RECTIFIABILITY OF MEASURES WITH BOUNDED RIESZ TRANSFORM OPERATOR: FROM SINGULAR OPERATORS TO GEOMETRIC MEASURE THEORY

ALEXANDER VOLBERG

1. Lecture 7: Almost orthogonality of flat layers contributions. The preparation for the estimate from below of each flat layer contribution.

Recall that our goal is to prove that the family of all non-BAUP cells $P \in \mathcal{D}$ is Carleson. In view of the just proved **abundance of flat cells in several fixed directions**, it obviously suffices to show the following statement:

We can choose $A, \alpha > 0$ such that for every fixed linear hyperplane H and for every integer N, the family $\mathcal{F} = \mathcal{F}(A, \alpha, H, N)$ of all non-BAUP cells $P \in D$ containing an (H, A, α)-flat cell Q at most N levels down from P is Carleson.

Then all non-BAUP cells are Carleson, and we are completely done by referring to David–Semmes theorem from the book [DS].

The idea of how we will prove this statement. Suppose the statement above is false. Then there will be P from $\mathcal{F}$ (family of non-BAUP cells containing a flat cell in a fixed direction at most N generations down) **such that it can be tiled (up to tiny measure) by arbitrarily large number of layers of non-BAUP cells.** Use now the abundance of flat cells. We can also tile the same cell P by layers of (H, A, α)-flat cells Q (up to tiny measure) also with as many layers as we wish. Moreover we can alternate layers. Namely:

Alternating non-BAUP and flat layers.

Lemma 1. *If $\mathcal{F}$ is not Carleson, then for every positive integer K and every $\eta > 0$, there exist a cell $P \in \mathcal{F}$ and $K+1$ alternating pairs of finite layers $\mathfrak{P}_k, \mathfrak{Q}_k \subset \mathcal{D}$ ($k = 0, \dots, K$) such that*

- $\mathfrak{P}_0 = \{P\}$.
- $\mathfrak{P}_k \subset \mathcal{F}_P$ *(that is they are from $\mathcal{F}$ and lie inside P) for all $k = 0, \dots, K$.*
- *All layers $\mathfrak{Q}_k$ consist of (H, A, α)-flat cells only.*

Date: April 6, 2014.

- *Each individual layer (either $\mathfrak{P}_k$, or $\mathfrak{Q}_k$) consists of pairwise disjoint cells.*
- *If $Q \in \mathfrak{Q}_k$, then there exists $P' \in \mathfrak{P}_k$ such that $Q \subset P'$ ($k = 0, \dots, K$).*
- *If $P' \in \mathfrak{P}_{k+1}$, then there exists $Q \in \mathfrak{Q}_k$ such that $P' \subset Q$ ($k = 0, \dots, K-1$).*
- $\sum_{Q \in \mathfrak{Q}_K} \mu(Q) \geqslant (1-\eta)\mu(P)$.

Sketch of the proof. Suppose $\mathcal{F}$ is not Carleson. For every $\eta' > 0$ and every positive integer M, we can find a cell $P \in \mathcal{F}$ and $M+1$ layers $\mathcal{L}_0, \dots, \mathcal{L}_M \subset \mathcal{F}_P$ that have the desired Cantor-type hierarchy and satisfy $\sum_{P' \in \mathcal{L}_M} \mu(P') \geqslant (1-\eta')\mu(P)$.

We will go now from the layer $\mathcal{L}_m$ to $\mathcal{L}_{m+SN}$, where $S = S(N)$ will be large and $M \approx KSN$, where K is from above. We take $P' \in \mathcal{L}_m$ and choose $Q(P')$ less than N generations down, which is flat. Those $P'' \in \mathcal{L}_{m+N}$ that are inside such $Q(P')$ we color white, the collection of $Q(P')$ we color blue. Notice that at this moment the mass of all non-colored $P'' \in \mathcal{L}_{m+N}$ is $\leqslant (1 - c4^{-4dN})\mu(P)$.

In those $P'' \in \mathcal{L}_{m+N}$ that are not colored again we will have $Q(P'')$ less than N generations down that are flat, color them blue, color white those $P''' \in \mathcal{L}_{m+2N}$ that are in some of $Q(P'')$. Notice that at this moment the mass of all non-colored $P''' \in \mathcal{L}_{m+2N}$ is $\leqslant (1 - c4^{-4dN})^2\mu(P)$.

Non-colored follow non-colored, and in S steps (if $S = S(N)$ is sufficiently large) the portion of $\mu(P)$ of non-colored cells become very small. Then we stop and put $m_{new} := m + SN$, we consider only the part of layer $\mathcal{L}_{m+SN}$, namely those cells of it that lie in some white colored cells. Call it $\mathcal{L}'_{m_{new}}$. So if $\mathcal{L}_m$ was $\mathfrak{P}_k$, then $\mathcal{L}'_{m_{new}}$ will be our $\mathfrak{P}_{k+1}$.

Consider all blue cells we have on the road. Take the family of maximal blue cells out of those which we just constructed. This will be layer of disjoint (H, A, α)-flat cells and this will be our layer $\mathfrak{Q}_k$.

Given K, we choose S very large, η' very small. Then we make the error in tiling $\mu(P)$ only of the order $(K+1)[\eta' + (1 - c4^{-4dN})^S]$, which is as small as we wish. Lemma is proved.

Towards almost orthogonality of flat layers. From now on, for a while (in this lecture at least) we will be interested only in the cells Q from the flat layers $\mathfrak{Q}_k$. With each such cell Q we will associate the corresponding approximating plane $L(Q)$ containing its center z_Q and parallel to H and the approximating measure $\nu_Q = a_Q \varphi_Q m_{L(Q)}$. These approximate measures we need to explain.

