Lecture 1: The Hardy Space on the Disc

In this first lecture we will focus on the Hardy space $H^2(\mathbb{D})$. We will have a "crash course" on the necessary theory for the Hardy space. Part of the reason for first introducing this space before the Dirichlet space, is that many of the ideas and results from this space serve as motivation and guide us when studying other spaces of holomorphic functions.

The following texts were instrumental in preparing the lectures on this. The interested student can consult them for more information.

[1] Jim Agler and John E. McCarthy, *Pick interpolation and Hilbert function spaces*, Graduate Studies in Mathematics, vol. 44, American Mathematical Society, Providence, RI, 2002.

[2] John B. Garnett, *Bounded analytic functions*, 1st ed., Graduate Texts in Mathematics, vol. 236, Springer, New York, 2007.

[3] Nikolai K. Nikolski, *Operators, functions, and systems: an easy reading. Vol. 1*, Mathematical Surveys and Monographs, vol. 92, American Mathematical Society, Providence, RI, 2002. Hardy, Hankel, and Toeplitz; Translated from the French by Andreas Hartmann.

[4] N. K. Nikol′skiĭ, *Treatise on the shift operator*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 273, Springer-Verlag, Berlin, 1986. Spectral function theory; With an appendix by S. V. Hruščev [S. V. Khrushchëv] and V. V. Peller; Translated from the Russian by Jaak Peetre.

[5] Eric T. Sawyer, *Function theory: interpolation and corona problems*, Fields Institute Monographs, vol. 25, American Mathematical Society, Providence, RI, 2009.

[6] Kristian Seip, *Interpolation and sampling in spaces of analytic functions*, University Lecture Series, vol. 33, American Mathematical Society, Providence, RI, 2004.

[7] Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy; Monographs in Harmonic Analysis, III.

1. Basic Definitions

We now introduce the space $H^2(\mathbb{D})$. Let $f \in \mathrm{Hol}(\mathbb{D})$, then we say that $f \in H^2(\mathbb{D})$ if

$$\sup_{0<r<1} \int_{\mathbb{T}} |f(re^{i\theta})|^2 dm(\theta) := \|f\|^2_{H^2(\mathbb{D})} < \infty. \tag{1.1}$$

A related space that will play a distinguished role in our space is the Hardy space $H^\infty(\mathbb{D})$

$$\sup_{z\in\mathbb{D}} |f(z)| := \|f\|_{H^\infty(\mathbb{D})} < \infty. \tag{1.2}$$

We will see that with the norms we have introduced, the space $H^2(\mathbb{D})$ is a Hilbert space, while the space $H^\infty(\mathbb{D})$ is a Banach space.

Exercise 1.1. *Show that it is possible to replace the* $\sup_{0<r<1}$ *by* $\lim_{r\to 1}$ *in the definition of* $H^2(\mathbb{D})$.

We now show other norms that can be used to study the functions in $H^2(\mathbb{D})$. First, recall that the Fourier transform of a function $f \in L^2(\mathbb{T})$ is given by

$$\hat{f}(n) = \int_{\mathbb{T}} f(e^{i\theta}) e^{-in\theta} dm(\theta).$$

Then, a simple computation shows that

$$\int_{\mathbb{T}} e^{i(n-m)\theta} dm(\theta) = \begin{cases} 1 & : & n = m \\ 0 & : & n \neq m \end{cases}$$

Using this, we see that for $f(z)=\sum_{n=0}^\infty a_n z^n$ that

$$\begin{aligned}\|f\|_{H^2(\mathbb{D})}^2 &= \sup_{0<r<1}\int_{\mathbb{T}} |f(re^{i\theta})|^2 dm(\theta)\\ &= \sup_{0<r<1}\int_{\mathbb{T}} \left|\sum_{n=0}^\infty a_n r^n e^{in\theta}\right|^2 dm(\theta)\\ &= \sup_{0<r<1}\sum_{n,m=0}^\infty a_n\overline{a_m} r^n r^m \int_{\mathbb{T}} e^{i(n-m)\theta} dm(\theta)\\ &= \sum_{n=0}^\infty |a_n|^2 = \|f\|_{H^2(\mathbb{D})}^2 .\end{aligned}$$

Note that this norm says that it is possible to study the behavior of the functions in $H^2(\mathbb{D})$ via their Fourier coefficients.

