RECTIFIABILITY OF MEASURES WITH BOUNDED RIESZ TRANSFORM OPERATOR: FROM SINGULAR OPERATORS TO GEOMETRIC MEASURE THEORY

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1. Lecture 6: The abundance of flat cells

We want to show the abundance of flat cells. First we will show the abundance of geometrically flat cells.

The word "abundance" will be used in a very concrete sense. This word will **almost** mean that the cells which are not flat can be only rare (Carleson). But in fact, abundance means that we relax the meaning of pleasure, in what follows we will be glad and it will be pleasant to us if we meet any cell (not geometrically flat for that matter) such that a fixed number of generations down one of its descendant cell is geometrically flat. "Abundance" will mean that unpleasant cells are rare (Carleson).

So, fix $A' > 1, \alpha' \in (0,1), \beta > 0$ to be chosen later. We want to show first that if $N > N_0(A', \alpha', \beta)$, then there exists a Carleson family $\mathcal{F}_1 \subset \mathcal{D}$ and a finite set $\mathcal{H}$ of linear hyperplanes such that every cell $P \in \mathcal{D} \setminus \mathcal{F}_1$ contains a **geometrically** $(H, 5A', \alpha')$**-flat** cell $Q \subset P$ at most N levels down from P for some linear hyperplane $H \in \mathcal{H}$ that may depend on P.

Let $R = \frac{1}{16}\ell(P)$. According to the Geometric Flattening Lemma, we can choose $\rho > 0$ so that either

First Case. There is a scale $\ell > \rho R$ and a point $z \in B(z_P, R - 16[(5A'+5)+\frac{\alpha'}{3}]\ell) \subset P$ such that μ is geometrically $(H', 16(5A'+5), \frac{\alpha'}{3})$-flat at z on the scale ℓ for some linear hyperplane H',

Second case. There exist $\Delta \in (0, \frac{1}{2})$, $\delta \in (\rho, \Delta)$ and a point $z \in B(z_P, (1-2\Delta)R)$ with $\operatorname{dist}(z, \operatorname{supp}\mu) < \frac{\delta}{4}R$ such that $|[R(\psi_{z,\delta R,\Delta R}\mu)](z)| > \beta$, where $\psi_{z,\delta R,\Delta R}$ is

$$\psi_{z,r,R}(x) = \psi_0\left(\frac{|x-z|}{R}\right) - \psi_0\left(\frac{|x-z|}{r}\right).$$

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In the first case, take any point $z' \in \operatorname{supp}\mu$ such that $|z-z'| < \frac{\alpha'}{3}\ell$ and choose the cell Q with $\ell(Q) \in [\ell, 16\ell)$ that contains z'. Since $z' \subset B(z_P, R) \subset P$ and $\ell(Q) < \ell(P)$, we must have $Q \subset P$. Also, since $|z_Q - z'| \leqslant 4\ell(Q)$, we have $|z - z_Q| < 4\ell(Q) + \frac{\alpha'}{3}\ell < 5\ell(Q)$.

Note now that, if μ is geometrically $(H, 16A, \alpha)$-flat at z on the scale ℓ, then it is geometrically (H, A, α)-flat at z on every scale $\ell' \in [\ell, 16\ell)$.

Note also that the geometric flatness is a stable condition with respect to shifts of the point and rotations of the plane.

Applying these observations with $\ell' = \ell(Q)$, $z' = z_Q$, $\varepsilon = \frac{\alpha'}{3A}$, and choosing any finite ε-net Y on the unit sphere, we see that μ is geometrically $(H, 5A', \alpha')$-flat at z_Q on the scale $\ell(Q)$ with some H whose unit normal belongs to Y. Note also that the number of levels between P and Q in this case is

