RECTIFIABILITY OF MEASURES WITH BOUNDED RIESZ TRANSFORM OPERATOR: FROM SINGULAR OPERATORS TO GEOMETRIC MEASURE THEORY

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1. Lecture 5: Geometry of cells. The sparsity of bad cells (a.k.a. the Main Theorem) as the reduction to David–Semmes Result

Let μ be a d-dimensional AD regular measure in $\mathbb{R}^{d+1}$. Let $E = \operatorname{supp}\mu$. Then there exists a family $\mathcal{D}$ of sets $Q \subset \mathbb{R}^{d+1}$ with the following properties:

- The family $\mathcal{D}$ is the union of families $\mathcal{D}_k$ (families of level k cells), $k \in \mathbb{Z}$.
- If $Q', Q'' \in \mathcal{D}_k$, then either $Q' = Q''$ or $Q' \cap Q'' = \varnothing$.
- Each $Q' \in \mathcal{D}_{k+1}$ is contained in some $Q \in \mathcal{D}_k$ (necessarily unique due to the previous property).
- The cells of each level cover E, i.e., $\cup_{Q\in\mathcal{D}_k} Q \supset E$ for every k.
- For each $Q \in \mathcal{D}_k$, there exists $z_Q \in Q \cap E$ (the "center" of Q) such that
$$B(z_Q, 2^{-4k-3}) \subset Q \subset B(z_Q, 2^{-4k+2})\,.$$
- For each $Q \in \mathcal{D}_k$ and every $\varepsilon > 0$, we have
$$\mu\{x \in Q : \operatorname{dist}(x, \mathbb{R}^{d+1} \setminus Q) < \varepsilon 2^{-4k}\} \leqslant C\varepsilon^{\gamma}\mu(Q);$$
$$C = C(d, reg),\ \gamma = \gamma(d, reg)\,.$$

Since all cells in $\mathcal{D}_k$ have approximately the same size 2^{-4k}, it will be convenient to introduce the notation $\ell(Q) = 2^{-4k}$ where k is the unique index for which $Q \in \mathcal{D}_k$.

Let Z_k be a maximal 2^{-4k}-separated set in $E = \operatorname{supp}\mu$. Then $\{B(z, 2^{-4k})\}_{z\in Z_k}$, cover E. For each $z \in Z_k$ consider the Voronoi cell
$$V_z := \{x \in \mathbb{R}^{d+1} \colon |x - z| = \min_{z'\in Z_k} |x - z'|\}\,.$$

Then 1) $V_z \subset B(z, 2^{-4k})$, 2) $\{V_z\}_{z\in Z_k}$ cover E,

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3) $\mathrm{dist}(z, \bigcup_{z'\in Z_k, z'\neq z} V_{z'}) \geqslant 2^{-4k-1}$. The last one because Z_k is 2^{-4k}-separated, the first one because Z_k is maximal such. Also

4) There are only finitely many $w \in Z_{k-1}$ such that $V_z \cap V_w \neq \emptyset$.

We say that $w \in Z_k$ is a descendant of $z \in Z_\ell$, $\ell \geqslant k$, if there is a chain $z_k = z, z_\ell = w$, $z_j \in Z_j$ such that $V_{z_j} \cap V_{z_{j+1}} \neq \emptyset$. $D(z)$ is the set of all descendants of z and

$$\widetilde{V}_z := \bigcup_{w\in D(z)} V_w\,.$$

Note that $\widetilde{V}_z$ contains V_z and is contained in the $2\sum_{\ell>k} 2^{-4\ell} = \frac{2}{15}2^{-4k}$-neighborhood of V_z. Thus,

$$\mathrm{dist}(z, \cup_{z'\in Z_k\setminus\{z\}} \widetilde{V}_{z'}) \geqslant 2^{-4k-1} - \frac{2}{15}2^{-4k} > 2^{-4k-2}\,. \tag{1}$$

Nobility order. There exists a partial order $\prec$ on $\cup_k Z_k$ such that each Z_k is linearly ordered under $\prec$ and the ordering of Z_{k+1} is consistent with that of Z_k in the sense that if $z', z'' \in Z_{k+1}$ and $z' \prec z''$, then for every $w' \in Z_k$ such that $V_{w'} \cap V_{z'} \neq \varnothing$, there exists $w'' \in Z_k$ such that $V_{w''} \cap V_{z''} \neq \varnothing$ and $w' \preceq w''$. In other words, the ordering we are after is analogous to the classical "nobility order" in the society: comparing maximally "noble" ancestors one generation up defines "nobility".