Fix K. Choose $\varepsilon > 0$, $A, \alpha > 0$, $\eta > 0$ in this order. Construct layers as it has been just done. Consider flat layers $\mathfrak{Q}_k$ ignoring the non-BAUP layers $\mathfrak{P}_k$ almost entirely.

For a cell $Q \in \mathcal{D}$ and $t > 0$, define

$$Q_t = \{x \in Q : \operatorname{dist}(x, \mathbb{R}^{d+1} \setminus Q) \geqslant t\ell(Q)\}\,. \tag{1}$$

Note that

$$\mu(Q \setminus Q_t) \leqslant C t^\gamma \mu(Q)\,, \tag{2}$$

for some $\gamma > 0$. This is stated at the beginning of Lecture 5.

Let us consider $\varphi_0 \in C^\infty$ supported on $B(0,1)$ and such that $\int \varphi_0\, dm = 1$ where m is the Lebesgue measure in $\mathbb{R}^{d+1}$. Put

$$\varphi_Q = \chi_{Q_{2\varepsilon}} * \frac{1}{(\varepsilon\ell(Q))^d}\, \varphi_0\Big(\frac{\cdot}{\varepsilon\ell(Q)}\Big)\,.$$

Then $\varphi_Q = 1$ on $Q_{3\varepsilon}$ and $\operatorname{supp}\varphi_Q \subset Q_\varepsilon$. In particular, the diameter of $\operatorname{supp}\varphi_Q$ is at most $8\ell(Q)$.

In addition,

$$\|\varphi_Q\|_{L^\infty} \leqslant 1, \quad \|\nabla\varphi_Q\|_{L^\infty} \leqslant \frac{C}{\varepsilon\ell(Q)}, \quad \|\nabla^2\varphi_Q\|_{L^\infty} \leqslant \frac{C}{\varepsilon^2\ell(Q)^2}\,.$$

The approximating measure is $\nu_Q = a_Q\varphi_Q m_{L(Q)}$, where a_Q is chosen so that

$$\nu_Q(\mathbb{R}^{d+1}) = \int \varphi_Q\, d\mu\,.$$

Both integrals $\int \varphi_Q\, dm_{L(Q)}$ and $\int \varphi_Q\, d\mu$ are comparable to $\ell(Q)^d$, provided that $\varepsilon < \frac{1}{48}$, say. In particular, in this case, the normalizing factors a_Q are bounded by some constant.

To formulate the orthogonality of layers we need to define

$$G_k = \sum_{Q \in \mathfrak{Q}_k} \varphi_Q R^H[\varphi_Q\mu - \nu_Q]\,, \quad k = 0, \dots, K\,.$$

Now put

$$F_k = G_k - G_{k+1}, \text{ when } k = 0, \dots, K-1, \qquad F_K = G_K\,.$$

Note that

$$\sum_{m=k}^{K} F_m = G_k\,.$$

"Orthogonality" of telescopic layers. This is almost orthogonality of "errors" (errors between genuine and flat situations).

Lemma 2. *Assuming that* $\varepsilon < \frac{1}{48}$, $A > 5$, *and* $\alpha < \varepsilon^8$, *we have*

$$|\langle F_k, G_{k+1}\rangle| \leqslant \sigma(\varepsilon,\alpha)\mu(P)$$

for all $k = 0, \dots, K-1$, *where* $\sigma(\varepsilon,\alpha)$ *is some positive function such that*

$$\lim_{\varepsilon\to 0+}[\lim_{\alpha\to 0+}\sigma(\varepsilon,\alpha)] = 0\,.$$

The proof is long and and not easy. Lemmas 1 and 2 from Lecture 5 are constantly used. And the boundedness of R_μ in $L^2(\mu)$ is used. The detailed proof is in [NToV1].

How almost orthogonality works?

We have the identity

$$\|G_0\|^2_{L^2(\mu)} = \left\|\sum_{k=0}^{K} F_k\right\|^2_{L^2(\mu)} = \sum_{k=0}^{K}\|F_k\|^2_{L^2(\mu)} + 2\sum_{k=0}^{K-1}\langle F_k, G_{k+1}\rangle_\mu\,.$$

As we will see in a minute, $\|G_0\|^2_{L^2(\mu)} \leqslant C\mu(P)$, and the scalar products can be made arbitrarily small by first choosing $\varepsilon > 0$ small enough and then taking a sufficiently small $\alpha > 0$ depending on ε.

At this point, we need to know that the non-BAUPness condition depends on a positive parameter δ. We will fix that δ from now on in addition to fixing the measure μ. Note that despite the fact that we need to prove that the family of non-BAUP cells is Carleson for every $\delta > 0$, the David-Semmes uniform rectifiability criterion does not require any particular rate of growth of the corresponding Carleson constant as a function of δ.

So we will get a contradiction if we are able to bound $\|F_k\|^2_{L^2(\mu)}$ for $k = 0, \dots, K-1$ from below by $\tau^2\mu(P)$, with some $\tau = \tau(\delta) > 0$ (as usual, the dependence on the dimension d and the regularity constants of μ is suppressed).