$$\log_{16} \frac{\ell(P)}{\ell(Q)} \leqslant \log_{16} \rho^{-1} + C\,.$$

Explanation of shifting and rotating. More precisely, if μ is geometrically $(H', A+5, \alpha)$-flat at z on the scale ℓ, then it is geometrically $(H, A, 2\alpha + A\varepsilon)$-flat at z' on the scale ℓ for every $z' \in B(z, 5\ell) \cap \operatorname{supp}\mu$ and every linear hyperplane H with unit normal vector n such that the angle between n and the unit normal vector n' to H' is less than ε. To see it, it is important to observe first that, despite the distance from z to z' may be quite large, the distance from z' to the affine hyperplane L' containing z and parallel to H' can be only $\alpha\ell$, so we do not need to shift L' by more than this amount to make it pass through z'. Combined with the inclusion $B(z', A\ell) \subset B(z, (A+5)\ell)$, this allows us to conclude that μ is $(H', A, 2\alpha)$-flat at z' on the scale ℓ. After this shift, we can rotate the plane L' around the $(d-1)$-dimensional affine plane containing z' and orthogonal to both n and n' by an angle less than ε to make it parallel to H. Again, no point of $L' \cap B(z, A\ell)$ will move by more than $A\varepsilon\ell$ and the desired conclusion follows.

If the **Second case** stated at page 1 of this lecture in fact happens, then there is a point $z \in B(z_P, (1-2\Delta)R)$ and a point $z' \in \operatorname{supp}\mu$, such that

$$|[R(\psi_{z,\delta R,\Delta R}\mu)](z)| > \beta\,, |z - z'| < \frac{\delta}{4}R\,,$$

where

$$\psi_{z,r,R}(x) = \psi_0\left(\frac{|x-z|}{R}\right) - \psi_0\left(\frac{|x-z|}{r}\right)\,.$$

Let now Q and Q' be the largest cells containing z' under the restrictions that $\ell(Q) < \frac{\Delta}{32}R$ and $\ell(Q') < \frac{\delta}{32}R$. Since both bounds are less than $\ell(P)$ and the first one is greater than the second one, we have $Q' \subset Q \subset P$.

Now we want to show that the difference of averages of $R_\mu \chi_E$ over Q and Q' is at least $\beta - C$ in absolute value for every set $E \supset B(z, 2R)$ and, hence, for every set $E \supset B(z_P, 5\ell(P))$. Here C depends only on the norm of operator R_μ in $L^2(\mu)$.

Estimate of the difference of averages $\langle R_\mu \chi_E\rangle_Q - \langle R_\mu \chi_E\rangle_{Q'}$. The function $\psi_{z,\delta R,\Delta R}$ is roughly characteristic function of the annulus. We can write $\chi_E = \psi_{z,\delta R,\Delta R} + f_1 + f_2$ where $|f_1|, |f_2| \leqslant 1$ and $\operatorname{supp} f_1 \subset \bar{B}(z, 2\delta R)$, $\operatorname{supp} f_2 \cap B(z, \Delta R) = \varnothing$. So

$$\int |R_\mu f_1|^2 \, d\mu \leqslant C \int |f_1|^2 \, d\mu \leqslant C(\delta R)^d \leqslant C\ell(Q')^d \leqslant C\mu(Q') .$$

Hence$|\langle R_\mu f_1\rangle_Q|, |\langle R_\mu f_1\rangle_{Q'}|$ are bounded by some C.

Note also that $Q \subset B(z', 8\ell(Q)) \subset B(z', \frac{\Delta}{4}R) \subset B(z, \frac{\Delta}{2}R)$, so the distance from Q to $\operatorname{supp} f_2$ is at least $\frac{\Delta}{2}R > \ell(Q)$. Thus,

$$\|R_\mu f_2\|_{\mathrm{Lip}(Q)} \leqslant C\ell(Q)^{-1}$$

the difference of the averages of $R_\mu f_2$ over Q and Q' is bounded by some C.