For $0<r<1$ and $z\in\mathbb{D}$ let $f_r(z)=f(rz)$. Then the computations done above, prove that the following proposition.

Proposition 1.2. *Suppose that $f\in H^2(\mathbb{D})$. Then, the sequence $\{f_r\}$ is Cauchy in $L^2(\mathbb{T})$.*

Proof. Using the computations from above, and obvious modifications, we see

$$\begin{aligned}\|f_r-f_s\|_{L^2(\mathbb{T})}^2 &= \int_{\mathbb{T}} \left|\sum_{n=0}^\infty a_n(r^n-s^n)e^{in\theta}\right|^2 dm(\theta)\\ &= \sum_{n=0}^\infty |(r^n-s^n)|^2|a_n|^2.\end{aligned}$$

But, as $r,s\to 1$ and the dominated convergence theorem, since $\sum_{n=0}^\infty |a_n|^2<\infty$, we can conclude that this last summand goes to zero. □

Now note that since $L^2(\mathbb{T})$ is a complete space, then we have an element $f^*\in L^2(\mathbb{T})$ given by $f^*=\lim_{r\to 1} f_r$ also in $L^2(\mathbb{T})$. Since $f^*\in L^2(\mathbb{T})$ we can compute the Fourier coefficients to be

$$\begin{aligned}\widehat{f^*}(n) &= \int_{\mathbb{T}} f^*(e^{i\theta})e^{-in\theta} dm(\theta)\\ &= \lim_{r\to 1}\int_{\mathbb{T}} f_r(e^{i\theta})e^{-in\theta} dm(\theta)\\ &= \left\{\begin{array}{ccc} a_n & : & n\geq 0\\ 0 & : & n<0\end{array}\right. .\end{aligned}$$

Note that the computations we have done thus far proves the following proposition.

Proposition 1.3. *Suppose that $f\in H^2(\mathbb{D})$ and $f^*(e^{i\theta})=\lim_{r\to 1} f(re^{i\theta})$ then*

$$\|f\|_{H^2(\mathbb{D})}^2=\sum_{n=0}^\infty |a_n|^2=\|f^*\|_{L^2(\mathbb{T})}^2 .$$

The only fact that remains to complete the proof of this proposition is that

$$\sum_{n=0}^{\infty} |a_n|^2 = \|f^*\|_{L^2(\mathbb{T})}^2$$

which holds by Parseval's Theorem.

This also shows that the inner product on $H^2(\mathbb{D})$ will satisfy

$$\langle f, g\rangle_{H^2(\mathbb{D})} = \int_{\mathbb{T}} f^*(e^{i\theta})\overline{g^*(e^{i\theta})}dm(\theta) = \sum_{n=0}^{\infty} a_n\overline{b_n}$$

where we have $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$.

Exercise 1.4. *Let H_0^2 denote the collection of functions in $H^2(\mathbb{D})$ such that $f(0) = 0$. Show that $L^2(\mathbb{T}) = H^2(\mathbb{D}) \oplus \overline{H_0^2(\mathbb{D})}$. Hint: This follows easily from the Fourier coefficients.*

1.1. **The Reproducing Kernel for $H^2(\mathbb{D})$.** The Hardy space $H^2(\mathbb{D})$ also has an additional property of being a reproducing kernel Hilbert space. This means that for each point $z \in \mathbb{D}$ there is a special function $k_z \in \mathbb{D}$ such that

$$\langle f, k_z\rangle_{H^2(\mathbb{D})} = f(z)$$

We now turn to determining what this function is.

Proposition 1.5. *Let $z \in \mathbb{D}$, then*

$$|f(z)| \leq \|f\|_{H^2(\mathbb{D})} \frac{1}{\sqrt{1-|z|^2}}.$$

Proof. Note that we have via taking limits in Cauchy's formula that

$$\begin{aligned} f(z) &= \lim_{r\to 1} f_r(z) \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f^*(w)}{w-z}dw \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f^*(e^{i\theta})}{e^{i\theta}-z} ie^{i\theta}d\theta \\ &= \int_{\mathbb{T}} \frac{f^*(e^{i\theta})}{1-ze^{-i\theta}}dm(\theta). \end{aligned}$$

