundedness. This will afford us our first opportunity to use the "calculus" of positive semidefinite forms. If (1.2) holds then

$$\overline{\varphi(\zeta)}^* f(\zeta) = \mathcal{M}_\varphi f = \sqrt{\delta},$$

and (1.11) holds:

$$\{I_{\mathbb{C}^N} - f(\zeta) f(\lambda)^*\} k(\zeta,\lambda) \succeq 0.$$

These two relations imply the positivity of the kernel function,

$$\begin{aligned}&\{\langle\varphi(\zeta),\varphi(\lambda)\rangle_{\mathbb{C}^N} - \delta\} k(\zeta,\lambda)\\ =\ &\left\{\langle\varphi(\zeta),\varphi(\lambda)\rangle_{\mathbb{C}^N} - \sqrt{\delta}\sqrt{\delta}\right\} k(\zeta,\lambda)\\ =\ &\left\{\langle\varphi(\zeta),\varphi(\lambda)\rangle_{\mathbb{C}^N} - \overline{\varphi(\zeta)}^* f(\zeta) f(\lambda)^* \overline{\varphi(\lambda)}\right\} k(\zeta,\lambda)\\ =\ &\overline{\varphi(\zeta)}^* \left[\{I_{\mathbb{C}^N} - f(\zeta) f(\lambda)^*\} k(\zeta,\lambda)\right] \overline{\varphi(\lambda)} \succeq 0.\end{aligned}$$

By (1.7) this is equivalent to (1.5), and hence to (1.4).

Conversely, normalize k at a fixed point $\lambda_0 \in \Omega$ so that $k_{\lambda_0} \equiv 1$. Since k is an irreducible complete Nevanlinna-Pick kernel, we can find a Hilbert space $\mathcal{K}$ and a map $b : \Omega \to \mathcal{K}$ with $b(\lambda_0) = 0$ and such that

$$k(\zeta,\lambda) = \frac{1}{1 - \langle b(\zeta), b(\lambda)\rangle_{\mathcal{K}}}. \tag{1.12}$$

This theorem has a long history and is not easy to prove (Theorem 7.31 in [1]). In fact, one can take $\mathcal{K}$ to be the Drury-Arveson Hardy space H_m^2 for some cardinal number m (Theorem 8.2 in [1]), but we will not need this. From (1.4) we now obtain (1.7):

$$K(\zeta,\lambda) \equiv \{\langle\varphi(\zeta),\varphi(\lambda)\rangle_{\mathbb{C}^N} - \delta\} k(\zeta,\lambda) \succeq 0.$$

By a kernel-valued version of the Lax-Milgram Theorem, we can factor the left hand side $K(\zeta,\lambda)$ as $\langle G(\zeta), G(\lambda)\rangle_{\mathcal{H}}$ where $G : \Omega \to \mathcal{H}$ for some auxiliary space $\mathcal{H}$. Indeed, define $F : \Omega \to X_K$ by $F(\zeta) = K_\zeta$ so that

$$K(\zeta,\lambda) = \langle K_\lambda, K_\zeta\rangle_{X_K} = \langle F(\lambda), F(\zeta)\rangle_{X_K}.$$

Now fix an orthonormal basis $\{\mathbf{e}_\alpha\}_\alpha$ for X_K and define a conjugate linear operator Γ by

$$\Gamma\left(\sum_\alpha c_\alpha \mathbf{e}_\alpha\right) = \sum_\alpha \overline{c_\alpha}\mathbf{e}_\alpha.$$

Then $G = \Gamma \circ F$ satisfies

$$K(\zeta,\lambda) = \langle F(\lambda), F(\zeta)\rangle_{X_K} = \langle \Gamma\circ F(\zeta), \Gamma\circ F(\lambda)\rangle_{X_K} = \langle G(\zeta), G(\lambda)\rangle_{X_K},$$

with $\mathcal{H} = X_K$ as required. Hence

$$\langle \varphi(\zeta), \varphi(\lambda) \rangle_{\mathbb{C}^N} - \delta = [1 - \langle b(\zeta), b(\lambda) \rangle_{\mathcal{K}}] \langle G(\zeta), G(\lambda) \rangle_{\mathcal{H}},$$

or equivalently,

$$\langle \varphi(\zeta), \varphi(\lambda) \rangle_{\mathbb{C}^N} + \langle b(\zeta), b(\lambda) \rangle_{\mathcal{K}} \langle G(\zeta), G(\lambda) \rangle_{\mathcal{H}} = \delta + \langle G(\zeta), G(\lambda) \rangle_{\mathcal{H}}. \tag{1.13}$$