We are left to to estimate the difference of averages $\langle R_\mu \psi_{z,\delta R,\Delta R}\rangle_Q - \langle R_\mu \psi_{z,\delta R,\Delta R}\rangle_{Q'}$. For this

$$\|R_\mu \psi_{z,\delta R,\Delta R}\|^2_{L^2(\mu)} \leqslant C\|\psi_{z,\delta R,\Delta R}\|^2_{L^2(\mu)} \leqslant C(\Delta R)^d \leqslant C\ell(Q)^d \leqslant C\mu(Q) ,$$

so the average over Q is bounded by a constant.

On the other hand,

$$Q' \subset B(z', 8\ell(Q')) \subset B(z', \frac{\delta}{4}R) \subset B(z, \frac{\delta}{2}R) .$$

Again $\operatorname{dist}(\operatorname{supp} \psi_{z,\delta R,\Delta R}, Q') \geqslant \frac{\delta}{2}R$. Therefore,

$$\|R(\psi_{z,\delta R,\Delta R}\mu)\|_{\mathrm{Lip}(B(z,\frac{\delta}{2}R))} \leqslant C(\delta R)^{-1} .$$

But this means that $|\langle R_\mu \psi_{z,\delta R,\Delta R}\rangle_{Q'} - R_\mu \psi_{z,\delta R,\Delta R}(z)| \leqslant C(\delta)$.

The quantity $|R_\mu \psi_{z,\delta R,\Delta R}(z)|$ is large! It is bigger than β.

We finally get

$$|\langle R_\mu \chi_E\rangle_Q - \langle R_\mu \chi_E\rangle_{Q'}| \geqslant \beta - 3C .$$

This conclusion can be rewritten as

$$\mu(P)^{-\frac{1}{2}}|\langle R_\mu \chi_E, \psi_P\rangle_\mu| \geqslant c\rho^{\frac{d}{2}}(\beta - C) \tag{1}$$

where

$$\psi_P = [\rho\ell(P)]^{\frac{d}{2}}\left(\frac{1}{\mu(Q)}\chi_Q - \frac{1}{\mu(Q')}\chi_{Q'}\right). \tag{2}$$

This is for any $E, B(z_P, 5\ell(P)) \subset E$.

Let us show that the set of cells satisfying the latter property can be only **rare**, that is **a Carleson family**.

Carlesonness of cells P satisfying the Second case of page 1 of this lecture.

Fix any cell P_0 and consider the cells P satisfying the **Second case** of page 1 of this lecture. This means that there exists a point z such that $|R_\mu \psi_{z,\delta R,\Delta R}(z)| > \beta$ with certain position of z inside P, $R \approx \ell(P)$, and large β. Then $B(z_P, 5\ell(P)) \subset B(z_{P_0}, 50\ell(P_0))$. Also we saw in (1) that for such P one has

$$\mu(P) \leqslant C(\rho,\beta)|\langle R_\mu \chi_{B(z_{P_0}, 50\ell(P_0))}, \psi_P\rangle_\mu|^2 . \tag{3}$$

Here ψ_P form **Haar system of depth** $N \approx \log\frac{\ell(P)}{\ell(Q')}$, $\ell(Q') \approx \delta\ell(P), \delta \in (\rho, 1/2)$, so $N \leqslant c\log\frac{1}{\rho}$ (see the definition below).

Any Haar system of depth N is **a Riesz system**. By the property of **Riesz system** (see below) we get that

$$\sum_{P\subset P_0}\mu(P) \leqslant C(\rho,\beta)\sum_{P\subset P_0}|\langle R_\mu \chi_{B(z_{P_0}, 50\ell(P_0))}, \psi_P\rangle_\mu|^2 \leqslant \tag{4}$$

$$C\|R_\mu \chi_{B(z_{P_0}, 50\ell(P_0))}\|^2_\mu . \tag{5}$$

The latter is smaller than $C\mu(B(z_{P_0}, 50\ell(P_0))) \leqslant C'\mu(P_0)$, and we established Carleson property of P's as above.

Abundance of geometrically flat cells is already obtained up to the understanding "what is the Riesz system and what is Haar systems of depth N, and why the latter is an example of the former."

