RECTIFIABILITY OF MEASURES WITH BOUNDED RIESZ TRANSFORM OPERATOR: FROM SINGULAR OPERATORS TO GEOMETRIC MEASURE THEORY

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1. Lecture 8: The estimate from below for each flat layer contribution and the end of the proof of David–Semmes conjecture in co-dimension 1

Recall that we claimed to have built the following vector function ψ with the following properties, where $(R^H)^*(\psi m) = \langle R^H, \psi m\rangle = \eta$ and all claims below are satisfied ($m := m_{d+1}$):

- $\psi = \sum_{P'\in\mathfrak{P}'} \psi_{P'}$, $\text{supp}\psi \subset S$, $\text{dist}(\text{supp}\psi, L) \geqslant \Delta\ell(Q) = \varepsilon^3\ell(Q)$.
- $\psi_{P'}$ is supported in the $2\ell(P')$-neighborhood of P' and satisfies

$$\int \psi_{P'} = 0, \quad \|\psi_{P'}\|_{L^\infty} \leqslant \frac{C}{\delta\ell(P')}, \quad \|\psi_{P'}\|_{\text{Lip}} \leqslant \frac{C}{\delta^2\ell(P')^2}\,.$$

- $\int |\psi|\, dm \leqslant C\delta^{-1}\mu(Q)$.
- $(R^H)^*(\psi m) = \eta$.
- $\|T^*(\psi m)\|_{L^\infty(\text{supp}\nu)} \leqslant C\alpha\delta^{-2}\varepsilon^{-3d-3}$.
- $\|\widetilde{R}^H(|\psi| m)\|_{L^2(\nu)} \leqslant C\delta^{-1}\sqrt{\mu(Q)}\,$.

Let us prove some of these claims. First,

$$\int |\psi|\, dm = \sum_{P'\in\mathfrak{P}'} \int |\psi_{P'}|\, dm \leqslant C \sum_{P'\in\mathfrak{P}'} [\delta\ell(P')]^{-1} m(B(z_{P'}, 6\ell(P')))$$
$$\leqslant C\delta^{-1} \sum_{P'\in\mathfrak{P}'} \ell(P')^d \leqslant C\delta^{-1} \sum_{P'\in\mathfrak{P}'} \mu(P') \leqslant C\delta^{-1}\mu(Q)\,.$$

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To get the uniform estimate for $T^*(\psi m)$, note that

$$|[T^*(\psi_{P'} m)](x)| = \left|\int \langle R^H(x^*-\cdot), \psi_{P'}\rangle\, dm\right| \leqslant C\delta^{-1}\|R^H(x^*-\cdot)\|_{\mathrm{Lip}(S)}$$
$$\cdot\,\ell(P')^{d+1} \leqslant C\delta^{-1}\Delta^{-d-1}\frac{\ell(P')^{d+1}}{\ell(Q)^{d+1}} \leqslant C\alpha\delta^{-2}\Delta^{-d-1}\frac{\mu(P')}{\mu(Q)}$$

for every $x\in \mathrm{supp}\nu$ (we remind the reader that $\ell(P')\leqslant 2\alpha\delta^{-1}\ell(Q)$). Adding up and recalling our choice $\Delta=\varepsilon^3$:

$$\|T^*\psi\|_{L^\infty(\mathrm{supp}\nu)} \leqslant C\alpha\delta^{-2}\varepsilon^{-3d-3}\sum_{P'\in\mathfrak{P}'}\frac{\mu(P')}{\mu(Q)} \leqslant C\alpha\delta^{-2}\varepsilon^{-3d-3}\,.$$

The bound of $\|\widetilde{R}^H(|\psi|\,dm)\|_{L^2(\nu)}$. First we estimate $\|\widetilde{R}^H(|\psi|\,dm)\|_{L^2(\mu_Q)}$. And then use our transfer estimates modifying the measure μ to ν as it has been already done many times before.