Now we rewrite (1.13) in terms of inner products of direct sums of Hilbert spaces,

$$\begin{aligned} &\langle \varphi(\zeta), \varphi(\lambda) \rangle_{\mathbb{C}^N} + \langle b(\zeta) \otimes G(\zeta), b(\lambda) \otimes G(\lambda) \rangle_{\mathcal{K} \otimes \mathcal{H}} \\ &= \left\langle \sqrt{\delta}, \sqrt{\delta} \right\rangle_{\mathbb{C}} + \langle G(\zeta), G(\lambda) \rangle_{\mathcal{H}}, \end{aligned}$$

so that it can be interpreted as saying that the map from $\mathcal{N}_1$ to $\mathcal{N}_2$ that sends the element

$$(\varphi(\lambda) u, b(\lambda) \otimes G(\lambda) u) \in \mathbb{C}^N \oplus (\mathcal{K} \otimes \mathcal{H}) \tag{1.14}$$

with $u \in \mathbb{C}$ to the element

$$\left(\sqrt{\delta} u, G(\lambda) u \right) \in \mathbb{C} \oplus \mathcal{H} \tag{1.15}$$

is an *isometry*! Here the spaces $\mathcal{N}_1$ and $\mathcal{N}_2$ are given by

$$\begin{aligned} \mathcal{N}_1 &= Span \left\{ \begin{pmatrix} \varphi(\lambda) \\ b(\lambda) \otimes G(\lambda) \end{pmatrix} u : u \in \mathbb{C}, \lambda \in \Omega \right\} \subset \mathbb{C}^N \oplus (\mathcal{K} \otimes \mathcal{H}), \\ \mathcal{N}_2 &= Span \left\{ \begin{pmatrix} \sqrt{\delta} \\ G(\lambda) \end{pmatrix} u : u \in \mathbb{C}, \lambda \in \Omega \right\} \subset \mathbb{C} \oplus \mathcal{H}. \end{aligned}$$

Thus using (1.4) we have obtained (1.13) that defines a linear isometry V' from the linear span $\mathcal{N}_1$ of the elements $\varphi(\lambda) u \oplus (b(\lambda) \otimes G(\lambda)) u$ in the direct sum $\mathbb{C}^N \oplus (\mathcal{K} \otimes \mathcal{H})$ onto a subspace $\mathcal{N}_2$ of the direct sum $\mathbb{C} \oplus \mathcal{H}$: the element in (1.14) goes to the element in (1.15). Now extend this isometry V' to an isometry V from all of $\mathbb{C}^N \oplus (\mathcal{K} \otimes \mathcal{H})$ *onto* $\mathbb{C} \oplus \mathcal{H}$, where we add an infinite-dimensional summand to $\mathcal{H}$ if necessary. Indeed, V' extends by continuity to an isometry from $\overline{\mathcal{N}_1}$ onto $\overline{\mathcal{N}_2}$, and provided the orthogonal complements of $\overline{\mathcal{N}_1}$ and $\overline{\mathcal{N}_2}$ have the same dimension, we can then trivially extend the isometry from all of $\mathbb{C}^N \oplus (\mathcal{K} \otimes \mathcal{H})$ *onto* $\mathbb{C} \oplus \mathcal{H}$. But the dimensions of the complements can be made equal by adding an infinite-dimensional summand to $\mathcal{H}$.

Decompose the extended isometry V as a block matrix

$$V = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathbb{C}^N \\ \mathcal{K} \otimes \mathcal{H} \end{bmatrix} \to \begin{bmatrix} \mathbb{C} \\ \mathcal{H} \end{bmatrix}. \tag{1.16}$$

Since V is an onto isometry we obtain the formulas,

$$\begin{aligned} \begin{bmatrix} A^*A + C^*C & A^*B + C^*D \\ B^*A + D^*C & B^*B + D^*D \end{bmatrix} &= \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ &= V^*V = I_{\mathbb{C}^N \oplus (\mathcal{K} \otimes \mathcal{H})} = \begin{bmatrix} I_{\mathbb{C}^N} & 0 \\ 0 & I_{\mathcal{K} \otimes \mathcal{H}} \end{bmatrix}. \end{aligned} \tag{1.17}$$

Then (1.16) on the subspace $\mathcal{N}_1$ becomes

$$\begin{aligned} A\varphi(\lambda) + B[b(\lambda) \otimes G(\lambda)] &= \sqrt{\delta}, \\ C\varphi(\lambda) + D[b(\lambda) \otimes G(\lambda)] &= G(\lambda). \end{aligned} \tag{1.18}$$

Now define $f:\Omega\to\mathcal{B}\left(\mathbb{C},\mathbb{C}^N\right)$ (which is of course isomorphic to $\mathbb{C}^N$) by

$$\overline{f(\lambda)}^{*}=A+B\left\{b(\lambda)\otimes\left(I-DE_{b(\lambda)}\right)^{-1}C\right\}, \tag{1.19}$$

where E_b is the map $E_b:\mathcal{H}\to\mathcal{K}\otimes\mathcal{H}$ given by

$$E_b v=b\otimes v,\qquad v\in\mathcal{H}. \tag{1.20}$$

Note that this formula for $\overline{f(\lambda)}^{*}$ is obtained by solving the second line in (1.18) for $G(\lambda)=\left(I-DE_{b(\lambda)}\right)^{-1}C\varphi(\lambda)$, and then substituting this in the first line and dropping $\varphi(\lambda)$. Observe that $E_b^{*}(c\otimes w)=\langle c,b\rangle_{\mathcal{K}}w$, so that

$$E_b^{*}E_c=\langle c,b\rangle_{\mathcal{K}}I_{\mathcal{H}}. \tag{1.21}$$

From this we conclude that $I-DE_{b(\lambda)}$ is invertible. Indeed, (1.12) shows that $\langle b,b\rangle_{\mathcal{K}}<1$ and (1.21) then implies that $E_{b(\lambda)}$ is a *strict* contraction. From the equation $B^{*}B+D^{*}D=I_{\mathcal{K}\otimes\mathcal{H}}$ in (1.17) we see that D is a contraction, which altogether implies $\left\|DE_{b(\lambda)}\right\|<1$. Thus $\overline{f(\lambda)}^{*}$ satisfies