We choose very large K, then we choose $A > A_0(\delta), \varepsilon < \varepsilon_0(\delta), \eta < \eta_0(\varepsilon), \alpha < \alpha_0(\varepsilon,\delta)$. Then we come to contradiction in

$$(-small(\varepsilon,\alpha,\eta) + K\tau^2)\mu(P) \leqslant \sum_{k=0}^{K}\|F_k\|^2_{L^2(\mu)} - small(\varepsilon,\alpha,\eta)\mu(P)$$

$$\leqslant \|G_0\|^2_{L^2(\mu)} \leqslant C\mu(P)\,.$$

The estimate from above: $\|G_0\|^2_{L^2(\mu)} \leqslant C\mu(P)$. Notice that $G_0 = \sum_{Q\in\mathfrak{Q}_0} \varphi_Q R^H[\varphi_Q\mu - \nu_Q]$. As the summands have pairwise disjoint supports, it will suffice to prove the inequality

$$\|\varphi_Q R^H(\varphi_Q\mu - \nu_Q)\|^2_{L^2(\mu)} \leqslant C\mu(Q)$$

for each individual $Q \in \mathfrak{Q}_0$ and then observe that $\sum_{Q\in\mathfrak{Q}_k} \mu(Q) \leqslant \mu(P)$. Of course $\|\varphi_Q R^H(\varphi_Q\mu)\|^2_{L^2(\mu)} \leqslant C\mu(Q)$ by the boundedness of R_μ. But the estimate $\|\varphi_Q R^H(\nu_Q)\|^2_{L^2(\mu)} \leqslant C\mu(Q)$ is not so trivial because we start with flat measure $\nu_Q := a_Q\varphi_Q m_L$ but we send it by R^H into $L^2(\mu)$. Such an estimate can be obtained by using an error estimate Lemmas 1 and 2 from Lecture 5. Details can be seen in Lemma 9 [NToV1].

The estimate from below. Densely packed cells.
Fix $k \in \{0, 1, \dots, K-1\}$. We can write the function F_k as

$$F_k = \sum_{Q\in\mathfrak{Q}_k} F^Q$$

where

$$F^Q = \varphi_Q R^H(\varphi_Q\mu - \nu_Q) - \sum_{Q'\in\mathfrak{Q}_{k+1}, Q'\subset Q} \varphi_{Q'} R^H(\varphi_{Q'}\mu - \nu_{Q'})\,.$$

We shall call a cell $Q \in \mathfrak{Q}_k$ densely packed if $\sum_{Q'\in\mathfrak{Q}_{k+1}, Q'\subset Q} \mu(Q') \geqslant (1-\varepsilon)\mu(Q)$. Otherwise we shall call the cell Q loosely packed. The loosely packed cells constitute a tiny minority of all cells in $\mathfrak{Q}_k$ if $\eta \leqslant \varepsilon^2$. Indeed, we have

$$\begin{aligned}
\sum_{\substack{Q\in\mathfrak{Q}_k \\ Q \text{ is packed loosely}}} \mu(Q) &\leqslant \varepsilon^{-1} \sum_{Q\in\mathfrak{Q}_k} \mu\left(Q \setminus \left(\bigcup_{Q'\in\mathfrak{Q}_{k+1}, Q'\subset Q} Q'\right)\right) \\
&= \varepsilon^{-1}\left[\sum_{Q\in\mathfrak{Q}_k} \mu(Q) - \sum_{Q'\in\mathfrak{Q}_{k+1}} \mu(Q')\right] \\
&\leqslant \varepsilon^{-1}\left[\mu(P) - \sum_{Q'\in\mathfrak{Q}_{k+1}} \mu(Q')\right] \leqslant \frac{\eta}{\varepsilon}\mu(P) \leqslant \varepsilon\mu(P)\,.
\end{aligned}$$

We can immediately conclude from here that

$$\sum_{\substack{Q\in\mathfrak{Q}_k\\ Q \text{ is densely packed}}} \mu(Q) = \sum_{Q\in\mathfrak{Q}_k} \mu(Q) - \sum_{\substack{Q\in\mathfrak{Q}_k\\ Q \text{ is loosely packed}}} \mu(Q)$$
$$\geqslant (1-\eta)\mu(P) - \varepsilon\mu(P) \geqslant (1-2\varepsilon)\mu(P)\,.$$

From now on, we will fix the choice $\eta = \varepsilon^2$.

We claim now that to estimate $\|F_k\|^2_{L^2(\mu)}$ from below by $\tau^2\mu(P)$, it suffices to show that for every densely packed cell $Q \in \mathfrak{Q}_k$, we have

$$\|F^Q\|^2_{L^2(\mu)} \geqslant 2\tau^2\mu(Q)\,. \tag{3}$$

To see it, just write

$$\|F_k\|^2_{L^2(\mu)} = \sum_{Q\in\mathfrak{Q}_k} \|F^Q\|^2_{L^2(\mu)} \geqslant \sum_{\substack{Q\in\mathfrak{Q}_k\\ Q \text{ is densely packed}}} \|F^Q\|^2_{L^2(\mu)}$$
$$\geqslant \sum_{\substack{Q\in\mathfrak{Q}_k\\ Q \text{ is densely packed}}} 2\tau^2\mu(Q) \geqslant 2(1-2\varepsilon)\tau^2\mu(P) \geqslant \tau^2\mu(P)\,,$$

provided that $\varepsilon < \frac{1}{4}$.

To prove (3) one uses "the process of modification of measure". It consists of several stages.
Stage 1: $\varphi_Q\mu$ to piecewise flat ν. The goal of this modification is to show

Lemma 3. *There exists a subset $\mathfrak{Q}'$ of $\mathfrak{Q}$ such that*

$$\sum_{Q'\in\mathfrak{Q}'} \mu(Q') \geqslant (1-C\varepsilon)\mu(Q), \tag{4}$$

and

$$\|F^Q\|_{L^2(\mu)} \geqslant \frac{1}{2}\|R^H(\nu-\nu_Q)\|_{L^2(\nu)} - \sigma(\varepsilon,\alpha)\sqrt{\mu(Q)}\,,$$

where $\nu = \sum_{Q'\in\mathfrak{Q}'} \nu_{Q'}$ and $\sigma(\varepsilon,\alpha)$ is some positive function such that $\lim_{\varepsilon\to 0+}[\lim_{\alpha\to 0+}\sigma(\varepsilon,\alpha)] = 0$.