Put now for each $z \in Z_k$ the cell

$$E_z := \widetilde{V}_z \setminus \bigcup_{z'\in Z_k, z\prec z'} \widetilde{V}_{z'}\,.$$

By (1) we have the left inclusion (the right one is clear too)

$$B(z, 2^{-4k-2}) \subset E_z \subset B(z, 2^{-4k+1}),\ \text{for all } z \in Z_k.$$

Next goal is to show the tiling: that for every $z \in Z_{k+1}$ there exists $w \in Z_k$ such that $E_z \subset E_w$. For a given $z \in Z_{k+1}$ choose w to be the largest in $\prec$ element of Z_k. Let $w' \in Z_k$ be such that $w \prec w'$. Let $z' \in Z_{k+1}, z' \in D(w')$. Automatically $z \prec z'$. And so $\{z' \in Z_{k+1} \colon z' \in D(w')\} \subset \{z' \in Z_{k+1} \colon z \prec z'\}$. On the other hand, by definition $V_{w'} \subset \bigcup_{z'\in Z_{k+1}\colon z'\in D(w')} V_{z'}$, and so

$$\widetilde{V}_{w'} = \bigcup_{z'\in Z_{k+1}\colon z'\in D(w')} \widetilde{V}_{z'} \subset \bigcup_{z'\in Z_{k+1}, z\prec z'} \widetilde{V}_{z'}$$

$$\widetilde{V}_z \setminus \bigcup_{z'\in Z_{k+1}, z\prec z'} \widetilde{V}_{z'} \subset \widetilde{V}_w \setminus \bigcup_{w'\in Z_k, w\prec w'} \widetilde{V}_{w'}$$

This is exactly $E_z \subset E_w$.

The notion of sparsity via Carleson measure condition.

For us this will be the right notion of sparse, rare family of cells.

From now on, we will fix a good AD regular in the entire space $\mathbb{R}^{d+1}$ measure μ and a David-Semmes lattice $\mathcal{D}$ associated with it.

Definition 1. A family $\mathcal{F} \subset \mathcal{D}$ is called Carleson with Carleson constant $C > 0$ if for every $P \in \mathcal{D}$, we have

$$\sum_{Q \in \mathcal{F}_P} \mu(Q) \leqslant C\mu(P)\,,$$

where

$$\mathcal{F}_P = \{Q \in \mathcal{F} : Q \subset P\}\,.$$

Bad cells: Non-BAUP cells.

We will start with the definition [DS, pg. 139] of a δ-non-BAUP cell. In this book there are many abbreviations for various kind of geometric badness of cells, here it stands for "bilateral approximation by unions of d-planes."

Definition 2. Let $\delta > 0$. We say that a cell $P \in \mathcal{D}$ is δ-non-BAUP if there exists a point $x \in P \cap \operatorname{supp} \mu$ such that for every hyperplane L passing through x, there exists a point $y \in B(x, \ell(P)) \cap L$ for which $B(y, \delta\ell(P)) \cap \operatorname{supp} \mu = \varnothing$.

Note that in this definition the plane L can go in any direction. In what follows, we will need only planes parallel to certain H but, since H is determined by the flatness direction of some unknown cell P, we cannot fix the direction of the plane L in the definition of non-BAUPness from the very beginning.

Theorem 1 (David–Semmes)**.** *Let μ be AD-regular. If for all $\delta > 0$ the family of δ-non-BAUP cells is a Carleson family, then μ is rectifiable.*

The Main Theorem

Theorem 2. *Let μ be an AD regular measure of dimension d in $\mathbb{R}^{d+1}$. If the associated d-dimensional Riesz transform operator*

$$f \mapsto R * (f\mu), \quad \text{where } R(x) = \frac{x}{|x|^{d+1}}\,,$$

is bounded in $L^2(\mu)$, then the non-BAUP cells in the David-Semmes lattice associated with μ form a Carleson family.

Proposition 3.18 of David–Semmes [DS] (page 141) asserts that this condition "implies the WHIP and the WTP" and hence, by Theorem 3.9 (pages 137), the uniform rectifiability of μ.

Hence the plan is to prove the main theorem, it reduces everything to the above mentioned Theorem of David–Semmes (whose proof is rather long (see [DS]), and the combination of these two results proves finally the Uniform Rectifiability of $E = \operatorname{supp}\mu$.

The Idea of the proof of the Main Theorem.

Using the boundedness of R_μ in $L^2(\mu)$ we will establish **the abundance of flat cells**. On the other hand, if non-BAUP cells are not rare (not Carleson) they will be also abundant. Then we will be able to build intermitting layers of flat and non-BAUP cells. This will allow us to construct an analog of the vector field ψ of Lecture 4 on non-BAUP scales. This is because non-BAUP cell have **holes** in $\operatorname{supp}\mu$ in it! Flat cells will play the role of the set E of Lecture 4 (which was totally flat). Then Riesz energy concentrated on each flat layer will be sufficiently large (the non-BAUP layer encompassing a flat layer and ψ of this non-BAUP layer ensures that). Then we will need that flat layers are **almost orthogonal**. Adding a huge amount of not-so-small Riesz energies we get estimate from below on $\int |R_\mu \mathbf{1}|^2\, d\mu$ as large as we wish. This leads to a contradiction.

The flatness condition and its consequences We shall fix a linear hyperplane $H \subset \mathbb{R}^{d+1}$. Let $z \in \mathbb{R}^{d+1}$, $A, \alpha, \ell > 0$ (we view A as a large number, α as a small number, and ℓ as a scale parameter). We want the measure μ to be close inside the ball $B(z, A\ell)$ to a multiple of the d-dimensional Lebesgue measure m_L on the hyperplane L containing z and parallel to H.