This then implies by an application of Cauchy-Schwarz and then computing that

$$\int_{\mathbb{T}} \frac{1}{|1-ze^{-i\theta}|^2}dm(\theta) = \frac{1}{1-|z|^2}.$$

$$|f(z)| \leq \|f^*\|_{L^2(\mathbb{T})} \frac{1}{\sqrt{1-|z|^2}} = \|f\|_{H^2(\mathbb{D})} \frac{1}{\sqrt{1-|z|^2}}.$$

□

With this proposition proved, we see that pointwise evaluation of functions $f \in H^2(\mathbb{D})$ is a bounded operator. So, by the Riesz representation theorem, we know that there is a *unique* function $k_z \in H^2(\mathbb{D})$ such that

$$f(z) = \langle f, k_z\rangle_{H^2(\mathbb{D})}.$$

We now turn to determining what this function in fact is. Observe that the following property holds since the functions $k_z \in H^2(\mathbb{D})$

$$k_\lambda(\xi) = \langle k_\lambda, k_\xi \rangle_{H^2(\mathbb{D})}.$$

We now collect a general fact about reproducing kernel Hilbert spaces. Good references for the facts presented here are [1–3]. These are Hilbert spaces $\mathcal{H}$ of functions over some domain X. For each point $\lambda \in X$ we have that point evaluation of the functions $f \in \mathcal{H}$ is a continuous operation. Therefore, we have a special function $k_\lambda \in \mathcal{H}$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle \quad \forall f \in \mathcal{H}.$$

This vector k_λ is called the reproducing kernel for the space $\mathcal{H}$. We also have the following property holding:

$$k_\lambda(\xi) = k(\lambda, \xi) = \langle k_\lambda, k_\xi \rangle.$$

This function is called the kernel function for the Hilbert space $\mathcal{H}$.

A simple fact is then the following.

Proposition 1.6. *Let $\mathcal{H}$ be a Hilbert function space on X and let $\{e_i\}$ be an orthonormal basis for $\mathcal{H}$. Then*

$$k_\lambda(\xi) = \sum_i \overline{e_i(\lambda)} e_i(\xi)$$

The proof of this fact is just an application of Parseval's Identity in a general Hilbert space, and two applications of the reproducing kernel property for $\mathcal{H}$. When we specialize to the case $H^2(\mathbb{D})$, then one can show that the functions $\{z^k\}$ are an orthonormal system of functions in $H^2(\mathbb{D})$. Thus, appropriately normalizing we have that

$$k_\lambda(\xi) = \sum_{k=0}^{\infty} \overline{\lambda}^k \xi^k.$$

Using this formula, we see that the function

$$k_z(w) = \frac{1}{1 - \overline{z}w}$$

is the reproducing kernel for the Hardy space $H^2(\mathbb{D})$.

Exercise 1.7. *Determine the norm of the function $k_z \in H^2(\mathbb{D})$.*

Exercise 1.8. *Using Proposition 1.5 determine the reproducing kernel for $H^2(\mathbb{D})$.*

Exercise 1.9. *Show that*

$$\int_{\mathbb{T}} \frac{1}{|1 - ze^{-i\theta}|^2} dm(\theta) = \frac{1}{1 - |z|^2}.$$

Exercise 1.10. *Let $A^2(\mathbb{D})$ denote the Bergman space of functions,*

$$A^2(\mathbb{D}) := \left\{ f \in \operatorname{Hol}(\mathbb{D}) : \|f\|^2_{A^2(\mathbb{D})} := \int_{\mathbb{D}} |f(z)|^2 \, dA(z) < \infty \right\}.$$

Show that the reproducing kernel for the Bergman space is given by

$$k_\lambda(z) = \frac{1}{(1 - \overline{\lambda} z)^2}.$$

2. Littlewood-Paley Identities and $H^2(\mathbb{D})$

We now show how it is possible to obtain another norm on $H^2(\mathbb{D})$ using information about the function on the disc $\mathbb{D}$. This equivalent norm will prove useful when we study the space of Carleson measures for $H^2(\mathbb{D})$ since it will allow us to generate a natural family of examples of functions which generate Carleson measures. Also, this new norm will allow us to place the Hardy space in a scale of Besov-Sobolev spaces.

First, we begin by recalling Green's formula in the case of the unit disc $\mathbb{D}$ and its boundary $\mathbb{T}$. Then Green's formula takes the form:

$$\int_{\mathbb{T}} u(\xi)dm(\xi) - u(0) = \int_{\mathbb{D}} \Delta u(z) \log \frac{1}{|z|} dA(z)$$

Note that we can move the point 0 to any other point $z \in \mathbb{D}$ by a Möbius map of the form $\varphi_z(w) = \frac{w-z}{1-\overline{z}w}$.