The proof is long and technical, see [NToV1], pages 62–66, but looking at

$$F^Q = \varphi_Q R^H(\varphi_Q\mu - \nu_Q) - \sum_{Q'\in\mathfrak{Q}_{k+1}, Q'\subset Q} \varphi_{Q'} R^H(\varphi_{Q'}\mu - \nu_{Q'})$$

we see that the claim is at least natural, as it says that the value of $R^H(\varphi_Q\mu - \varphi_{Q'}\mu)$ on each $Q' \in \mathfrak{Q}'$ almost cancels out the value of $R^H(\nu_Q - \nu_{Q'})$ on Q'. This boils down to saying that for $Q', Q'' \in \mathfrak{Q}', Q' \neq Q''$, one can see that the value of $R^H(\varphi_{Q'}\mu - \nu_{Q'})$ on Q'' is negligible if α and ε are small.

Which is believable because of flatness. And we also replace μ outside in $\|F^Q\|_{L^2(\mu)}$ by the measure ν consisting of flat pieces parallel and close to flat ν_Q.

For the choice of $\mathfrak{Q}'$ one uses Marcinkiewicz function in the following way: For $Q' \in \mathfrak{Q}$, define

$$g(Q') = \sum_{Q'' \in \mathfrak{Q}} \left[\frac{\ell(Q'')}{D(Q', Q'')} \right]^{d+1}$$

where

$$D(Q', Q'') = \ell(Q') + \ell(Q'') + \operatorname{dist}(Q', Q'')$$

is the "long distance" between Q' and Q''.

We have

$$\begin{aligned}
\sum_{Q' \in \mathfrak{Q}} g(Q')\mu(Q') &= \sum_{Q', Q'' \in \mathfrak{Q}} \ell(Q'')^{d+1} \frac{\mu(Q')}{D(Q', Q'')^{d+1}} \\
&\leqslant C \sum_{Q', Q'' \in \mathfrak{Q}} \ell(Q'')^{d+1} \int_{Q'} \frac{d\mu(x)}{[\ell(Q'') + \operatorname{dist}(x, Q'')]^{d+1}} \\
&\leqslant C \sum_{Q'' \in \mathfrak{Q}} \ell(Q'')^{d+1} \int \frac{d\mu(x)}{[\ell(Q'') + \operatorname{dist}(x, Q'')]^{d+1}} \\
&\leqslant C \sum_{Q'' \in \mathfrak{Q}} \ell(Q'')^{d} \leqslant C \sum_{Q'' \in \mathfrak{Q}} \mu(Q'') \leqslant C\mu(Q)\,.
\end{aligned}$$

Let $\mathfrak{Q}_* = \{Q' \in \mathfrak{Q} : g(Q') > \varepsilon^{-1}\}$, $\mathfrak{Q}' = \mathfrak{Q} \setminus \mathfrak{Q}_*$. Then, by Chebyshev's inequality,

$$\sum_{Q' \in \mathfrak{Q}_*} \mu(Q') \leqslant C\varepsilon\mu(Q)\,,$$

so

$$\sum_{Q' \in \mathfrak{Q}'} \mu(Q') \geqslant (1 - C\varepsilon)\mu(Q)\,,$$

which is (4).

In estimating the value of $R^H(\varphi_{Q'}\mu - \nu_{Q'})$ on Q'', one uses Lemmas 1,2 from Lecture 5. Then one can see that $\sum_{Q' \in \mathfrak{Q}', Q' \neq Q''} |R^H(\varphi_{Q'}\mu - \nu_{Q'})|$ can be estimated from above by $C\alpha\varepsilon^{-d-2} \sum_{Q' \in \mathfrak{Q}', Q' \neq Q''} \left[\frac{\ell(Q')}{D(Q', Q'')}\right]^{d+1} \leqslant C\alpha\varepsilon^{-d-2} g(Q'')$.

Stage 2. The next measure modification. Reflection trick. Fix a hyperplane L parallel to H at the distance $2\Delta\ell(Q)$ from $\mathrm{supp}\mu\cap Q$. Number Δ is small compared to ε and large compared to α. Let S be the (closed) half-space bounded by L that contains $\mathrm{supp}\mu\cap Q$. For $x\in S$, denote by x^* the reflection of x about L. Define the kernels

$$\widetilde{R}^H(x,y)=R^H(x-y)-R^H(x^*-y),\qquad x,y\in S$$

and denote by $\widetilde{R}^H$ the corresponding operator. We will assume that $\alpha << \Delta$, so the approximating hyperplanes $L(Q')$ $(Q'\in\mathfrak{Q}')$ and $L(Q)$, which lie within the distance $\alpha\ell(Q)$ from $\mathrm{supp}\mu\cap Q$ are contained in S and lie at the distance $\Delta\ell(Q)$ or greater from the boundary hyperplane L.

