Lecture 10: Carleson Measures for the Dirichlet Space

We now turn to characterizing the Carleson measures for the Dirichlet space $\mathcal{D}$. Recall that we defined a non-negative measure μ to be $\mathcal{D}$-Carleson if the following embedding holds

$$\int_{\mathbb{D}} |f(z)|^2 \, d\mu(z) \leq C(\mu) \|f\|_{\mathcal{D}}^2 \quad \forall f \in \mathcal{D}.$$

Again we want to characterize these measures in a geometric fashion via some testing conditions. Our tool this time will be to use harmonic analysis to understand the characterization, and the method of proof will be very similar to how we understood the Carleson measures for $H^2(\mathbb{D})$.

This proof will be more at the level of a detailed sketch of the exact argument. Part of the reason for this is that the proof depends heavily on capacity theory and would require a full lecture to develop the correct machinery to handle everything correctly. Instead, we will accept many of the facts needed in capacity theory as just given, and these will be stated in the lecture notes when needed.

It turns out that our method of proof will work to characterize the the Carleson measures for the Besov-Sobolev spaces $B_\sigma^2(\mathbb{D})$ for $0 \leq \sigma < \frac{1}{2}$. Recall that these spaces were defined as the space of holomorphic functions on the disc such that

$$|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{2\sigma} dA(z) := \|f\|_{B_\sigma^2(\mathbb{D})}^2 < \infty.$$

Recall also that we had an equivalent norm on functions $f \in B_\sigma^2(\mathbb{D})$ given by

$$\|f\|_{B_\sigma^2(\mathbb{D})}^2 \approx \sum_{n=0}^{\infty} (n+1)^{1-2\sigma} |a_n|^2 \approx \sum_{n=0}^{\infty} (n^2+1)^{\frac{1}{2}-\sigma} |a_n|^2 .$$

A simple computation shows that these are reproducing kernel Hilbert spaces with the reproducing kernel given by

$$k_\lambda^\sigma(z) = \frac{1}{(1-\overline{\lambda}z)^\sigma}.$$

As above, we define a non-negative measure μ to be $B_\sigma^2(\mathbb{D})$-Carleson if for all $f \in B_\sigma^2(\mathbb{D})$ we have

$$\int_{\mathbb{D}} |f(z)|^2 \, d\mu(z) \leq C(\mu) \|f\|_{B_\sigma^2(\mathbb{D})}^2 .$$

Since $k_\lambda^\sigma \in B_\sigma^2(\mathbb{D})$ we have an obvious necessary condition for a measure μ to be $B_\sigma^2(\mathbb{D})$-Carleson. Indeed, simply testing on the family of reproducing kernels gives the following geometric condition.

Lemma 0.1. *Let $0 \leq \sigma \leq \frac{1}{2}$. Suppose that μ is a $B_\sigma^2(\mathbb{D})$-Carleson measure. Then μ must satisfy*

$$\mu(T(I)) \leq C(\mu) |I|^{2\sigma} .$$

Here

$$T(I) = \{z = re^{i\theta} \in \mathbb{D} : 0 < 1-r \leq |I|, \theta \in I\}.$$

is the "tent" over the interval I.

Proof. We handle the case when $0 < \sigma < \frac{1}{2}$ and leave the case $\sigma = 0$ as an exercise. Testing on the kernel $k_\lambda^\sigma(z)$ we see that the following must hold

$$\int_{\mathbb{D}} \frac{1}{\left|1-\overline{\lambda}z\right|^{4\sigma}} d\mu(z) \leq C(\mu) \frac{1}{(1-|\lambda|^2)^{2\sigma}}$$

So now observe that if $\lambda, z \in T(I)$ then we have that

$$\left|1-\overline{\lambda}z\right| \approx |I|$$

and

$$1-|\lambda|^2 \approx |I|$$

The second estimate clearly follows from the first. While the first follows from obvious geometric estimates (simply draw the picture for $T(I)$ and estimate).