In fact, fix $A, \alpha > 0$. We shall say that a cell $Q \in \mathcal{D}$ is (geometrically) (H, A, α)-flat if the measure μ is (geometrically) (H, A, α)-flat at z_Q on the scale $\ell(Q)$.

We have just shown (modulo estimates of Riesz system that follows) that there exists an integer N, a finite set $\mathcal{H}$ of linear hyperplanes in $\mathbb{R}^{d+1}$, and a Carleson family $\mathcal{F} \subset \mathcal{D}$ (depending on A, α) such that for every cell $P \in \mathcal{D} \setminus \mathcal{F}$, there exist $H \in \mathcal{H}$ and a **geometrically** (H, A, α)**-flat** cell $Q \subset P$ that is at most N levels down from P.

Riesz systems: Haar system of a given depth, Lipschitz wavelet system.

Let ψ_Q $(Q \in D)$ be a system of $L^2(\mu)$ functions.

Definition 1. The functions ψ_Q form a Riesz family with Riesz constant $C > 0$ if

$$\left\| \sum_{Q\in\mathcal{D}} a_Q \psi_Q \right\|^2_{L^2(\mu)} \leqslant C \sum_{Q\in\mathcal{D}} a_Q^2$$

for any real coefficients a_Q.

Note that if the functions ψ_Q form a Riesz family with Riesz constant C, then for every $f \in L^2(\mu)$, we have

$$\sum_{Q\in\mathcal{D}} |\langle f, \psi_Q\rangle_\mu|^2 \leqslant C\|f\|^2_{L^2(\mu)}\,.$$

Assume next that for each cell $Q \in D$ we have a set Ψ_Q of $L^2(\mu)$ functions associated with Q.

Definition 2. The family Ψ_Q $(Q \in \mathcal{D})$ of sets of functions is a Riesz system with Riesz constant $C > 0$ if for every choice of functions $\psi_Q \in \Psi_Q$, the functions ψ_Q form a Riesz family with Riesz constant C.

Riesz systems are useful because of the following Lemma.

Lemma 1. *Suppose R_μ is bounded in $L^2(\mu)$. Suppose that Ψ_Q is any Riesz system. Fix $A > 1$. For each $Q \in \mathcal{D}$, define*

$$\xi(Q) = \inf_{E: B(z_Q, A\ell(Q))\subset E, \mu(E)<+\infty} \sup_{\psi\in\Psi_Q} \mu(Q)^{-1/2} |\langle R_\mu \chi_E, \psi\rangle_\mu|\,.$$

Then $\forall \delta > 0, \mathcal{F}_\delta := \{Q \in \mathcal{D} : \xi(Q) \geqslant \delta\}$ is Carleson.

Proof. Fix a cell P_0. Then $E = B(z_{P_0}, (50A + 50)\ell(P_0))$ satisfies $B(z_P, A\ell(P)) \subset B(z_{P_0}, (50A + 50)\ell(P_0))$. Therefore,

$$\sum_{P\subset P_0} \mu(P) \leqslant \delta^{-2} \sum_{P\subset P_0} |\langle R_\mu \chi_{B(z_{P_0},(50A+50)\ell(P_0))}, \psi_P\rangle_\mu|^2 \leqslant$$
$$C\|R_\mu \chi_{B(z_{P_0},(50A+50)\ell(P_0))}\|^2_\mu \leqslant C\mu(P_0)\,.$$

□

It is important that there are two natural classes of Riesz systems: **Haar systems of fixed depth** $\Psi^H(N)$, and **Lipschitz wavelet systems** $\Psi^L(A)$.