Lemma 4. *The goal of this section is to show that, for some appropriately chosen $\Delta=\Delta(\alpha,\varepsilon)>0$, and under our usual assumptions about ε, A, and α, we have*

$$\|R^H(\nu-\nu_Q)\|_{L^2(\nu)}\geqslant\|\widetilde{R}^H\nu\|_{L^2(\nu)}-\sigma(\varepsilon,\alpha)\sqrt{\mu(Q)}$$

where, again, $\sigma(\varepsilon,\alpha)$ is some positive function such that

$$\lim_{\varepsilon\to0+}[\lim_{\alpha\to0+}\sigma(\varepsilon,\alpha)]=0\,.$$

Again we refer the reader to [NToV1], this lemma has a proof on pages 69–72 of this article.

Now we choose $\Delta=\varepsilon^3$ and $\alpha=\varepsilon^C$ with large C. We come to the point that we need to estimate from below **the Riesz Energy**

$$\|\widetilde{R}^H\nu\|_{L^2(\nu)}\,,$$

where $\nu:=\sum_{Q'\in\mathfrak{Q}',Q'\subset Q}\nu_{Q'}$. It is **truly desirable** to have $\|\widetilde{R}^H\nu\|_{L^2(\nu)}\geqslant$ ... using another than ε constant. **To give a δ-breath.** The subset $\mathfrak{Q}'$ of the set $\{Q'\colon Q'\subset Q, Q'\in\mathfrak{Q}_{k+1}\}$ is chosen above. In fact, it is almost the whole $\{Q'\colon Q'\subset Q, Q'\in\mathfrak{Q}_{k+1}\}$, the difference being the use of a certain Marcinkiewicz function to choose $\mathfrak{Q}'$.

To estimate Riesz Energy we need function ψ, $\widetilde{R}^H(\psi m)=1, m+m_{d+1}$, on ν and such that: see below. For that we need first non-BAUP layer $\mathfrak{P}_{k+1}$ tiling Q and tiled by $\mathfrak{Q}_{k+1}$ and special family of cells in it.

This is almost the last effort before the end of this Wagnerian proof. We need to choose some non-BAUP cells and associated with them vector functions ψ to be able to estimate the Riesz energy $\|\widetilde{R}^H\nu\|_{L^2(\nu)}$, from below.

A collection of P's (inside Q) of non-BAUP layer $\mathfrak{P}_{k+1}$. One can construct (under our usual assumptions of ε is sufficiently small in terms of δ, A is sufficiently large in terms of δ, α is sufficiently small in terms of ε and δ), a family $\mathfrak{P}' \subset \mathfrak{P}_{k+1}$ such that

- Every cell $P' \subset \mathfrak{P}'$ is contained in Q_ε and satisfies $\ell(P') \leqslant 2\alpha\delta^{-1}\ell(Q)$.
- $\sum_{P'\in\mathfrak{P}'} \mu(P') \geqslant c\mu(Q)$.
- The balls $B(z_{P'}, 10\ell(P'))$, $P' \in \mathfrak{P}'$ are pairwise disjoint.
- The function
$$h(x) = \sum_{P'\in\mathfrak{P}'} \left[\frac{\ell(P')}{\ell(P') + \operatorname{dist}(x, P')}\right]^{d+1}$$
satisfies $\|h\|_{L^\infty} \leqslant C$.

We start with showing that every δ-non-BAUP cell P' contained in Q has much smaller size than Q. Indeed, we know that $\operatorname{supp}\mu \cap B(z_Q, A\ell(Q))$ is contained in the $\alpha\ell(Q)$-neighborhood of $L(Q)$ and that $B(y, \alpha\ell(Q)) \cap \operatorname{supp}\mu \neq \varnothing$ for every $y \in B(z_Q, A\ell(Q)) \cap L(Q)$. Suppose that $P' \subset Q$ is δ-non-BAUP. If $A > 5$, then

$$B(x_{P'}, \ell(P')) \subset B(z_Q, 5\ell(Q)) \subset B(z_Q, A\ell(Q)) .$$

Moreover, since $y_{P'} - x_{P'} \in H$, we have

$$\operatorname{dist}(y_{P'}, L(Q)) = \operatorname{dist}(x_{P'}, L(Q)) \leqslant \alpha\ell(Q) .$$

Let $y^*_{P'}$ be the projection of $y_{P'}$ to $L(Q)$. Then $|y^*_{P'} - y_{P'}| \leqslant \alpha\ell(Q)$ and $|y^*_{P'} - z_Q| \leqslant |y_{P'} - z_Q| < A\ell(Q)$. Thus, the ball $B(y_{P'}, 2\alpha\ell(Q)) \supset B(y^*_{P'}, \alpha\ell(Q))$ intersects $\operatorname{supp}\mu$, so $\delta\ell(P') < 2\alpha\ell(Q)$, i.e.,

$$\ell(P') \leqslant 2\alpha\delta^{-1}\ell(Q),$$

which is desired smallness of $\mathfrak{P}_{k+1}$ cells inside $Q \in \mathfrak{Q}_k$.

Let now $\mathfrak{P} = \{P' \in \mathfrak{P}_{k+1} : P' \subset Q\}$. Consider the Marcinkiewicz function again

$$g(P') = \sum_{P''\in\mathfrak{P}} \left[\frac{\ell(P'')}{D(P', P'')}\right]^{d+1}$$

The standard argument with integration it over Q shows that

$$\sum_{P'\in\mathfrak{P}} g(P')\mu(P') \leqslant C_1\mu(Q)$$

for some $C_1 > 0$ depending on the dimension d and the goodness parameters of μ only. Define

$$\mathfrak{P}^* = \{P' \in \mathfrak{P} : P' \subset Q_\varepsilon, g(P') \leqslant 3C_1\}\,.$$