Exercise 2.1. *Work out Green's formula for the point $z \in \mathbb{D}$.*

We will begin with a function $g \in L^1(\mathbb{T})$ and, as usual, let $g(z)$ denote the Poisson extension of the function g. The gradient of a function g is given by $\nabla g = (\partial_x g, \partial_y g)$ and we have

$$|\nabla g(z)|^2 = |\partial_x g(x,y)|^2 + |\partial_y g(x,y)|^2 .$$

In the case when g happens to be an analytic function we have that

$$|\nabla g(z)|^2 = |\partial g(z)|^2 = |g'(z)|^2 .$$

Lemma 2.2 (Littlewood-Paley Identity)**.** *Suppose that $g \in L^1(\mathbb{T})$ and if $g(0) = \int_{\mathbb{T}} g dm$ then*

$$2\int_{\mathbb{D}} |\nabla g(z)|^2 \log \frac{1}{|z|} dA(z) = \int_{\mathbb{T}} |g - g(0)|^2 \, dm = \int_{\mathbb{T}} |g|^2 \, dm - |g(0)|^2 .$$

Proof. With out loss of generality we may assume that $g(0) = 0$, since we can reduce to this case by considering the function $\tilde{g} = g - g(0)$. We will apply Green's Theorem with the function $u = |g|^2$. Since $g(0) = 0$ we have that $u(0) = |g(0)|^2 = 0$. Now observe that

$$\begin{aligned}
\partial\overline{\partial} |g(z)|^2 &= \partial \left(\overline{\partial} g \overline{g} + g \overline{\partial}\overline{g}\right) \\
&= \partial\overline{\partial} g \overline{g} + g \partial\overline{\partial}\overline{g} + \overline{\partial} g \partial \overline{g} + \partial g \overline{\partial}\overline{g} \\
&= \overline{\partial} g \partial \overline{g} + \partial g \overline{\partial}\overline{g} = |\partial g|^2 + \left|\overline{\partial} g\right|^2 \\
&= \frac{1}{2} |\nabla g(z)|^2 .
\end{aligned}$$

Here the last equality follows from the definitions of the operators ∂ and $\overline{\partial}$. Using this we see that

$$\Delta |g(z)|^2 = 2 |\nabla g(z)|^2 .$$

Substituting into Green's formula we have

$$\int_{\mathbb{T}} |g(\xi)|^2 \, dm = \int_{\mathbb{D}} \Delta(|g(z)|^2) \log \frac{1}{|z|} dA(z) = 2 \int_{\mathbb{D}} |\nabla g(z)|^2 \log \frac{1}{|z|} dA(z)$$

proving the Lemma. □

Using this lemma, we have another way to compute the norm of a function in $H^2(\mathbb{D})$.

Proposition 2.3. *Suppose that $g \in H^2(\mathbb{D})$ then we have*

$$\|g\|_{H^2(\mathbb{D})}^2 = |g(0)|^2 + 2\int_{\mathbb{D}} |g'(z)|^2 \log\frac{1}{|z|} dA(z).$$

The proof of this follows by simple rearrangement of the above Lemma.

We give a slightly different way to see the resulting norm that in some cases is easier to use. More importantly for us, it will allow us to place the Hardy space in a scale of analytic function spaces that are very interesting.

Lemma 2.4. *If $g \in L^1(\mathbb{T})$ then*

$$\int_{\mathbb{D}} |\nabla g(z)|^2 (1-|z|^2) dA(z) \leq 2\int_{\mathbb{D}} |\nabla g(z)|^2 \log\frac{1}{|z|} dA(z) \leq C\int_{\mathbb{D}} |\nabla g(z)|^2 (1-|z|^2) dA(z).$$

Proof. First note that $1-t \leq 2\log\frac{1}{t}$ if $0 \leq t < 1$. So we have that

$$\int_{\mathbb{D}} |\nabla g(z)|^2 (1-|z|^2) dA(z) \leq 2\int_{\mathbb{D}} |\nabla g(z)|^2 \log\frac{1}{|z|} dA(z).$$