Recall that for every $P'\in\mathfrak{P}'$, we have $\int|\psi_{P'}|\,dm\leqslant C\delta^{-1}\ell(P')^d$. Hence, we can choose constants $b_{P'}\in(0,C\delta^{-1})$ so that $|\psi_{P'}|m - b_{P'}\chi_{P'}\mu$ is a balanced signed measure, i.e.,

$$\int|\psi_{P'}|\,dm = b_{P'}\int\chi_{P'}\,d\mu\,.$$

Let

$$f=\sum_{P'\in\mathfrak{P}'} b_{P'}\chi_{P'}\,.$$

Our goal is to first prove a pointwise estimate

$$|\widetilde{R}^H(|\psi|m)| \leqslant C\delta^{-1} + |\widetilde{R}^H(f\mu)| + \sum_{P'\in\mathfrak{P}'}\chi_{V(P')}|\widetilde{R}^H(b_{P'}\chi_{P'}\mu)|\,, \tag{1}$$

where for each $P'\in\mathfrak{P}'$, denote by $V(P')$ the set of all points $x\in\mathbb{R}^{d+1}$ such that $\mathrm{dist}(x,P')\leqslant\mathrm{dist}(x,P'')$ for all $P''\in\mathfrak{P}'$.

This estimate of $|\widetilde{R}^H(|\psi|m)|$ above now gives

$$\|\widetilde{R}^H(|\psi|m)\|^2_{L^2(\mu_Q)} \leqslant$$
$$\leqslant C\left[\delta^{-2}\mu(Q) + \|f\|^2_{L^2(\mu)} + \sum_{P'\in\mathfrak{P}'}\|b_{P'}\chi_{P'}\|^2_{L^2(\mu)}\right] \leqslant C\delta^{-2}\mu(Q)\,,$$

which we wanted. To get the pointwise estimate (1) we write for $x\in V(P')$:

$$(2)\quad [\widetilde{R}^H(|\psi|m - f\mu)](x) = [\widetilde{R}^H(|\psi_{P'}|m)](x) - [\widetilde{R}^H(b_{P'}\chi_{P'}\mu)](x) + \sum_{P''\in\mathfrak{P}', P''\neq P'} [\widetilde{R}^H(|\psi_{P''}|m - b_{P''}\chi_{P''}\mu)](x)\,.$$

If $x \in V(P')$ and cells are Vitali disjoint, then $\mathrm{dist}(x, P'') \geqslant c\ell(P'')$ and so

$$\Big|R^H(|\psi_{P''}|m - b_{P''}\chi_{P''}\mu)](x)\Big| = \Big|\int K^H(x-\cdot)\,d(|\psi_{P''}|m - b_{P''}\chi_{P''}\mu)\Big|$$
$$= \Big|\int [K^H(x-\cdot) - K^H(x - z_{P''})]\,d(|\psi_{P''}|m - b_{P''}\chi_{P''}\mu)\Big|$$
$$\leqslant 2\|K^H(x-\cdot) - K^H(x-z_{P''})\|_{L^\infty(P'')}\int |\psi_{P''}|\,dm$$
$$\leqslant \frac{C\ell(P'')}{\mathrm{dist}(x,P'')^{d+1}}\delta^{-1}\ell(P'')^d \leqslant C\delta^{-1}\left[\frac{\ell(P'')}{\ell(P'') + \mathrm{dist}(x,P'')}\right]^{d+1},$$

and the same for $R^H(x^* - y)$. Hence all this huge sum in (2) is $\leqslant \delta^{-1}h(x) \leqslant C/\delta$ by the Marcinkiewicz choice of $\mathfrak{P}'$, see Lecture 7 under the title "**A collection of P's (inside Q) of non-BAUP layer $\mathfrak{P}_{k+1}$.**"

Note also that

$$\|\widetilde{R}^H(|\psi_{P'}|m)\|_{L^\infty} \leqslant C\delta^{-1}$$

(this is just the trivial bound $C\ell(P')$ for the integral of the absolute value of the kernel over a set of diameter $12\ell(P')$ multiplied by the bound $\frac{C}{\delta\ell(P')}$ for the maximum of $|\psi_{P'}|$). Therefore,

$$\|\widetilde{R}^H(|\psi|m)\|^2_{L^2(\mu_Q)} \leqslant C\delta^{-2}\mu(Q)$$

is proved, and then we (non-trivially, but habitually) transfer this into

$$\|\widetilde{R}^H(|\psi|m)\|^2_{L^2(\nu)} \leqslant C\delta^{-2}\mu(Q)$$

by using (as we did several times before) Lemmas 1, 2 of Lecture 5.