Note that

$$\sum_{P'\in\mathfrak{P}^*} \mu(P') \geqslant \sum_{P'\in\mathfrak{P}} \mu(P') - \sum_{P'\in\mathfrak{P}:P'\not\subset Q_\varepsilon} \mu(P') - \sum_{P'\in\mathfrak{P}:g(P')>3C_1} \mu(P')\,.$$

However,

$$\sum_{P'\in\mathfrak{P}} \mu(P') \geqslant \sum_{Q'\in\mathfrak{Q}} \mu(Q') \geqslant (1-\varepsilon)\mu(Q)\,.$$

Further, since the diameter of each $P' \in \mathfrak{P}$ is at most $8\ell(P') \leqslant 8\alpha\delta^{-1}\ell(Q)$, every cell $P' \in \mathfrak{P}$ that is not contained in Q_ε is contained in $Q \setminus Q_{2\varepsilon}$, provided that $\alpha < \frac{1}{8}\varepsilon\delta$. Thus, under this restriction,

$$\sum_{P'\in\mathfrak{P}:P'\not\subset Q_\varepsilon} \mu(P') \leqslant \mu(Q\setminus Q_{2\varepsilon}) \leqslant C\varepsilon^\gamma\mu(Q)\,.$$

Further, since the diameter of each $P' \in \mathfrak{P}$ is at most $8\ell(P') \leqslant 8\alpha\delta^{-1}\ell(Q)$, every cell $P' \in \mathfrak{P}$ that is not contained in Q_ε is contained in $Q \setminus Q_{2\varepsilon}$, provided that $\alpha < \frac{1}{8}\varepsilon\delta$. Thus, under this restriction,

$$\sum_{P'\in\mathfrak{P}:P'\not\subset Q_\varepsilon} \mu(P') \leqslant \mu(Q\setminus Q_{2\varepsilon}) \leqslant C\varepsilon^\gamma\mu(Q)\,.$$

Finally, by Chebyshev's inequality,

$$\sum_{P'\in\mathfrak{P}:g(P')>3C_1} \mu(P') \leqslant \frac{\mu(Q)}{3}\,.$$

Bringing these three estimates together, we get the inequality $\sum_{P'\in\mathfrak{P}^*}\mu(P') \geqslant \frac{1}{2}\mu(Q)$, provided that A, ε, α satisfy some restrictions of the admissible type.

Vitali's lemma sparseness. Now we will rarefy the family $\mathfrak{P}^*$ a little bit more. Consider the balls $B(z_{P'}, 10\ell(P'))$, $P' \in \mathfrak{P}^*$. By the classical Vitali covering theorem, we can choose some subfamily $\mathfrak{P}' \subset \mathfrak{P}^*$ such that the balls $B(z_{P'}, 10\ell(P'))$, $P' \in \mathfrak{P}'$ are pairwise disjoint but

$$\bigcup_{P'\in\mathfrak{P}'} B(z_{P'}, 30\ell(P')) \supset \bigcup_{P'\in\mathfrak{P}^*} B(z_{P'}, 10\ell(P')) \supset \bigcup_{P'\in\mathfrak{P}^*} P'\,.$$

Then we will still have

$$\sum_{P'\in\mathfrak{P}'}\mu(P')\geqslant c\sum_{P'\in\mathfrak{P}'}\ell(P')^d$$
$$\geqslant c\sum_{P'\in\mathfrak{P}'}\mu(B(z_{P'},30\ell(P')))\geqslant c\sum_{P'\in\mathfrak{P}^*}\mu(P')\geqslant c\mu(Q)\,.$$

The estimate on $h=\sum_{P'\in\mathfrak{P}'}\left[\frac{\ell(P')}{\ell(P')+\mathrm{dist}(x,P')}\right]^{d+1}$ follows from the Marcinkiewicz choice of $\mathfrak{P}^*$.

The preparation for building of vector function ψ that one needs for the estimate of Riesz energy from below.

First functions η_p. Fix the non-BAUPness parameter $\delta\in(0,1)$. Fix any C^∞ radial function η_0 supported in $B(0,1)$ such that $0\leqslant\eta_0\leqslant 1$ and $\eta_0=1$ on $B(0,\frac12)$. For every $P'\in\mathfrak{P}'$, define

$$\eta_{P'}(x)=\eta_0\left(\frac{1}{\delta\ell(P')}(x-x_{P'})\right)-\eta_0\left(\frac{1}{\delta\ell(P')}(x-y_{P'})\right)\,.$$

Here $x_{P'}$ is the point of cell P' belonging to $\mathrm{supp}\mu$, and $y_{P'}$ is a point that is the center of the δ-hole (a ball disjoint with $\mathrm{supp}\mu$) in P' that exists by the definition of δ-nonBAUPness.

Note that $\eta_{P'}$ is supported on the ball $B(z_{P'},6\ell(P'))$. This ball is contained in Q, provided that $12\alpha\delta^{-1}<\varepsilon$ (recall that $\ell(P')\leqslant 2\alpha\delta^{-1}\ell(Q)$ and $P'\subset Q_\varepsilon$). Also $\eta_{P'}\geqslant 1$ on $B(x_{P'},\frac{\delta}{2}\ell(P'))$ and the support of the negative part of $\eta_{P'}$ is disjoint with $\mathrm{supp}\mu$. Put

$$\eta=\sum_{P'\in\mathfrak{P}'}\eta_{P'}\,.$$

Since even the balls $B(z_{P'},10\ell(P'))$ corresponding to different $P'\in\mathfrak{P}'$ are disjoint, we have $-1\leqslant\eta\leqslant 1$.

We want to show that $\int\eta\,d\nu\geqslant c(\delta)\mu(Q)$ with some $c(\delta)>0$.