$$\begin{aligned}\overline{f(\lambda)}^{*}\varphi(\lambda) &= A\varphi(\lambda)+B\left[b(\lambda)\otimes\left(I-DE_{b(\lambda)}\right)^{-1}C\varphi(\lambda)\right]\\ &= A\varphi(\lambda)+B\left[b(\lambda)\otimes G(\lambda)\right]\\ &= \sqrt{\delta},\end{aligned} \tag{1.22}$$

which is the second line in (1.2).

To see that the first line in (1.2) holds, we must show that $f(\lambda)$ is a contractive multiplier, i.e. that (1.11) holds:

$$\left\{I_{\mathbb{C}^N}-f(\zeta)f(\lambda)^{*}\right\}k(\zeta,\lambda)\succeq 0. \tag{1.23}$$

For this we use (1.17). We compute with

$$\begin{aligned}\overline{f(\lambda)}^{*} &= A+BE_{b(\lambda)}\left(I-DE_{b(\lambda)}\right)^{-1}C,\\ \overline{f(\zeta)} &= A^{*}+C^{*}\left(I-E_{b(\zeta)}^{*}D^{*}\right)^{-1}E_{b(\zeta)}^{*}B^{*},\end{aligned}$$

that

$$\begin{aligned}I_{\mathbb{C}^N}-\overline{f(\zeta)f(\lambda)^{*}} &= I-\left[A^{*}+C^{*}\left(I-E_{b(\zeta)}^{*}D^{*}\right)^{-1}E_{b(\zeta)}^{*}B^{*}\right]\\ &\quad\times\left[A+BE_{b(\lambda)}\left(I-DE_{b(\lambda)}\right)^{-1}C\right]\\ &= I-A^{*}A-A^{*}BE_{b(\lambda)}\left(I-DE_{b(\lambda)}\right)^{-1}C\\ &\quad -C^{*}\left(I-E_{b(\zeta)}^{*}D^{*}\right)^{-1}E_{b(\zeta)}^{*}B^{*}A\\ &\quad -C^{*}\left(I-E_{b(\zeta)}^{*}D^{*}\right)^{-1}E_{b(\zeta)}^{*}B^{*}BE_{b(\lambda)}\left(I-DE_{b(\lambda)}\right)^{-1}C,\end{aligned}$$

and then using (1.17) we obtain

$$
\begin{aligned}
(1.24)\qquad & I_{\mathbb{C}^N} - \overline{f(\zeta) f(\lambda)^*} \\
= \; & C^*C + C^*DE_{b(\lambda)}\left(I - DE_{b(\lambda)}\right)^{-1} C \\
& + C^*\left(I - E^*_{b(\zeta)}D^*\right)^{-1} E^*_{b(\zeta)} D^* C \\
& + C^*\left(I - E^*_{b(\zeta)}D^*\right)^{-1} E^*_{b(\zeta)} \left(D^*D - I\right) E_{b(\lambda)} \left(I - DE_{b(\lambda)}\right)^{-1} C \\
= \; & C^*\left(I - E^*_{b(\zeta)}D^*\right)^{-1} \\
& \times \left\{\left(I - E^*_{b(\zeta)}D^*\right)\left(I - DE_{b(\lambda)}\right) + \left(I - E^*_{b(\zeta)}D^*\right) DE_{b(\lambda)} \right. \\
& \left. + E^*_{b(\zeta)}D^*\left(I - DE_{b(\lambda)}\right) - E^*_{b(\zeta)}E_{b(\lambda)} + E^*_{b(\zeta)}D^*DE_{b(\lambda)}\right\} \\
& \times \left(I - DE_{b(\lambda)}\right)^{-1} C \\
= \; & C^*\left(I - E^*_{b(\zeta)}D^*\right)^{-1}\left(I - E^*_{b(\zeta)}E_{b(\lambda)}\right)\left(I - DE_{b(\lambda)}\right)^{-1} C \\
= \; & \left(1 - \overline{\langle b(\zeta), b(\lambda)\rangle_{\mathcal{K}}}\right) C^*\left(I - E^*_{b(\zeta)}D^*\right)^{-1}\left(I - DE_{b(\lambda)}\right)^{-1} C,
\end{aligned}
$$

where the last line follows from (1.21). Thus using (1.12) the left side of (1.23), which is an $N \times N$ matix-valued kernel function, has its complex conjugate equal to

$$
\begin{aligned}
& C^*\left(I - E^*_{b(\zeta)}D^*\right)^{-1}\left(I - DE_{b(\lambda)}\right)^{-1} C \\
& = \left\langle \left(I - DE_{b(\lambda)}\right)^{-1} C, \left(I - DE_{b(\zeta)}\right)^{-1} C \right\rangle_{\mathcal{H}},
\end{aligned}
$$

which is an $N \times N$ matix-valued Grammian, hence a positive kernel as required.