Using this we see that

$$\mu\left(T(I)\right) \leq C(\mu) \left|I\right|^{2\sigma}.$$

□

Exercise 0.2. *Carry out the similar computations necessary to prove that same Lemma for the case of the Dirichlet space $\mathcal{D}$. In this case we have that a necessary condition for a measure μ to be $\mathcal{D}$-Carleson is that for any $I \subset \mathbb{T}$ we have*

$$\mu\left(T(I)\right) \leq C \frac{1}{\log \frac{1}{|I|}}.$$

Note that we have seen this computation and idea before when studying the Carleson measures for $H^2(\mathbb{D})$. However, for the space $H^2(\mathbb{D})$ this condition is also sufficient. Lets give a reason now why this condition can not be sufficient for the space $B_\sigma^2(\mathbb{D})$. Suppose we had an interval I and the corresponding tent $T(I)$. Then we know that the measure must satisfy

$$\mu(T(I)) \leq C \left|I\right|^{2\sigma}.$$

However, if we were to split I into n disjoint pieces I_j then estimating

$$\mu\left(\cup_j T(I_j)\right) \leq \sum_{j=1}^{n} \left|I_j\right|^{2\sigma} \leq n^{1-2\sigma} \left(\sum_{j=1}^{n} \left|I_j\right|\right)^{2\sigma} = n^{1-2\sigma} \left|I\right|^{2\sigma}.$$

These estimates highlight the problem that we need to over come.

Before we state the characterization of Carleson measures for $B_\sigma^2(\mathbb{D})$, we recall some necessary background. For $0 < \sigma < \frac{1}{2}$. Let $k_\sigma(\theta) = |\theta|^{-\frac{1}{2}-\sigma}$ for $|\theta| \leq \pi$ and then extend by periodicity to the whole real line.

Then we have that the Fourier coefficients $\hat{k}_\sigma(n)$ are of the form $a_n(1+n^2)^{-\frac{1}{4}+\frac{\sigma}{2}}$ with $0 \leq \delta \leq a_n < \delta^{-1}$. Recall that we showed that a function $f \in H^2(\mathbb{D})$ was determined by its Taylor (Fourier) coefficients and could be recovered by the harmonic extension of its $L^2(\mathbb{T})$ boundary values. The harmonic extension of a function $f \in L^2(\mathbb{T})$ was given by the Poisson integral of the function,

$$P(f)(z) = \int_{\mathbb{T}} f(\xi) \frac{1-|z|^2}{\left|1-\overline{\xi}z\right|^2} dm(\xi) = \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta}.$$

The last equality follows by a simple computation.

Exercise 0.3. *Show that the Fourier coefficients of k_σ have the claimed form.*

Using this we have an isomorphism between $H^2(\mathbb{D})$ and $B^2_\sigma(\mathbb{D})$ given by sending f the harmonic extension of $f * k_\sigma$

Lemma 0.4. *A function $g \in B^2_\sigma(\mathbb{D})$ for $0 < \sigma < \frac{1}{2}$ if and only if $g = P(f*k_\sigma)$ for $f \in H^2(\mathbb{D})$. Moreover, the $\|g\|_{B^2_\sigma(\mathbb{D})}$ is comparable to $\|f\|_{H^2(\mathbb{D})}$.*

Proof. Let $f = \sum_{n=0}^\infty b_n z^n$. Then we have that

$$P(f * k_\sigma)(z) = \sum_{n=0}^\infty a_n b_n (1+n^2)^{-\frac{1}{4}+\frac{\sigma}{2}}.$$

This computation uses that Fourier coefficients of f are supported only on the non-negative integers and that the Fourier coefficient of the convolution is the product of the Fourier coefficients.

Then, we have

$$\delta^2 \|f\|_{H^2}^2 \leq \sum_{n=0}^\infty (1+n^2)^{\frac{1}{2}-\sigma} \left| a_n b_n (1+n^2)^{-\frac{1}{4}+\sigma} \right|^2 \leq \delta^{-2} \|f\|_{H^2}$$

which shows the isomorphism property. Also note that the middle term is comparable to $\|g\|_{B^2_\sigma(\mathbb{D})}$.