Qualitative step: smearing of the measure ν without the estimate of its density We replace the measure ν by a compactly supported measure $\widetilde{\nu}$ that has a bounded density with respect to the $(d+1)$-dimensional Lebesgue measure m in $\mathbb{R}^{d+1}$. We use the notations from Lecture 7 under the title "**Stage 2. The next measure modification. Reflection trick.**" We want just "thicken" the support of measure ν to make it $(d+1)$-dimensional "thickening" of the actual d-dimensional support of ν.

For every $\varkappa > 0$, we will construct a measure $\widetilde{\nu}$ with the following properties:

- $\widetilde{\nu}$ is absolutely continuous and has bounded density with respect to m.
- $\mathrm{supp}\widetilde{\nu} \subset S$ and $\mathrm{dist}(\mathrm{supp}\widetilde{\nu}, L) \geqslant \Delta\ell(Q)$.
- $\widetilde{\nu}(S) = \nu(S) \leqslant \mu(Q)$.
- $\int \eta \, d\widetilde{\nu} \geqslant \int \eta \, d\nu - \varkappa$.
- $\int |\widetilde{R}^H(|\psi|m)|^2 \, d\widetilde{\nu} \leqslant \int |\widetilde{R}^H(|\psi|m)|^2 \, d\nu + \varkappa$.
- $\int |\widetilde{R}^H\widetilde{\nu}|^2 \, d\widetilde{\nu} \leqslant \int |\widetilde{R}^H\nu|^2 \, d\nu + \varkappa$.

Here is how $\widetilde{\nu}$ is constructed. It is important to note that this step will be purely qualitative. The boundedness of the density $\frac{d\widetilde{\nu}}{dm}$ will be used to show the existence of a minimizer in a certain extremal problem and the continuity of the corresponding Riesz potential but the supremum bound of $\frac{d\widetilde{\nu}}{dm}$ itself will not enter any final estimates.

Fix some radial non-negative C^∞-function φ_1 with $\mathrm{supp}\varphi_1 \subset B(0,1)$ and $\int \varphi_1 \, dm = 1$. For $0 < s \leqslant 1$, define

$$\varphi_s(x) = s^{-d-1}\varphi_1(s^{-1}x)$$

and

$$\nu_s = \nu * \varphi_s \,.$$

Clearly, all the supports of the measures ν_s are contained in some compact set and ν_s converge to ν weakly as $s \to 0+$. If s is much less than $\Delta\ell(Q)$, we have $\mathrm{supp}\nu_s \subset S$ and $\mathrm{dist}(\mathrm{supp}\nu_s, L) > \Delta\ell(Q)$. Also, the total mass of ν_s is the same as the total mass of ν for all s.

Note that both η and $|\widetilde{R}^H(|\psi|m)|^2$ are continuous functions in S, so the weak convergence is enough to establish the convergence of the corresponding integrals. What is less obvious is that the integrals $\int |\widetilde{R}^H\nu_s|^2 \, d\nu_s$ also converge to the integral $\int |\widetilde{R}^H\nu|^2 \, d\nu$ because formally it is a trilinear form in the measure argument with a singular kernel.

Note, however, that for every finite measure σ, we have $\widetilde{R}^H\sigma = R^H(\sigma - \sigma^*)$ where σ^* is the reflection of the measure σ about the boundary hyperplane L of S, i.e., $\sigma^*(E) = \sigma(E^*)$ where $E^* = \{x^* : x \in E\}$. Moreover, R^H commutes with shifts and, since φ_s is radial (all we really need is the symmetry about H), we have $(\nu * \varphi_s)^* = \nu^* * \varphi_s$.

Hence,

$$\widetilde{R}^H\nu_s = R^H[\nu * \varphi_s - \nu^* * \varphi_s] = R^H[(\nu - \nu^*) * \varphi_s] = [R^H(\nu - \nu^*)] * \varphi_s \,.$$

Lemma 1. *Suppose that f is a C^2 smooth compactly supported function on L. Then the functions $R^H_\delta(f\,m_L)$ converge to some limit $R^H(f\,m_L)$ uniformly on the entire space $\mathbb{R}^{d+1}$ as $\delta \to 0+$. Moreover, $R^H(f\,m_L)$ is a Lipschitz function in $\mathbb{R}^{d+1}$ harmonic outside* $\operatorname{supp}(f\,m_L)$*, and we have*

$$\sup |R^H(f\,m_L)| \leqslant CD^2 \sup_L |\nabla^2_H f|$$

and

$$\|R^H(f\,m_L)\|_{\mathrm{Lip}} \leqslant CD \sup_L |\nabla^2_H f|$$

where D is the diameter of $\operatorname{supp}(f\,m_L)$ *and ∇_H is the partial gradient involving only the derivatives in the directions parallel to H.*