2. The $\overline{\partial}$-equation in the Dirichlet Space

As is well-known there is an intimate connection between the Corona Theorem and $\overline{\partial}$-problems. In our context, a $\overline{\partial}$-problem will be to solve the following differential equation

$$
\overline{\partial} b = \mu \tag{2.1}
$$

where μ is a Carleson measure for the space $\mathcal{D}$ and b is some unknown function. Now solving this problem is an easy application of Cauchy's formula, however we will need to obtain estimates of the solutions. Tho obtain these estimates, we will provide a different solution operator to the $\overline{\partial}$-problem more appropriately suited to our contexts.

In [11] Xiao's constructed a non-linear solution operator for (2.1) that is well adapted to solve (2.1) and obtain estimates. We note that in the case of $H^\infty(\mathbb{D})$ that this result was first obtained by P. Jones, [5]. First, note that

$$
F(z) = \frac{1}{2\pi i} \int_{\mathbb{D}} \frac{d\mu(\zeta)}{\zeta - z} dA(\zeta)
$$

satisfies $\overline{\partial} F = \mu$ in the sense of distribution.

Exercise 2.1. *Prove this claim.*

The difficulty with this solution kernel is that it does not allow for one to obtain good estimates on the solution. To rectify this, following Jones [5], we are now going to define a new non-linear kernel that will overcome this difficulty.

Theorem 2.2 (Jones, [5]). *Let μ be a complex $H^2(\mathbb{D})$ Carleson measure on $\mathbb{D}$. Then with $S(\mu)(z)$ given by*

$$S(\mu)(z) = \int_{\mathbb{D}} K(\sigma, z, \zeta)\, d\mu(\zeta) \tag{2.2}$$

where $\sigma = \frac{|\mu|}{\|\mu\|_{CM(H^2)}}$ and

$$K(\sigma, z, \zeta) \equiv \frac{2i}{\pi} \frac{1-|\zeta|^2}{(z-\zeta)\left(1-\overline{\zeta}z\right)} \exp\left\{\int_{|\omega|\geq|\zeta|} \left(-\frac{1+\overline{\omega}z}{1-\overline{\omega}z} + \frac{1+\overline{\omega}\zeta}{1-\overline{\omega}\zeta}\right) d\sigma(\omega)\right\},$$

we have that:

1. $S(\mu) \in L^1_{loc}(\mathbb{D})$.
2. $\overline{\partial} S(\mu) = \mu$ *in the sense of distributions.*
3. $\int_{\mathbb{D}} \left|K\left(\frac{|\mu|}{\|\mu\|_{CM(H^2)}}, x, \zeta\right)\right| d|\mu|(\zeta) \lesssim \|\mu\|_{CM(H^2)}$ *for all $x \in \mathbb{T} = \partial\mathbb{D}$,*

   so $\|S(\mu)\|_{L^\infty(\mathbb{T})} \lesssim \|\mu\|_{CM(H^2)}$.

With this set-up, we now prove the follow theorem due to Xiao, extending Theorem 2.2, about estimates for $\overline{\partial}$-problems in the Dirichlet space.

Theorem 2.3 (Xiao, [11]). *If $|g(z)|^2\, dA(z)$ is a $\mathcal{D}$-Carleson measure then there is a function f such that $\overline{\partial} f = g$ and*

$$\|f\|_{\mathcal{M}_{\mathcal{W}^{1/2}(\mathbb{T})}} \lesssim \|\, |g(z)|^2\, dA(z)\|_{CM(\mathcal{D})}.$$

Proof. First note that the measure $\mu = |g| dA$ satisfies the the following estimate

$$\begin{aligned} \left(\int_{S(I)} d\mu(z)\right)^2 &\leq \int_{S(I)} |g(z)|^2\, dA(z) \int_{S(I)} dA(z) \\ &\lesssim |I|^2, \end{aligned}$$

so the measure $d\mu$ is a $H^2(\mathbb{D})$ Carleson measure. Thus, by Theorem 2.2, we have a solution given by

$$S(\mu)(z)$$

that satisfies $\overline{\partial} S(\mu) = g$ for all $z \in \mathbb{D}$ and $\|S(\mu)\|_{L^\infty(\mathbb{T})} \lesssim \||g|^2 dA\|_{CM(\mathcal{D})}$.

We must now also show that $S(\mu)\,|_{\mathbb{T}} \in X_2^{\frac{1}{2}}(\mathbb{T})$, i.e. that

$$d\mu_{S(\mu)} \equiv |\nabla S(\mu)(z)|^2\, dA(z)$$

is also a $\mathcal{D}$-Carleson measure. Of course, since $\overline{\partial} S(\mu) = g$ in $\mathbb{D}$ and we obtain immediately that $\left|\frac{\partial}{\partial \overline{z}} S(\mu)(z)\right|^2 dA(z) = |g(z)|^2 dA(z)$ is a $\mathcal{D}$-Carleson measure. It is possible to take the derivative ∂ of $S(\mu)(z)$ and then use the theory of singular integral operators and the singular Beurling transform to estimate $\partial S(\mu)$.