The other direction is also as easy. Given $g \in B^2_\sigma(\mathbb{D})$ with $g(z) = \sum_{n=0}^\infty c_n z^n$ then we simply set

$$f(z) = \sum_{n=0}^\infty \frac{c_n}{a_n}(1+n^2)^{\frac{1}{4}-\sigma} z^n.$$

Then the same computations as above show that $g = P(f * k_\sigma)(z)$ and that $f \in H^2(\mathbb{D})$ with

$$\|f\|_{H^2(\mathbb{D})} \approx \|g\|_{B^2_\sigma(\mathbb{D})}.$$

□

We next give a lemma that is useful in understanding the Carleson measures for $B^2_\sigma(\mathbb{D})$ given this new equivalent norm we have.

Lemma 0.5. *A non-negative measure μ is a $B^2_\sigma(\mathbb{D})$ Carleson measure for $0 < \sigma < \frac{1}{2}$ if and only if*

$$\int_{\mathbb{D}} |P(f * k_\sigma)(z)|^2 \, d\mu(z) \leq C(\mu) \|f\|^2_{L^2(\mathbb{T})}$$

for all $f \in L^2(\mathbb{T})$ such that $f \geq 0$.

Proof. First suppose that the embedding condition holds. Then since the Poisson kernel and the kernel k_σ are both positive we can have the same result (with possibly a worse constant) for all $f \in L^2(\mathbb{T})$. So in particular the embedding condition holds for all $f \in H^2(\mathbb{D})$, i.e, for any $f \in H^2(\mathbb{D})$ we have

$$\int_{\mathbb{D}} |P(f * k_\sigma)(z)|^2 \, d\mu(z) \leq C(\mu) \|f\|^2_{H^2(\mathbb{D})}.$$

However, by the previous Lemma, we have that the integrand recovers all $g \in B^2_\sigma(\mathbb{D})$ and the norm on the right hand side is equivalent to the norm of $g \in B^2_\sigma(\mathbb{D})$. Thus, we have that

$$\int_{\mathbb{D}} |g(z)|^2 \, d\mu(z) \leq C(\mu) \|g\|^2_{B^2_\sigma(\mathbb{D})}$$

and so μ is a $B^2_\sigma(\mathbb{D})$ Carleson measure.

Conversely, suppose that μ is a $B^2_\sigma(\mathbb{D})$ Carleson measure. Again, by the previous lemma, we have that for all $f \in H^2(\mathbb{D})$ that

$$\int_{\mathbb{D}} |P(f * k_\sigma)(z)|^2 \, d\mu(z) \leq C(\mu) \|f\|^2_{H^2(\mathbb{D})}.$$

However, a general function $f \in L^2(\mathbb{T})$ can be written as $f = f_1 + f_2$ with $f_1, \overline{f_2} \in H^2(\mathbb{D})$, $f_2(0) = 0$ and $\|f\|^2_{L^2} = \|f_1\|^2_{H^2} + \|f_2\|^2_{H^2}$. Using this decomposition it is trivial to see that for all $f \in L^2(\mathbb{T})$ we have

$$\int_{\mathbb{D}} |P(f * k_\sigma)(z)|^2 \, d\mu(z) \leq C(\mu) \|f\|^2_{L^2(\mathbb{T})}.$$

So in particular the embedding result holds for all $f \in L^2(\mathbb{T})$ such that $f \geq 0$. □

Based on the observation that the "single-box" condition $\mu(T(I)) \lesssim |I|^{2\sigma}$ fails to characterize the $B^2_\sigma(\mathbb{D})$ Carleson measures, we need a finer geometric method by which we can measure the size of an interval or set in $\mathbb{T}$. This requires us to introduce the notion of capacity.

Let $E \subset \mathbb{T}$ then let $\tau(E)$ denote the identification of the corresponding subset of $[-\pi, \pi]$ on the real line $\mathbb{R}$. For $0 \leq \sigma < \frac{1}{2}$ let

$$\mathrm{Cap}_\sigma(E) = \inf \left\{ \|f\|^2_{L^2(\mathbb{T})} : f \geq 0, k_\sigma * f \geq 1 \text{ on } \tau(E) \right\}.$$

Recalling the harmonic analysis proof of the characterization of $H^p(\mathbb{D})$ Carleson measures, one can anticipate that a lemma of the following type will be necessary to obtain the characterization.