Obviously, $\int\eta\,d\mu\geqslant c(\delta)\mu(Q)$ with some $c(\delta)>0$. This is because of the choice of $\mathfrak{P}'$ and because, where η is negative does not carry any mass μ.

Moreover,

$$\int\eta\,\Phi\,d\mu\geqslant\int\eta_+\,d\mu-\left|\int(\chi_Q-\Phi)\,d\mu\right|\geqslant c(\delta)\mu(Q)-\varepsilon^\gamma\mu(Q)\geqslant\frac{c}{2}\mu(Q)\,.$$

So we need to estimate as a small thing $\int\eta\,(d\Phi\mu-\nu)$, which is the sum over $Q'\in\mathfrak{Q}'$ of $\int\eta\,(d\varphi_{Q'}\mu-\nu_{Q'})$. By Lemmas 1, 2 of Lecture 5 we have

$$|\int \eta\,(d\varphi_{Q'}\mu - \nu_{Q'})| \leqslant C\alpha\ell(Q')^{d+2}\|\varphi_{Q'}\|_{\mathrm{Lip}}\|\eta\|_{\mathrm{Lip}(\mathrm{supp}\varphi_{Q'})} \leqslant$$

$$C\alpha\varepsilon^{-1}\ell(Q')^{d+1}\|\eta\|_{\mathrm{Lip}(\mathrm{supp}\varphi_{Q'})}\leqslant C\alpha\varepsilon^{-1}\mu(Q') \sup_{P' :\, B(z_{P'},6\ell(P'))\cap Q'_\varepsilon\neq\emptyset} \frac{\ell(Q')}{\delta\ell(P')}.$$

For $Q'\subset P'$, fine. Otherwise $Q'\cap P'=\emptyset, B(z_{P'}, 6\ell(P'))\cap Q'_\varepsilon\neq\emptyset$ give $C\ell(P')\geqslant \varepsilon\ell(Q')$. And again smallness of α kills all $\varepsilon^{-2}\delta^{-1}$.

Now vector function ψ: $\psi=\Delta\int^x\eta$. Fix $P'\in\mathfrak{P}'$. Let $e_{P'}$ be the unit vector in the direction $y_{P'}-x_{P'}$. Put

$$u_{P'}(x)=\int_{-\infty}^0 \eta_{P'}(x+te_{P'})\,dt\,.$$

Let us think that H is parallel to $x_{d+1}=0$ and that $e_1=e_{P'}$ (this is without loss of generality). Then $\partial_1 u_{P'}=\eta_{P'}$. But

$$R^H=(\partial_1,\dots,\partial_d)\frac{1}{|x|^{d-1}}.$$

Therefore,

$$R^H\Delta u_{P'} = R^H\Delta\int\eta_{P'} = (\partial_1,\dots,\partial_d)\frac{1}{|x|^{d-1}}\star\Delta\int\eta_{P'} =$$

$=(\partial_1,\dots,\partial_d)\int\eta_{P'}$. We showed that

$$R^{H,1}\Delta u_{P'}=\partial_1\int\eta_{P'}=\eta_{P'}\Rightarrow\langle R^H,\Delta u_{P'}\cdot e_1\rangle=\eta_{P'}\,.$$

Since the restriction of $\eta_{P'}$ to any line parallel to $e_{P'}$ consists of two opposite bumps, the support of $u_{P'}$ is contained in the convex hull of $B(x_{P'},\delta\ell(P'))$ and $B(y_{P'},\delta\ell(P'))$. Also, since

$$\|\nabla^j\eta_{P'}\|_{L^\infty}\leqslant C(j)[\delta\ell(P')]^{-j}$$

and since $\mathrm{supp}\eta_{P'}$ intersects any line parallel to $e_{P'}$ over two intervals of total length $4\delta\ell(P')$ or less, we have

$$|\nabla^j u_{P'}(x)|\leqslant\int_{-\infty}^0|(\nabla^j\eta_{P'})(x+te_{P'})|\,dt\leqslant\frac{C(j)}{[\delta\ell(P')]^{j-1}}$$

for all $j\geqslant 0$. Define the vector fields

$$\psi_{P'}=(\Delta u_{P'})e_{P'},\qquad \psi=\sum_{P'\in\mathfrak{P}'}\psi_{P'}\,.$$

Vector function ψ is built. Properties of ψ. Then, clearly, $(R^H)(\psi m) = \langle R^H, \psi m\rangle = \eta$ and all claims below are satisfied ($m := m_{d+1}$):

- $\psi = \sum_{P'\in\mathfrak{P}'} \psi_{P'}$, $\mathrm{supp}\psi \subset S$, $\mathrm{dist}(\mathrm{supp}\psi, L) \geqslant \Delta\ell(Q) = \varepsilon^3\ell(Q)$.
- $\psi_{P'}$ is supported in the $2\ell(P')$-neighborhood of P' and satisfies
$$\int \psi_{P'} = 0, \quad \|\psi_{P'}\|_{L^\infty} \leqslant \frac{C}{\delta\ell(P')}, \quad \|\psi_{P'}\|_{\mathrm{Lip}} \leqslant \frac{C}{\delta^2\ell(P')^2}\,.$$
- $\int |\psi|\, dm \leqslant C\delta^{-1}\mu(Q)$.
- $(R^H)^*(\psi m) = \eta$.
- $\|T^*(\psi m)\|_{L^\infty(\mathrm{supp}\nu)} \leqslant C\alpha\delta^{-2}\varepsilon^{-3d-3}$.
- $\|\widetilde{R}^H(|\psi| m)\|_{L^2(\nu)} \leqslant C\delta^{-1}\sqrt{\mu(Q)}\,.$