Let now N be any positive integer. For each $Q \in \mathcal{D}$, define the set of Haar functions $\Psi^h_Q(N)$ of depth N as the set of all functions ψ that are supported on Q, are constant on every cell $Q' \in \mathcal{D}$ that is N levels down from Q, and satisfy $\int \psi\, d\mu = 0$, $\int \psi^2\, d\mu \leqslant C$. The Riesz property follows immediately from the fact that $\mathcal{D}$ can be represented as a finite union of the sets $\mathcal{D}^{(j)} = \cup_{k:k\equiv j \mod N}\mathcal{D}_k$ ($j = 0, \dots, N-1$) and that for every choice of $\psi_Q \in \Psi^h_Q(N)$, the functions ψ_Q corresponding to the cells Q from a fixed $\mathcal{D}^{(j)}$ form a bounded orthogonal family.

Our $\psi_P = [\rho\ell(P)]^{\frac{d}{2}}\left(\frac{1}{\mu(Q)}\chi_Q - \frac{1}{\mu(Q')}\chi_{Q'}\right)$ from (2) are obviously from $\Psi^H(N)$ with $N \leqslant c \log\frac{1}{\rho}$, so the second inequality in (4) is done, and **the abundance of geometrically flat cells is completely established**.

Lipschitz wavelet systems. We will need then to establish the abundance of (H, A, α) flat (not just geometrically flat) cells.

In the Lipschitz wavelet system, the set $\Psi^\ell_Q(A)$ consists of all Lipschitz functions ψ supported on $B(z_Q, A\ell(Q))$ such that $\int \psi\, d\mu = 0$ and $\|\psi\|_{\text{Lip}} \leqslant C\ell(Q)^{-\frac{d}{2}-1}$. Since μ is nice, we automatically have $\int |\psi|^2\, d\mu \leqslant C(A)\ell(Q)^{-d}\mu(Q) \leqslant C(A)$ in this case.

If $Q, Q' \in \mathcal{D}$ and $\ell(Q') \leqslant \ell(Q)$, then, for any two functions $\psi_Q \in \Psi^\ell_Q(A)$ and $\psi_{Q'} \in \Psi^\ell_{Q'}(A)$, we can have $\langle\psi_Q, \psi_{Q'}\rangle_\mu \neq 0$ only if $B(z_Q, A\ell(Q)) \cap B(z_{Q'}, A\ell(Q')) \neq \varnothing$, in which case,

$$|\langle\psi_Q, \psi_{Q'}\rangle_\mu| \leqslant \|\psi_Q\|_{\text{Lip}}\text{diam}(Q')\|\psi_{Q'}\|_{L^1(\mu)} \leqslant C(A)\left[\frac{\ell(Q')}{\ell(Q)}\right]^{\frac{d}{2}+1}.$$

Then:

$$\begin{aligned}
\left\|\sum_{Q\in\mathcal{D}} a_Q\psi_Q\right\|^2_{L^2(\mu)} &\leqslant 2\sum_{Q,Q'\in\mathcal{D},\,\ell(Q')\leqslant\ell(Q)} |a_Q|\cdot|a_{Q'}|\cdot|\langle\psi_Q,\psi_{Q'}\rangle_\mu| \\
&\leqslant C(A)\sum_{\substack{Q,Q'\in\mathcal{D},\,\ell(Q')\leqslant\ell(Q)\\ B(z_Q,A\ell(Q))\cap B(z_{Q'},A\ell(Q'))\neq\varnothing}} \left[\frac{\ell(Q')}{\ell(Q)}\right]^{\frac{d}{2}+1}|a_Q|\cdot|a_{Q'}| \\
&\leqslant C(A)\sum_{\substack{Q,Q'\in\mathcal{D},\,\ell(Q')\leqslant\ell(Q)\\ B(z_Q,A\ell(Q))\cap B(z_{Q'},A\ell(Q'))\neq\varnothing}} \left\{\left[\frac{\ell(Q')}{\ell(Q)}\right]^{d+1}|a_Q|^2 + \frac{\ell(Q')}{\ell(Q)}|a_{Q'}|^2\right\}.
\end{aligned}$$