We now give some necessary facts about capacity that we will use in the proof of Lemma 0.6. These facts can be found in the book by Adams and Hedberg, [1]. In the interest of not deviating too far from the intended goal, we will just take these results and facts more or less as a black box. The results being stated below have been translated to the special case at hand. There is a much more general theory of capacities that follows much of the same constructions given below. We additionally are focusing on the case when $p = 2$ since much of the arguments become a little easier. The interested reader can attempt to extend the concepts below to $p \neq 2$.

We are given a kernel $g(x, y)$ on $\mathbb{R} \times \mathbb{R}$ and a measure ν and will now define two potentials that will play a role in what we are studing. Let μ be a non-negative Borel measure, (and we denote the class of all such measures by $\mathcal{M}^+(\mathbb{R})$) and let f be a ν-measurable function. The potentials are then

$$\begin{aligned} \mathcal{G}f(x) &= \int_{\mathbb{R}} g(x, y) f(y) d\nu(y); \\ \check{\mathcal{G}}\mu(y) &= \int_{\mathbb{R}} g(x, y) d\mu(x). \end{aligned}$$

Based on these two potentials, we also can define the mutual energy by

$$\mathcal{E}_g(f, \mu) = \int_{\mathbb{R}} \mathcal{G}f(x) d\mu(x) = \int_{\mathbb{R}} \check{\mathcal{G}}\mu(y) f(y) d\nu(y) = \int_{\mathbb{R}} \int_{\mathbb{R}} g(x, y) f(y) d\mu(x) d\nu(y).$$

For the physics minded, this notion of energy and potential is exactly what appears when studying the potential energy associated to a charge on a object.

As a definition, we take the capacity of a set with respect to the kernel g to be

$$\mathrm{cap}_g(E) = \inf\left\{\|f\|^2_{L^2(\nu)} : \mathcal{G}f(x) \geq 1 \text{ on } E\right\}.$$

It turns out that based just on this definition and then using standard measure theory arguments one can prove the capacity is subadditive and monotone. One can further extend this definition to more general sets (much as in standard measure theory where one first defines Lebesgue measure for "nice" sets and then extends to more general sets). Another important fact is that there is a unique function f^E such that $f \in L^p(\nu)$ and $\mathcal{G}f^E(x) \geq 1$ on E (technically up to a set of capacity zero) and

$$\int_{\mathbb{R}} (f^E)^2 = \mathrm{cap}_g(E).$$

This unique function is usually called the *capacitary function*.

A deeper fact that we will need, is that it is possible to compute the capacity of a set via a dual definition. In particular, for compact sets $K \subset \mathbb{R}$ we will have

$$\mathrm{cap}_g(K)^{\frac{1}{2}} = \sup\left\{\mu(K) : \mu \in \mathcal{M}^+(K), \left\|\check{\mathcal{G}}\mu\right\|_{L^2(\nu)} \leq 1\right\}$$

One direction is trivial, namely it is easy to show that $\mu(K) \leq \mathrm{cap}_g(K)^{\frac{1}{2}}$ just using the definitions and Hölder's Inequality. The other direction then uses von Neumann's minimax theorem applied to the mutual energy functional. Similarly to the existence of the capacitary function, there is a capacitary measure which plays an important role. For any $K \subset \mathbb{R}$ compact we have a measure μ^K such that $f^K = \check{\mathcal{G}}\mu^K$ and

$$\mu^K(K) = \int_{\mathbb{R}} (\check{\mathcal{G}}\mu^K)^2 d\nu = \int_{\mathbb{R}} \mathcal{G}f^K d\mu^K = \mathrm{cap}_g(K).$$

Notice that we are now in the setting where we are considering $g(x,y) = |x-y|^{-\sigma-\frac{1}{2}}$ and $d\nu = dx$.

Lemma 0.6 (Strong Capacitary Inequality)**.** *Let $f \in L^2(\mathbb{T})$ then*

$$2\int_0^\infty \lambda \mathrm{Cap}_\sigma\left(\{\xi \in \mathbb{T} : k_\sigma * f(\xi) > \lambda\}\right) d\lambda \leq C\|f\|^2_{L^2(\mathbb{T})}.$$

This inequality is called the strong capacitary inequality and will play the role of the distribution function that appeared when we were studying Carleson measures for $H^2(\mathbb{D})$.