We will instead use the following clever device of Xiao that greatly ameliorates the singularity. We introduce the function

$$\widetilde{S}(\mu)(z) = \int_{\mathbb{D}} \widetilde{K}(\sigma, z, \zeta)\, d\nu(\zeta)$$

where

$$\widetilde{K}(\sigma,z,\zeta)\equiv\frac{2i}{\pi}\frac{1-|\zeta|^2}{\left|1-\overline{\zeta}z\right|^2}\exp\left\{\int_{|\omega|\geq|\zeta|}\left(-\frac{1+\overline{\omega}z}{1-\overline{\omega}z}+\frac{1+\overline{\omega}\zeta}{1-\overline{\omega}\zeta}\right)d\sigma(\omega)\right\}.$$

Now, observe that

$$\frac{\widetilde{K}(\sigma,z,\zeta)}{K(\sigma,z,\zeta)}=\frac{z-\zeta}{1-\overline{z}\zeta}=\varphi_z(\zeta)$$

is the autormorphism of the disk that interchanges 0 and z, and moreover, $\varphi_z(\zeta)=\frac{z(1-\overline{z}\zeta)}{1-\overline{z}\zeta}=z$ for $z\in\mathbb{T}$. Using this we see that $\widetilde{S}(\mu)(z)=zS(\mu)(z)$ for $z\in\mathbb{T}$ and so it suffices to show instead that $\left|\nabla\widetilde{S}(\mu)(z)\right|^2dA(z)$ is a $\mathcal{D}$-Carleson measure. The advantage is that $\widetilde{K}(\sigma,z,\zeta)$ has a much milder singularity than $K(\sigma,z,\zeta)$.

We next show that the operator $\widetilde{S}(\mu)$ still provides a solution to the problem $\overline{\partial}b=\mu$. We sketch the details for this, and the interested reader can fill in the pieces. Observe that for $w\in\mathbb{D}$, the function $f_w(z)=\frac{1-|w|^2}{(1-\overline{w}z)^2}$ belongs to $H^1(\mathbb{D})$ with norm independent of $w\in\mathbb{D}$. Indeed, we have that

$$f_w(z)=\left(\frac{(1-|w|^2)^{1/2}}{1-z\overline{w}}\right)^2\in H^2(\mathbb{D})\cdot H^2(\mathbb{D})=H^1(\mathbb{D})$$

with $\|f_w\|_{H^1(\mathbb{D})}\leq\left\|\frac{(1-|w|^2)^{1/2}}{1-z\overline{w}}\right\|_{H^2(\mathbb{D})}^2=1$.

Also, since the Carleson measures for $H^2(\mathbb{D})$ coincide with the Carleson measures for $H^1(\mathbb{D})$, if μ is a $H^1(\mathbb{D})$ Carleson measure, then we have

$$\int_{\mathbb{D}}|f_w(z)|\,d\mu(z)\lesssim\|f_w\|_{H^1(\mathbb{D})}\|\mu\|_{CM(H^2)}. \tag{2.3}$$

This estimate then allows us to conclude

$$\begin{aligned}\mathrm{Re}\left(\int_{|w|\geq|\zeta|}\left(\frac{1+\overline{w}\zeta}{1-\overline{w}\zeta}\right)|g(w)|dA(w)\right) &= \int_{|w|\geq|\zeta|}\mathrm{Re}\left(\frac{1+\overline{w}\zeta}{1-\overline{w}\zeta}\right)|g(w)|dA(w)\\ &\leq 2\int_{\mathbb{D}}\frac{1-|\xi|^2}{|1-\overline{w}\xi|^2}|g(\xi)|dA(\xi)\\ &\leq 2\|f_w\|_{H^1(\mathbb{D})}=2.\end{aligned}$$

Using the computations above, it can then be shown that

$$\int_{\mathbb{D}}\frac{1-|z\overline{\xi}|^2}{|1-z\overline{\xi}|^2}\exp\left(-\int_{|w|\geq|\xi|}\frac{1-|z\overline{w}|^2}{|1-z\overline{w}|^2}|g(w)|dA(w)\right)|g(\xi)|dA(\xi)\leq 1. \tag{2.4}$$

Using these estimates and computations we arrive at the condition that

$$\left|\nabla\widetilde{S}(\mu)(z)\right|\lesssim\int_{\mathbb{D}}\frac{|g(w)|}{|1-\overline{w}z|^2}dA(w).$$

Finally, we then apply the transformation of Carleson measure result of Rochberg and Wu, see Lemma 6 in [7], to conclude that

$$\left|\nabla\widetilde{S}(\mu)(z)\right|^2dA(z)$$

is a $\mathcal{D}$-Carleson measure. We now state a result of Rochberg-Wu that shows it is possible to transform Carleson measures to Carleson measures via positive operators.

For $\alpha \in (-\infty, \frac{1}{2}]$ we let

$$B_\alpha(\mathbb{D}) = \left\{ f \in \mathrm{Hol} : |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{1-2\alpha} dA(z) < \infty \right\}.$$

Note that this is just a relabeling of the space $B_\sigma(\mathbb{D})$ with $\sigma = 1 - 2\alpha$.

Lemma 2.4 (Rochberg-Wu, [7]). *Let $\alpha \in (-\infty, \frac{1}{2}]$ and $\beta > \max\{-1, -1-2\alpha\}$ and $b > \max\{\frac{\beta+3}{2}, \frac{\beta+3}{2} - \alpha\}$. Also, let*

$$Tf(z) = \int_{\mathbb{D}} \frac{f(w)}{|1 - \overline{w}z|^b} (1-|w|^2)^{b-2} dA(w).$$

If $|f(z)|^2 (1-|z|^2)^\beta dA(z)$ is an B_α-Carleson measure, then $|Tf(z)|^2 (1-|z|^2)^\beta dA(z)$ is also an B_α-Carleson measure.