$$\sum_{\substack{Q'\in\mathcal{D}:\,\ell(Q')\leqslant\ell(Q)\\ B(z_Q,A\ell(Q))\cap B(z_{Q'},A\ell(Q'))\neq\varnothing}}\left[\frac{\ell(Q')}{\ell(Q)}\right]^{d+1}\leqslant C,\quad \sum_{\substack{Q\in\mathcal{D}:\,\ell(Q')\leqslant\ell(Q)\\ B(z_Q,A\ell(Q))\cap B(z_{Q'},A\ell(Q'))\neq\varnothing}}\frac{\ell(Q')}{\ell(Q)}\leqslant C\cdot\frac{\ell(Q')}{\ell(Q)}\,.$$

We recall Flattening Lemma from Lecture 5, that says how to get flat cell if it is already geometrically flat.

Lemma 2. *Fix four positive parameters $A,\alpha,\widetilde{c},\widetilde{C}$. $\exists A',\alpha'>0$ depending on $A,\alpha,\widetilde{c},\widetilde{C}$ and d such that: if H is a linear hyperplane in $\mathbb{R}^{d+1}$, $z\in\mathbb{R}^{d+1}$, L is the affine hyperplane containing z and parallel to H, $\ell>0$, and μ is a $\widetilde{C}$-good finite measure in $\mathbb{R}^{d+1}$ that is AD regular in $B(z,5A'\ell)$ with the lower regularity constant $\widetilde{c}$. Assume that μ is geometrically $(H,5A',\alpha')$-flat at z on the scale ℓ and, in addition, for every (vector-valued) Lipschitz function g with $\operatorname{supp} g\subset B(z,5A'\ell)$, $\|g\|_{\mathrm{Lip}}\leqslant\ell^{-1}$, and $\int g\,d\mu=0$, one has*

$$|\langle R^H_\mu 1,g\rangle_\mu|\leqslant\alpha'\ell^d\,.$$

Then μ is (H,A,α)-flat at z on the scale ℓ.

We already saw that all cells P (except for a rare (Carleson) family $\mathcal{F}_1$) are such that not more than N generation down inside P a cell Q lies, which is (H,A',α')-geometrically flat, where A',α' depend on A,α as Flattening Lemma requires, and $H=H_Q$ belongs to $\mathcal{H}$, a finite family (cardinality of it depends on A,α too).

Given P, we find such Q, and Flattening Lemma applied to any $\mu:=\mu\cdot 1_E, E\supset B(z_Q,100A'\ell(Q))$, shows that either Q is (H,A,α)-flat, or for each such E there exists a function $g=g_E$ such that it is supported on $B(z_Q,5A'\ell(Q))$, $\int g\,d\mu=0$, Lipschitz with norm at most $1/\ell(Q)\leqslant C(N)/\ell(P)$ and

$$\langle R^H_\mu 1_E,g\rangle\geqslant\alpha'\ell(Q)^d=c(N)\alpha'\ell(P)^d\,.$$

Consider $\psi_P=\psi_{P,E}=g/\ell(P)^{\frac{d}{2}}$. They form a Lipschitz wavelet system $\Psi^L(C)$, as on page 6 of this lecture. Therefore,

$$\xi(P)=\mu(P)^{-\frac12}\inf_{E:\,E\supset B(z_p,C\ell(P))}\ \sup_{\psi\in\Psi^L(C)}|\langle R_\mu 1_E,\psi\rangle|\geqslant C(N)\alpha'\,.$$

We know by Lemma 1 of this lecture that such P can form only a rare (Carleson) family if R_μ is a bounded operator in $L^2(\mu)$. Call it $\mathcal{F}_2$. So by the exception of two rare families $\mathcal{F}_1,\mathcal{F}_2$, any other cell $P\in\mathcal{D}$ will have inside it and not more than N (fixed number depending on A,α) generations down a sub-cell Q, which is (H,A,α)-flat. Here H will be chosen from a finite family $\mathcal{H}$ of hyperplanes (having fixed cardinality depending on A, α).

The abundance of flat cells is completely proved.

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