Proof. Note that it suffices to prove that

$$\sum_{k\in\mathbb{Z}} 2^{2k}\mathrm{Cap}_\sigma\left(\{\xi \in \mathbb{T} : k_\sigma * f(\xi) > 2^k\}\right) \leq C\|f\|^2_{L^2(\mathbb{T})}.$$

Without loss of generality, we may suppose that f is a non-negative, smooth compactly supported function. Set

$$E_k = \{\xi : k_\sigma * f(\xi) \geq 2^k\}$$

The hypotheses on f imply that E_k is compact and empty for k sufficiently large. Define

$$J := \sum_{k\in\mathbb{Z}} 2^{2k}\mathrm{Cap}_\sigma\left(\{\xi \in \mathbb{T} : k_\sigma * f(\xi) > 2^k\}\right)$$

and let μ_k be the extremal measure for E_k which exists by the discussion above. Then we have

$$\begin{aligned} J &= \sum_{k=-\infty}^{\infty} 2^{2k} \int d\mu_k \\ &\leq \sum_{k=-\infty}^{\infty} 2^{k} \int k_\sigma * f d\mu_k \\ &= \sum_{k=-\infty}^{\infty} 2^{k} \int f(k_\sigma * \mu_k) dx \\ &\leq \|f\|_{L^2} \left\| \sum_{k=-\infty}^{\infty} 2^k k_\sigma * \mu_k \right\|_{L^2} . \end{aligned}$$

We will set

$$L := \left\| \sum_{k=-\infty}^{\infty} 2^{2k-1} k_\sigma * \mu_k \right\|_{L^2}^2$$

and show that $L \lesssim J$, which would then give the result of the Lemma.

To see that $L \lesssim J$, first, set

$$\Lambda(x) = \sum_{k=-\infty}^{\infty} 2^k k_\sigma * \mu_k(x)$$

and

$$\Lambda_j(x) = \sum_{k=j}^{\infty} 2^k k_\sigma * \mu_k(x).$$

Clearly we have that $\Lambda_j \in L^2$ (because of the assumptions on f) and $\Lambda_j \to \Lambda$. Now, notice that

$$\begin{aligned} \Lambda(x)^2 &= \lim_{n\to-\infty} \Lambda_n(x)^2 = \lim_{n\to-\infty} \sum_{j=n}^{\infty} \left(\Lambda_j(x)^2 - \Lambda_{j+1}(x)^2\right) \\ &\leq 2 \sum_{j=-\infty}^{\infty} \Lambda_j(x) \left(\Lambda_j(x) - \Lambda_{j-1}(x)\right) \\ &= 2 \sum_{j=-\infty}^{\infty} \Lambda_j(x) 2^j k_\sigma * \mu_j(x). \end{aligned}$$

Using this we see that

$$\begin{aligned} L &\leq 2\int \sum_j 2^j \Lambda_j(x)(k_\sigma * \mu_j(x))dx \\ &= 2\sum_j \sum_{k\geq j} 2^k 2^j \int k_\sigma * \mu_k(x) k_\sigma * \mu_j(x) dx \\ &= 2\sum_j \sum_{k\geq j} 2^k 2^j \int (\check{\mathcal{G}}\mu^K)^2 dx \\ &= \sum_j \sum_{k\geq j} 2^k 2^j \operatorname{Cap}_\sigma(E_k) \\ &= \sum_k 2^{2k} \operatorname{Cap}_\sigma(E_k) = J. \end{aligned}$$

□

There is a minor issue here in that we had defined the capacity we were interested in via the periodization of the kernel k_σ. While, the proof we gave is technically correct for the case when the capacity is defined on the whole real line $\mathbb{R}$, it would require some modifications to work in the case at hand. In the interest of not making these notes to much longer and deviating too far from the goal, we will point the reader to the paper by Stegenga for the necessary changes, [3].

There are more general versions of this result for different potentials, different kernels, and different L^p spaces. The interested reader should consult the books [2] and [1].

With these preliminaries out of the way, we then have the following characterization of the Carleson measures for $B^2_\sigma(\mathbb{D})$ obtained by Stegenga.