Since $|g(z)|^2 dA(z)$ is a $\mathcal{D}$-Carleson measure, this then follows from Lemma 2.4. □

Exercise 2.5. *Show that the same result as Theorem 2.3 remains true when $0 \leq \sigma < \frac{1}{2}$.*

Exercise 2.6. *Show that $|\partial S(\mu)(z)|^2 dA(z)$ is a $\mathcal{D}$-Carleson measure by appealing to the theory of singular integral operators. Hint: By computing the derivative of $S(\mu)$ you are arriving at the Beurling Transform. Then one needs to observe that $(1-|w|^2)^b$ is a A_2 weight for certain values of b.*

Exercise 2.7. *The function $\widetilde{S}(\mu)(z)$ behaves essentially like the holomorphic projection $\Gamma_1 \mu(z)$ where,*

$$\Gamma_s g(z) \equiv \int_{\mathbb{D}} \frac{g(w)\left(1-|w|^2\right)^s}{(1-\overline{w}\cdot z)^{1+s}} dA(w).$$

One can see that if $\mu = |g| dA$ is a $\mathcal{D}$-Carleson measure, then $|\nabla \Gamma_1 \mu|^2 dA(z)$ is also a $\mathcal{D}$-Carleson measure. Using this idea and Lemma 2.4 show that $\left|\nabla \widetilde{S}(\mu)(z)\right|^2 dA(z)$ is a $\mathcal{D}$-Carleson measure.

2.1. **The Case of the Dirichlet Space.** This general set-up applies to our setting with $X = \mathcal{D}$ and $M_{\mathcal{D}}$ the multiplier algebra for the Dirichlet space. The Toeplitz corona theorem thus provides a useful tool for reducing the multiplier corona property (1.2) to the more tractable, but still very difficult, baby corona property (1.1) for the multiplier algebras $M_{\mathcal{D}}$.

We now state a simple proposition that will be useful in understanding the relationships between the Corona problems for $\mathcal{D}$ and $M_{\mathcal{D}}$.

Proposition 2.8. *Suppose that $g_1, \ldots, g_N \in M_{\mathcal{D}}$. Define the map*

$$M_{(g_1,\ldots,g_n)}(f_1, \ldots, f_n) := \sum_{k=1}^{N} g_k(z) f_k(z).$$

Then the following are equivalent

(i) $M_{(g_1,\ldots,g_n)} : M_{\mathcal{D}} \times \cdots \times M_{\mathcal{D}} \mapsto M_{\mathcal{D}}$ *is onto;*
(ii) $M_{(g_1,\ldots,g_n)} : \mathcal{D} \times \cdots \times \mathcal{D} \mapsto \mathcal{D}$ *is onto;*

(iii) *There exists a $\delta > 0$ such that for all $z \in \mathbb{D}$ we have*

$$\sum_{k=1}^{N} |g_k(z)|^2 \geq \delta > 0.$$

It is easy to see that both (i) and (ii) each individually imply (iii). We will show that condition (iii) implies both (i) and (ii). Note that by the Toeplitz Corona Theorem 1.1 it would suffice to prove that (iii) implies (ii) since the result then lifts to give the statement in (i). However, we feel that it is instructive to prove the theorems separately.

The equivalence between the conditions in Proposition 2.8 were originally proved by Tolokonnikov [9]. Below, we will give an alternate proof of this due to Xiao [11] of the corona theorem for the multiplier algebra $M_{\mathcal{D}}$ and for the Dirichlet space $\mathcal{D}$. An alternate proof for the case of the Dirichlet space $\mathcal{D}$, and then for $M_{\mathcal{D}}$ via the Toeplitz Corona Theorem was given by Trent [10]. The proof of Trent is of a different nature than what we present here.

Exercise 2.9. *Prove that (i) and (ii) each individually imply (iii).*

Exercise 2.10. *Show that if (i) or (ii) holds, then there are in fact estimates to the Corona problem as well.*

Exercise 2.11. *Show that the above Theorem is also true for the multiplier algebra of $B^2_\sigma(\mathbb{D})$ as well.*

2.2. **The Corona Theorem for the Algebra $M_{\mathcal{D}(\mathbb{D})}$.** Here we prove the implication that (iii) implies (i) in Proposition 2.8

Proof of (iii) *implies* (i) *in Proposition 2.8.* Note that is suffices to prove that there exists $f_1, \ldots, f_n \in M_{\mathcal{D}}$ such that

$$1 = \sum_{j=1}^{N} f_j(z) g_j(z) \quad \forall z \in \mathbb{D}.$$

Once we have found the $\{f_j\}_{j=1}^N$ then simply multiplying through by the desired function h proves the implication. For $j = 1, \ldots, N$, define

$$\varphi_j(z) = \frac{\overline{g_j(z)}}{|g(z)|^2},$$

where

$$|g(z)|^2 = \sum_{j=1}^{N} |g_j(z)|^2.$$

Then we have that

$$1 = \sum_{j=1}^{N} g_j(z)\varphi_j(z).$$

Suppose that we can find functions b_{jk} such that

(i) $\overline{\partial} b_{jk} = \varphi_j \overline{\partial} \varphi_k$;
(ii) $\|b_{jk}\|_{M_{\mathcal{D}}} \lesssim 1$.