By this lemma, $R^H(\nu-\nu^*)$ is a bounded Lipschitz function, so the convergence $[R^H(\nu-\nu^*)]*\varphi_s \to R^H(\nu-\nu^*)$ as $s \to 0+$ is uniform on compact sets and so is the convergence $|[R^H(\nu-\nu^*)]*\varphi_s|^2 \to |R^H(\nu-\nu^*)|^2$. Thus, despite all the singularities in the kernel, $|\widetilde{R}^H\nu_s|^2$ converges to $|\widetilde{R}^H\nu|^2$ uniformly, which is enough to ensure that

$$\int |\widetilde{R}^H\nu_s|^2\,d\nu_s \to \int |\widetilde{R}^H\nu|^2\,d\nu$$

as $s \to 0+$. So, we can take $\widetilde{\nu} = \nu_s$ with sufficiently small $s > 0$.

Now the crucial part of the proof comes. We are going to give the estimate from below of $\int |\widetilde{R}^H\nu|^2\,d\nu$.

Suppose $\|\widetilde{R}^H\nu\|_{L^2(\nu)} < \lambda\mu(Q)$ with tiny λ. Our goal is to bring this to contradiction (and to find how small is λ that gives the contradiction).

If this inequality holds, then, choosing sufficiently small smearing parameter we get very small $\varkappa > 0$ and we can ensure that the measure $\widetilde{\nu}$ constructed in the previous section, satisfies

$$\int |\widetilde{R}^H\widetilde{\nu}|^2\,d\widetilde{\nu} < \lambda\mu(Q)\,, \int \eta\,d\widetilde{\nu} \geqslant \theta\mu(Q)\,, \int |\widetilde{R}^H(|\psi|m)|^2\,d\widetilde{\nu} \leqslant \Theta\mu(Q)$$

where $\theta, \Theta > 0$ are two quantities depending only on δ (plus, of course, the dimension d and the goodness and AD-regularity constants of μ).

Our aim is to show that if $\lambda = \lambda(\delta) > 0$ is chosen small enough, then these three conditions are incompatible.

Then of course $\|\widetilde{R}^H\nu\|_{L^2(\nu)} \geqslant \lambda\mu(Q)$ with not-so-tiny λ, and almost orthogonality finishes the contradiction, as we have already shown in Lecture 7.

Extremal problem. For non-negative $a \in L^\infty(m)$, define $\widetilde{\nu}_a = a\widetilde{\nu}$ and consider the extremal problem

$$\Xi(a) = \lambda\mu(Q)\|a\|_{L^\infty(m)} + \int |\widetilde{R}^H \widetilde{\nu}_a|^2 d\widetilde{\nu}_a \to \min$$

under the restriction $\int \eta\, d\widetilde{\nu}_a \geqslant \theta\mu(Q)$. Note that since $\widetilde{\nu}$ is absolutely continuous and has bounded density with respect to $m = m_{d+1}$, the measure $\widetilde{\nu}_a$ is well defined and has the same properties.

The first goal is to show that the minimum is attained and for every minimizer a, we have $\|a\|_{L^\infty(m)} \leqslant 2$ and

$$|\widetilde{R}^H \widetilde{\nu}_a|^2 + 2(\widetilde{R}^H)^*[(\widetilde{R}^H \widetilde{\nu}_a)\widetilde{\nu}_a] \leqslant 6\lambda\theta^{-1}$$

everywhere in S.

This is done exactly as in Lecture 4. In fact, one should compare what follows very closely with the reasoning in Lecture 4.