Theorem 0.7 (Stegenga, [3]). *Suppose that* $0 \leq \sigma < \frac{1}{2}$ *and let* μ *be a non-negative Borel measure in the disc* $\mathbb{D}$. *Then the following are equivalent:*

(i) *The embedding*

$$B^2_\sigma(\mathbb{D}) \to L^2(\mathbb{D},\mu)$$

is bounded;

(ii) *There exists a constant* $C(\mu)$ *such that for all* $f \in B^2_\sigma(\mathbb{D})$ *we have*

$$\int_{\mathbb{D}} |f(z)|^2 \, d\mu(z) \leq C(\mu) \, \|f\|^2_{B^2_\sigma(\mathbb{D})};$$

(iii) *There is a constant* $C > 0$ *such that for all families* $\{I_j\}$ *of disjoint intervals in* $\mathbb{T}$ *we have*

$$\mu\left(\cup_j T(I_j)\right) \leq C \operatorname{Cap}_\sigma\left(\cup_j I_j\right)$$

where

$$T(I) = \{z = re^{i\theta} \in \mathbb{D} : 0 < 1 - r \leq |I|, \theta \in I\}.$$

Moreover the constants in (ii) *and* (iii) *are comparable.*

In either case we will call such a measure a $B^2_\sigma(\mathbb{D})$-Carleson measure. It is important to note that in this case that the reproducing kernel thesis does not hold. Namely, there exist measures μ such that (iii) holds for a single interval (as opposed to the family), but that (i) and (ii) fail.

Another way to view condition (iii) is that for any open set $\Omega \subset \mathbb{T}$ we have that

$$\mu\left(T(\Omega)\right) \leq C\,\mathrm{Cap}_\sigma\left(\Omega\right).$$

Proof. As we have seen, the equivalence between (i) and (ii) is immediate. We now focus on showing that (ii) and (iii) are equivalent. So, first suppose that μ is a $B^2_\sigma(\mathbb{D})$ Carleson measure. Let $E = \cup_{j=1}^n I_j$. Now select a test function for the capacity, namely competitor $f \in L^2(\mathbb{T})$ for the infimum such that $f \geq 0$ and $f * k_\sigma \geq 1$ on $\tau(E)$. Now, a simple computation shows that $P(\chi_{I_j})(z) \geq \frac{1}{4}$ for $z \in T(I_j)$. So, we further have that

$$P(f * k_\sigma)(z) \geq \frac{1}{4} \quad \forall z \in \bigcup_{j=1}^n T(I_j).$$

Thus, by Lemma 0.5 we have that

$$\begin{aligned}\mu\left(\bigcup_{j=1}^n T(I_j)\right) &\lesssim \int_{\mathbb{D}} \left|P(f * k_\sigma)(z)\right|^2 d\mu(z) \\ &\lesssim \|f\|^2_{L^2(\mathbb{T})}\end{aligned}$$

Since f was arbitrary, this implies that for any competitor for the infimum that we have

$$\mu\left(\bigcup_{j=1}^n T(I_j)\right) \lesssim \|f\|^2_{L^2(\mathbb{T})}.$$

So in particular we have that

$$\mu\left(\bigcup_{j=1}^n T(I_j)\right) \lesssim \mathrm{Cap}_\sigma\left(\cup_j I_j\right).$$

Conversely, suppose that for all disjoint collections of intervals $I_j \subset \mathbb{T}$ that we have

$$\mu\left(\bigcup_{j=1}^n T(I_j)\right) \lesssim \mathrm{Cap}_\sigma\left(\cup_j I_j\right).$$

We need to show that for all $f \in B^2_\sigma(\mathbb{D})$ that we have

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \lesssim \|f\|^2_{B^2_\sigma(\mathbb{D})}.$$

Equivalently, via Lemma 0.5 it suffices to show that

$$\int_{\mathbb{D}} \left|P(f * k_\sigma)(z)\right|^2 d\mu(z) \lesssim \|f\|^2_{L^2(\mathbb{T})}.$$

Set $u(z) = P(f * k_\sigma)(z)$. Recall that we defined the nontangential maximal function by

$$u^*(\xi) = \sup_{z \in \Gamma_a(\xi)} |u(z)|$$

Now, observe the following. First, the set

$$\{t : u^*(t) > \lambda\}$$

is an open set, and so is a disjoint union of open intervals $\{I_j\}$ with centers c_j. Let $T(I_j)$ denote the corresponding tent over I_j. By analogous computations from before, we have that if it is the case that $|u(z)| > \lambda$, then on the a sub-interval of I_j depending only on the information from I_j that $u^*(t) > \lambda$.