If we can find such functions, then we define

$$f_j(z) = \varphi_j(z) + \sum_{k=1}^{N} \left(b_{jk}(z) - b_{kj}(z)\right) g_k(z),$$

then the functions are analytic since,

$$\begin{aligned}
\overline{\partial} f_j &= \overline{\partial}\varphi_j + \sum_{k=1}^{N} \left(\overline{\partial} b_{jk} - \overline{\partial} b_{kj}\right) g_k \\
&= \overline{\partial}\varphi_j + \sum_{k=1}^{N} \left(\varphi_j \overline{\partial}\varphi_k - \varphi_k \overline{\partial}\varphi_j\right) g_k \\
&= \overline{\partial}\varphi_j - \left(\sum_{k=1}^{N} \varphi_k g_k\right) \overline{\partial}\varphi_j + \varphi_j \overline{\partial} \left(\sum_{k=1}^{N} g_k \varphi_k\right) \\
&= \overline{\partial}\varphi_j - \overline{\partial}\varphi_j + \varphi_j \overline{\partial} 1 = 0.
\end{aligned}$$

Moreover, because of the estimate b_{jk} it is clear that the functions $g_j \in M_{\mathcal{D}}$. Finally, with these functions we have

$$\begin{aligned}
\sum_{j=1}^{N} f_j(z) g_j(z) &= \sum_{j=1}^{N} g_j(z)\varphi_j(z) + \sum_{j=1}^{N}\sum_{k=1}^{N} \left(b_{jk}(z) - b_{kj}(z)\right) g_k(z) g_j(z) \\
&= 1 + 0 = 1
\end{aligned}$$

which solves the problem.

The only issue that remains is to demonstrate that it is in fact possible to find such solutions b_{jk}, and to do so, it suffices to demonstrate that the functions $\varphi_j \overline{\partial}\varphi_k$ are in fact Carleson measures for the Dirichlet space $\mathcal{D}$. We accomplish this by first noting that

$$\varphi_j \overline{\partial}\varphi_k = \frac{\overline{g_j}}{|g|^2} \frac{\sum_l g_l \left(\overline{g_l g_k' - g_k g_l'}\right)}{|g|^4}.$$

This then implies that

$$\left|\varphi_j \overline{\partial}\varphi_k\right|^2 \lesssim \frac{1}{|g|^6} \sum_{l=1}^{N} |g_l'|^2 \lesssim \sum_{l=1}^{N} |g_l'|^2.$$

Here we have used the properties that $0 < \delta \leq \sum_l |g_l(z)|$. Note also that the implied constants here depend upon the parameter δ and the number of generators N. However, since the functions $g_l \in M_{\mathcal{D}}$ we have that

$$|g_l'|^2 dA(z)$$

are $\mathcal{D}$-Carleson measures, and thus an application of Theorem 2.3 provides the solutions b_{jk} with estimates that we seek. □

We next turn to proving that (iii) implies (ii). The method of proof will be similar to what appears above, however, we can resort to the simpler solution operator to conclude the result.

Proof of (iii) *implies* (ii) *in Proposition 2.8.* In the proof of this implication, we must show directly that given an $f \in \mathcal{D}$ it is possible to find $f_1, \ldots, f_N \in \mathcal{D}$ such that

$$\sum_{j=1}^{N} f_j(z) g_j(z) = f(z) \quad \forall z \in \mathbb{D},$$

under the hypothesis that $\{g_j\}_{j=1}^N$ satisfies (iii). Again, define the functions

$$\varphi_j(z) = \frac{\overline{g_j(z)}}{|g(z)|^2}$$

where $|g(z)|^2 = \sum_{j=1}^N |g_j(z)|^2$. As we saw above in the proof that (iii) implies (i), it will suffice to find functions $b_{j,k}$ such that

(i) $\overline{\partial} b_{jk} = f \varphi_j \overline{\partial} \varphi_k$;
(ii) $\|b_{jk}\|_{\mathcal{D}} \lesssim 1$.

If we can find such functions, then we define

$$f_k(z) = f(z)\varphi_k(z) + \sum_{j=1}^{N} \left(b_{jk}(z) - b_{kj}(z)\right) g_k(z).$$

A similar computation to what appears above shows that we have

$$\overline{\partial} f_k = 0 \quad \forall 1 \leq k \leq N$$

and that

$$\sum_{k=1}^{N} f_k(z) g_k(z) = f(z) \quad \forall z \in \mathbb{D} \text{ and } \|f_k\|_{\mathcal{D}} \lesssim 1.$$

These three properties follow from the properties of the solution b_{jk} and the properties of $\{g_j\}_{j=1}^N$.