Contradiction: why this smallness is impossible? Integrate the last inequality against $|\psi|\, dm$, where ψ is the vector field constructed at the end of Lecture 7 and at the beginning of this Lecture. We then get

$$\int |\widetilde{R}^H \widetilde{\nu}_a|^2 \cdot |\psi|\, dm + 2\int \Big[(\widetilde{R}^H)^*[(\widetilde{R}^H \widetilde{\nu}_a)\widetilde{\nu}_a]\Big] \cdot |\psi|\, dm$$
$$\leqslant 6\lambda\theta^{-1} \int |\psi| dm \leqslant C\lambda\theta^{-1}\delta^{-1}\mu(Q)\,.$$

Rewrite the second integral on the left as

$$\int \Big\langle \widetilde{R}^H \widetilde{\nu}_a, \widetilde{R}^H(|\psi| m)\Big\rangle\, d\widetilde{\nu}_a\,.$$

Then, by the Cauchy inequality,

$$\int \Big[(\widetilde{R}^H)^*[(\widetilde{R}^H \widetilde{\nu}_a)\widetilde{\nu}_a]\Big] \cdot |\psi|\, dm$$
$$\leqslant \Big[\int |\widetilde{R}^H \widetilde{\nu}_a|^2\, d\widetilde{\nu}_a\Big]^{\frac{1}{2}} \Big[\int |\widetilde{R}^H(|\psi| m)|^2\, d\widetilde{\nu}_a\Big]^{\frac{1}{2}}$$
$$\leqslant \Xi(a)^{\frac{1}{2}} \Big[\int |\widetilde{R}^H(|\psi| m)|^2\, d\widetilde{\nu}_a\Big]^{\frac{1}{2}}\,.$$

Recall that $\|a\|_{L^\infty(m)} \leqslant 2$, so we can replace $\widetilde{\nu}_a$ by $\widetilde{\nu}$ in the last integral losing at most a factor of 2. Taking into account that

$$\int |\widetilde{R}^H(|\psi| m)|^2\, d\widetilde{\nu} \leqslant \Theta\mu(Q)\,,$$

we get

$$\left|\int \Big[(\widetilde{R}^H)^*[(\widetilde{R}^H\widetilde{\nu}_a)\widetilde{\nu}_a]\Big] \cdot |\psi|\, dm\right| \leqslant C\,[\lambda\Theta]^{\frac{1}{2}}\,\mu(Q)\,.$$

Thus,

(3)

$$\int |\widetilde{R}^H\widetilde{\nu}_a|\cdot|\psi|\, dm \leqslant \Big(\int |\widetilde{R}^H\widetilde{\nu}_a|^2\cdot|\psi|\, dm\Big)^{1/2}\big(\int |\psi|\, dm\big)^{1/2} \leqslant C(\delta)\lambda^{1/4}\mu(Q)\,.$$

In particular, (3) implies

$$\int \langle \widetilde{R}^H\widetilde{\nu}_a, \psi\rangle\, dm \leqslant C(\delta)\lambda^{\frac{1}{4}}\mu(Q)\,. \tag{4}$$

On the other hand,

$$\int \langle \widetilde{R}^H\widetilde{\nu}_a, \psi\rangle\, dm = \int [(\widetilde{R}^H)^*(\psi m)]\, d\widetilde{\nu}_a$$
$$= \int [(R^H)^*(\psi m)]\, d\widetilde{\nu}_a - \int [T^*(\psi m)]\, d\widetilde{\nu}_a \geqslant \int \eta\, d\widetilde{\nu}_a - \sigma(\varepsilon,\alpha)\widetilde{\nu}_a(S).$$

This yields

(5)

$$\int [(\widetilde{R}^H)^*(\psi m)]\, d\widetilde{\nu}_a \geqslant \theta\mu(Q) - \sigma(\varepsilon,\alpha)\widetilde{\nu}_a(S) \geqslant [\theta - 2\sigma(\varepsilon,\alpha)]\mu(Q) \geqslant \frac{\theta}{2}\mu(Q)\,,$$

if ε and α are chosen small enough (in this order). Thus, if λ has been chosen smaller than a certain constant depending on δ only, we get a contradiction between (4) and (5) (their left hand sides are identically equal).

The conclusion is that the estimate $\int |\widetilde{R}^H\nu|^2\, d\nu \geqslant \lambda\mu(Q)$ if λ has been chosen smaller than a certain constant depending on δ only. Then in Lecture 7 we explained that such estimates from below will sum up to too big a number for $\int |R_\mu 1|^2\, d\mu$ because of the almost orthogonality of flat layers. And we explained in Lecture 7 that this leads to a contradiction.

We started our abyss to this contradiction by assuming that δ-non-BAUP cells are not rare (do not form a Carleson sequence). See the beginning of Lecture 7. Henceforth, they have to be rare. But then David–Semmes main result of [DS] gives the rectifiability of μ.

The proof of David–Semmes conjecture in co-dimension 1 is completely done.

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