So by the computations above, it is the case that

$$\{z \in \mathbb{D} : |u(z)| > \lambda\} \subset \bigcup_j T(I_j)$$

Thus, we have

$$\begin{aligned} \mu\left(\{z \in \mathbb{D} : |u(z)| > \lambda\}\right) &\leq \mu\left(\bigcup_{j=1}^n T(I_j)\right) \\ &\leq C(\mu)\,\mathrm{Cap}_\sigma\left(\bigcup_{j=1}^n I_j\right) \\ &= C(\mu)\,\mathrm{Cap}_\sigma\left(\{t \in \mathbb{R} : u^*(t) > \lambda\}\right). \end{aligned}$$

We now apply all these observations with $u = P(f * k_\sigma)$. Doing so we see that

$$\begin{aligned} \int_{\mathbb{D}} |P(f * k_\sigma)(z)|^2\, d\mu(z) &= 2\int_0^\infty \lambda\mu\left(\{z : |P(f * k_\sigma)(z)| > \lambda\}\right) d\lambda \\ &\lesssim \int_0^\infty \lambda\,\mathrm{Cap}_\sigma\left(\{P(f * k_\sigma)^*(\xi) > \lambda\}\right) d\lambda \\ &\lesssim \int_0^\infty \lambda\,\mathrm{Cap}_\sigma\left(\{M(f * k_\sigma)(\xi) > \lambda\}\right) d\lambda \\ &\lesssim \int_0^\infty \lambda\,\mathrm{Cap}_\sigma\left(\{Mf * k_\sigma(\xi) > \lambda\}\right) d\lambda \\ &\lesssim \|Mf\|_{L^2(\mathbb{T})}^2 \\ &\lesssim \|f\|_{L^2(\mathbb{T})}^2. \end{aligned}$$

Here in the second to last inequality we used Lemma 0.6. Then applying Lemma 0.5 gives that μ is a $B^2_\sigma(\mathbb{D})$ Carleson measure. □

Stegenga further goes to show that the one-box condition is not sufficient for the characterization of Carleson measures.

Theorem 0.8 (Stegenga, [3])**.** *There exists a function $f \in H^\infty(\mathbb{D})$ such that*

$$\int_{T(I)} |f'(z)|^2\, dA(z) \lesssim \frac{1}{\log \frac{1}{|I|}}$$

*but is **not** a multiplier of $\mathcal{D}$.*

Exercise 0.9. *Look up the counterexample constructed by Stegenga and work through it.*

We can combine the observations from this lecture with the previous lecture to obtain a more geometric characterization of the multipliers for the space $B^2_\sigma(\mathbb{D})$.

Theorem 0.10 (Stegenga, [3])**.** *The following are equivalent*

(i) *The function $f \in \mathcal{M}_{B^2_\sigma(\mathbb{D})}$;*

(ii) *The function* $f \in H^\infty(\mathbb{D})$ *and*

$$\int_{\cup_{j=1}^n I_j} |f'(z)|^2 (1-|z|^2)^{2\sigma} dA(z) \lesssim \mathrm{Cap}_\sigma\left(\bigcup_{j=1}^n I_j\right)$$

for all disjoint collections of intervals $\{I_j\} \subset \mathbb{T}$.

References

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[2] V. G. Maz′ja and T. O. Šapošnikova, *Theory of multipliers in spaces of differentiable functions and their applications*, Theory of cubature formulas and numerical mathematics (Proc. Conf., Novosibirsk, 1978) (Russian), "Nauka" Sibirsk. Otdel., Novosibirsk, 1980, pp. 225–233, 255 (Russian). ↑7

[3] David A. Stegenga, *Multipliers of the Dirichlet space*, Illinois J. Math. **24** (1980), no. 1, 113–139. ↑7, 9