We now show how to find solutions b_{jk} satisfying the desired properties. Set

$$b_{jk}(z) = \frac{1}{2\pi i} \int_{\mathbb{D}} f(w) \frac{\varphi_j(w) \overline{\partial} \varphi_k(w)}{z - w} dA(w).$$

Then we clearly have that b_{jk} are smooth solutions on $\mathbb{D}$ that satisfy $\overline{\partial} b_{jk} = f \varphi_j \overline{\partial} \varphi_k$, and need only establish the estimates. First, note that

$$\int_{\mathbb{D}} \left|\overline{\partial} b_{jk}\right|^2 dA(z) \;\lesssim\; \|f\|_{\mathcal{D}}^2 . \tag{2.5}$$

Indeed, we have

$$\begin{aligned}
\int_{\mathbb{D}} \left|\overline{\partial} b_{jk}\right|^2 dA(z) &= \int_{\mathbb{D}} \left|\overline{\partial} f \varphi_j \overline{\partial}\varphi_k\right|^2 dA(z) \\
&\lesssim \sum_{j=1}^{N} \int_{\mathbb{D}} \left|f(z) g_j'(z)\right|^2 dA(z) \\
&\lesssim \sum_{j=1}^{N} \int_{\mathbb{D}} \left|(f(z)g_j(z))\right|^2 dA(z) + \sum_{j=1}^{N} \int_{\mathbb{D}} \left|f'(z)g_j(z)\right|^2 dA(z) \\
&\lesssim \|f\|_{\mathcal{D}}^2 .
\end{aligned}$$

Here we used that $g_j \in M_{\mathcal{D}}$ and so $g_j f \in \mathcal{D}$ for all j. We next need to show that the following estimate holds:

$$\int_{\mathbb{D}} \left|\partial b_{jk}\right|^2 dA(z) \lesssim \|f\|_{\mathcal{D}}^2 . \tag{2.6}$$

Once we have established (2.6), then this combined with (2.5) implies that we have

$$\|b_{jk}\|_{\mathcal{W}^{1/2}(\mathbb{T})} \lesssim \|f\|_{\mathcal{D}} .$$

To accomplish this, we consider a related solution operator that moves the singularity outside the disk. Let $\varphi_z(\zeta) = \frac{z-\zeta}{1-\overline{z}\zeta}$ and consider the anti-analytic function

$$\begin{aligned}
\widetilde{b}_{jk}(z) &= -\frac{1}{2\pi i} \int_{\mathbb{D}} \frac{f(\zeta)\varphi_j(\zeta)\overline{\partial}\varphi_k(\zeta)}{\zeta - z} \varphi_z(\zeta)\, dA(\zeta) \\
&= \frac{1}{2\pi i} \int_{\mathbb{D}} \frac{f(\zeta)\varphi_j(\zeta)\overline{\partial}\varphi_k(\zeta)}{1-\overline{z}\zeta} dA(\zeta).
\end{aligned}$$

For $z \in \mathbb{T}$ we have $\varphi_z(\zeta) = \frac{z(1-\overline{z}\zeta)}{1-\overline{z}\zeta} = z$ so that

$$b_{jk}(z) = -\overline{z}\widetilde{b}_{jk}(z), \qquad z \in \mathbb{T}. \tag{2.7}$$

It is easy to see that the operator of multiplication by $\overline{z}$, which is the backward shift operator on Fourier coefficients, is a bounded operator on $\mathcal{W}^{\frac{1}{2}}(\mathbb{T})$. Thus by (2.7) and the fact that $\overline{\widetilde{b}_{jk}(z)}$ is analytic, it suffices to establish that

$$\int_{\mathbb{D}} \left|\overline{\partial}\, \widetilde{b}_{jk}(z)\right|^2 dA(z) \lesssim \int_{\mathbb{D}} \left|f(z)\varphi_j(z)\overline{\partial}\varphi_k(z)\right|^2 dA(z) \tag{2.8}$$

since as we saw above the right hand side of the above expression is bounded by $\|f\|_{\mathcal{D}}^2$. For this we compute

$$\left|\frac{\partial}{\partial\overline{z}}\widetilde{b}_{jk}(z)\right| = \left|\frac{1}{2\pi i}\int_{\mathbb{D}} \frac{f(\zeta)\varphi_j(\zeta)\overline{\partial}\varphi_k(\zeta)}{(1-\overline{z}\zeta)^2}\zeta dA(\zeta)\right| \leq \int_{\mathbb{D}} K(z,\zeta) \left|f(\zeta)\varphi_j(\zeta)\overline{\partial}\varphi_k(\zeta)\right| dA(\zeta),$$

where $K(z,\zeta) = \frac{1}{\pi|1-\overline{z}\zeta|^2}$. Using the estimate

$$\int_{\mathbb{D}} \frac{\left(1-|w|^2\right)^{-\frac{1}{2}}}{|1-\overline{w}z|^2} dA(w) \approx \left(1-|z|^2\right)^{-\frac{1}{2}}$$

we obtain that the "Schur function" $h(\zeta) = \left(1 - |\zeta|^2\right)^{-\frac{1}{2}}$ satisfies

$$\int_{\mathbb{D}} K(z,\zeta)\, h(\zeta)\, dA(\zeta) \lesssim h(z).$$

A simple case of Schur's test now yields (2.8). $\square$

Exercise 2.12. *Give an alternate proof of the estimate* (2.6) *by using the theory of singular integral operators, in particular the Beurling transform.*

Exercise 2.13. *Assuming that (ii) holds in Proposition 2.8 use the Toeplitz Corona Theorem to give an alternate proof of (iii) implies (i).*

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