References

[Carl] L. Carleson, Selected Problems on Exceptional Sets, Van Nostrand, 1967.
[Ch] M. Christ, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral, Colloq. Math. 60/61, 1990, pp. 601-628.
[D] G. David, Wavelets and singular integrals on curves and surfaces, Lecture Notes in Mathematics, vol. 1465, Springer-Verlag, 1991.
[D1] G. David, Unrectifiable 1-sets have vanishing analytic capacity, Revista Mat. Iberoamericana, 14(2), 1998, pp. 369-479.
[Da4] G. David, *Wavelets and singular integrals on curves and surfaces,* Lecture Notes in Math. 1465, Springer-Verlag, Berlin, 1991.
[DM] G. David and P. Mattila. Removable sets for Lipschitz harmonic functions in the plane. Rev. Mat. Iberoamericana 16(1) (2000), 137–215.
[DS] G. David, S. Semmes, Analysis of and on uniformly rectifiable sets, Mathematical Surveys and Monographs, Volume 38, 1993, AMS, Providence, RI.
[ENV] V. Eiderman, F. Nazarov, A. Volberg, The s-Riesz transform of an s-dimensional measure in $\mathbb{R}^2$ is unbounded for $1 < s < 2$, available from http://arxiv.org/abs/1109.2260.
[Fa] H. Farag, The Riesz kernels do not give rise to higher-dimensional analogues of the Menger-Melnikov curvature, Publ. Mat. 43 (1999), no. 1, 251–260.
[Fe] H. Federer, Geometric Measure Theory, Springer 1969.
[HMM] S. Hofmann, J. M. Martell, S. Mayboroda, Uniform rectifiability and harmonic measure III: Riesz transform bounds imply uniform rectifiability of boundaries of 1-sided NTA domains, http://arxiv.org/abs/1207.1527.
[PJ] P.W. Jones, Rectifiable sets and the traveling salesman problem, Invent. Math. 102 (1990), 1–15.
[Law] Lawler, E.L.: The Traveling Salesman Problem. New York: Wiley-lnterscience, 1985.
[Le] J. C. Léger, Menger curvature and rectifiability, Ann. of Math. 149 (1999), 831–869.

[KO] K. Okikiolu, Characterization of subsets of rectifiable sets in $\mathbb{R}^n$, J. London Math. Soc., (2) 46 (1992), pp. 336–348.

[Ma] P. Mattila. *Geometry of sets and measures in Euclidean spaces,* Cambridge Stud. Adv. Math. 44, Cambridge Univ. Press, Cambridge, 1995.

[MMV] P. Mattila, M. Melnikov, J. Verdera, The Cauchy integral, analytic capacity, and uniform rectifiability, Ann. of Math. (2) 144, 1996, pp. 127-136.

[MPa] P. Mattila and P.V. Paramonov. On geometric properties of harmonic Lip1-capacity, Pacific J. Math. 171:2 (1995), 469490.

[NToV1] F. Nazarov, X. Tolsa and A. Volberg, On the uniform rectifiability of AD-regular measures with bounded Riesz transform operator: the case of codimension 1, arXiv:1212.5229, to appear in Acta Math.

[NToV2] F. Nazarov, X. Tolsa and A. Volberg, The Riesz transform, rectifiability, and removability for Lipschitz harmonic functions, arXiv:1212.5431, to appear in Publ. Mat.

[NTV] F. Nazarov, S. Treil and A. Volberg, The Tb-theorem on non- homogeneous spaces that proves a conjecture of Vitushkin. Preprint of 2000 vailable at www.crm.cat/Paginas/Publications/02/Pr519.pdf or arXiv:1401.2479.

[NTrV1] F. Nazarov, S. Treil and A. Volberg, Weak type estimates and Cotlar inequalities for Calderón-Zygmund operators in nonhomogeneous spaces, Int. Math. Res. Notices **9** (1998), 463–487.

[Paj] H. Pajot, Théorème de recouvrement par des ensembles Ahlfors-réguliers et capacité analytique. (French) C. R. Acad. Sci. Paris Sér. I Math. 323 (1996), no. 2, 133–135.

[Pr] L. Prat, Potential theory of signed Riesz kernels: capacity and Hausdorff measure. Int. Math. Res. Not. 2004, no. 19, 937981.

[PS] Preparata, F.P., Shamos, M.I.: Computational Geometry. Berlin Heidelberg New York: Springer 1985.

[RS] Subsets of Rectifiable curves in Hilbert Space-The Analyst's TSP, arXiv:math/0602675, J. Anal. Math. 103 (2007), 331375.

[T1] X. Tolsa, Painlevé's problem and the semiadditivity of analytic capacity, Acta Math. 190:1 (2003), 105–149.

[T2] X. Tolsa, Uniform rectifiability, Calderón–Zygmund operators with odd kernels, and quasiorthogonality, Proc. London Math. Soc. 98(2), 2009, pp. 393-426.

[T3] X. Tolsa, Principal values for Riesz transform and rectifiability, J. Func. Anal. vol. 254(7), 2008, pp. 1811-1863.

[T-b] X. Tolsa, *Analytic capacity, the Cauchy transform, and non-homogeneous Calderón-Zygmund theory.* To appear (2012).

[Vo] A. Volberg, *Calderón-Zygmund capacities and operators on nonhomogeneous spaces.* CBMS Regional Conf. Ser. in Math. 100, Amer. Math. Soc., Providence, 2003.

Alexander Volberg, Department of Mathematics, Michigan State University, East Lansing, Michigan, USA