To prove the alternate inequality, first, suppose that the integral on the right hand side is finite and then normalize it so that

$$\int_{\mathbb{D}} |\nabla g(z)|^2 (1-|z|^2) dA(z) = 1.$$

Now, if $|z| > \frac{1}{4}$ then we have that $\log\frac{1}{|z|} \leq C(1-|z|^2)$, and so we then have that

$$\int_{\frac{1}{4}\leq|z|\leq 1} |\nabla g(z)|^2 \log\frac{1}{|z|} dA(z) \leq C\int_{\mathbb{D}} |\nabla g(z)|^2 (1-|z|^2) dA(z).$$

In the case when $|z| < \frac{1}{4}$ we exploit the subharmonicity of $|\nabla g(z)|$. By subharmonicity we have that

$$\begin{aligned} |\nabla g(z)|^2 &\leq 16\int_{|\xi - z|<\frac{1}{4}} |\nabla g(\xi)|^2 dA(\xi) \\ &\leq 32\int_{|\xi|<\frac{1}{2}} |\nabla g(\xi)|^2 (1-|\xi|^2) dA(\xi) = 32 \end{aligned}$$

For the last inequality, we have used that for $|z| < \frac{1}{4}$ and $|\xi - z| < \frac{1}{4}$ that $|\xi| < \frac{1}{2}$. We then use the fact that when $|\xi| < \frac{1}{2}$ that $1 - |\xi|^2 \geq \frac{3}{4} \geq \frac{1}{2}$. Using this, we see that

$$\int_{|z|<\frac{1}{4}} |\nabla g(z)|^2 \log\frac{1}{|z|} dA(z) \leq C\int_{|z|<\frac{1}{4}} \log\frac{1}{|z|} dA(z) = C.$$

Combining the estimates we have obtained when $|z| \geq \frac{1}{4}$ and when $|z| \leq \frac{1}{4}$ then gives that

$$2\int_{\mathbb{D}} |\nabla g(z)|^2 \log\frac{1}{|z|} dA(z) \leq C\int_{\mathbb{D}} |\nabla g(z)|^2 (1-|z|^2) dA(z).$$

□

Again, by rearrangement of the above Lemma we have another equivalent norm on the space $H^2(\mathbb{D})$

$$|g(0)|^2 + \int_{\mathbb{D}} |g'(z)|^2 (1-|z|^2) dA(z) \leq \|g\|_{H^2(\mathbb{D})}^2 \leq C\left(|g(0)|^2 + \int_{\mathbb{D}} |g'(z)|^2 (1-|z|^2) dA(z)\right).$$

Exercise 2.5. *Give an alternate proof of the above equivalent norm on $H^2(\mathbb{D})$ using Fourier series. Doing this, you can obtain a better (in fact sharp) estimate of the constant C.*

2.1. **Besov-Sobolev Spaces.** We now (briefly) introduce the Besov-Sobolev spaces on the unit disc $\mathbb{D}$. We fix a parameter $0 \leq \sigma \leq \frac{1}{2}$ and define the Besov-Sobolev space $B^2_\sigma(\mathbb{D})$ as the collection of analytic functions on the disc such that

$$\|f\|^2_{B^2_\sigma(\mathbb{D})} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^{2\sigma} dA(z) < \infty.$$

Based on the Lemmas above, we have that when $\sigma = \frac{1}{2}$ that $H^2(\mathbb{D}) = B^2_{\frac{1}{2}}(\mathbb{D})$, with equivalent norms. When $\sigma = 0$, then we are looking at the functions that are analytic on $\mathbb{D}$ and such that its derivative is square integrable. This is nothing other than the Dirichlet space which we will study later on.

Exercise 2.6. *Show that an equivalent norm on the space $B^2_\sigma(\mathbb{D})$ is given by*

$$\sum_{n=0}^{\infty} n^{1-2\sigma} |a_n|^2$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

References

[1] Jim Agler and John E. McCarthy, *Pick interpolation and Hilbert function spaces*, Graduate Studies in Mathematics, vol. 44, American Mathematical Society, Providence, RI, 2002.

[2] Eric T. Sawyer, *Function theory: interpolation and corona problems*, Fields Institute Monographs, vol. 25, American Mathematical Society, Providence, RI, 2009.

[3] Kristian Seip, *Interpolation and sampling in spaces of analytic functions*, University Lecture Series, vol. 33, American Mathematical Society, Providence, RI, 2004.