We say that a measure μ is geometrically (H, A, α)-flat at the point z on the scale ℓ if every point of $\operatorname{supp}\mu \cap B(z, A\ell)$ lies within distance $\alpha\ell$ from the affine hyperplane L containing z and parallel to H and every point of $L \cap B(z, A\ell)$ lies within distance $\alpha\ell$ from $\operatorname{supp}\mu$. We say that a measure μ is (H, A, α)-flat at the point z on the scale ℓ if it is geometrically (H, A, α)-flat at the point z on the scale ℓ and, in addition, for every Lipschitz function f supported on $B(z, A\ell)$ such that $\|f\|_{\mathrm{Lip}} \leqslant \ell^{-1}$ and $\int f\, dm_L = 0$, we have

$$\left| \int f\, d\mu \right| \leqslant \alpha \ell^d .$$

Note that the geometric (H, A, α)-flatness is a condition on $\operatorname{supp}\mu$ only. It doesn't tell one anything about the distribution of the measure μ on its support. The latter is primarily controlled by the second, analytic, condition in the full (H, A, α)-flatness. These two conditions are not completely independent: if, say, μ is AD regular, then the analytic condition implies the geometric one with slightly worse parameters. However, it will be convenient for us just to demand them separately.

The flatness means the possibility of mass transporting $\mu \,|\, B(z, A\ell)$ to $c \cdot m_L \,|\, B(z, A\ell)$ with small cost α.

Flatness allows to switch integration over μ to that over $c \cdot m_L$. Below are technical but very useful lemmas estimating the error of such switching.

Two important elementary widely used technical lemmas.

Lemma 1. *Let μ be a nice measure (estimate from above). Assume that μ is (H, A, α)-flat at z on scale ℓ with some $A > 5$, $\alpha \in (0, 1)$. Let φ be any non-negative Lipschitz function supported on $B(z, 5\ell)$ with $\int \varphi \, dm_L > 0$. Put*

$$a = \left(\int \varphi \, dm_L \right)^{-1} \int \varphi \, d\mu, \qquad \nu = a\varphi m_L \,.$$

Let Ψ be any function with $\|\Psi\|_{\mathrm{Lip}(\operatorname{supp}\varphi)} < +\infty$. Then

$$\left| \int \Psi \, d(\varphi\mu - \nu) \right| \leqslant C\alpha\ell^{d+2} \|\Psi\|_{\mathrm{Lip}(\operatorname{supp}\varphi)} \|\varphi\|_{\mathrm{Lip}} \,.$$

As a corollary, for every $p \geqslant 1$, we have

$$\left| \int |\Psi|^p \, d(\varphi\mu - \nu) \right| \leqslant C(p)\alpha\ell^{d+2} \|\Psi\|^{p-1}_{L^\infty(\operatorname{supp}\varphi)} \|\Psi\|_{\mathrm{Lip}(\operatorname{supp}\varphi)} \|\varphi\|_{\mathrm{Lip}} \,.$$

Lemma 2. *Assume in addition to the conditions of Lemma 1 that $\varphi \in C^2$, and that the ratio of integrals a is bounded from above by some known constant. Then*

$$\left| \int \Psi\varphi[R^H(\varphi\mu - \nu)] \, d\mu \right| \leqslant C\alpha^{\frac{1}{d+2}} \ell^{d+2} \left[\|\Psi\|_{L^\infty(\operatorname{supp}\varphi)} + \ell \|\Psi\|_{\mathrm{Lip}(\operatorname{supp}\varphi)} \right] \|\varphi\|^2_{\mathrm{Lip}} \,.$$

where $C > 0$ may, in addition to the dependence on d, which goes without mentioning, depend also on the growth constant of μ and the upper bound for a.

Disclaimer: The integral should be understood first. Split it as $\int \Psi\varphi[R^H(\varphi\mu)]\,d\mu - \int \Psi\varphi[R^H\nu]\,d\mu$. Then $R^H\nu = a\,R^H(\varphi\,dm_L)$ and so is a smooth function as φ is smooth. The first term should be understood as a form by using anti-symmetry of R^H.

The first lemma is just by definition. In the second Lemma choose $\delta = \alpha^{\frac{1}{d+2}}$ and split $R^H = R^H_{\delta\ell} + R^{H,\delta\ell}$. Then

$$\int \Psi\varphi[R^H_{\delta\ell}(\varphi\mu - \nu)]\,d\mu = -\int R^H_{\delta\ell}(\Psi\varphi\,d\mu)\,d(\varphi\mu-\nu)$$

is estimated by the first Lemma using $\|R^H_{\delta\ell}\|_{\mathrm{Lip}} \leqslant \delta^{-(d+1)}\ell^{-(d+1)}$ and $\|R^H_{\delta\ell}(\Psi\varphi\,d\mu)\|_{\mathrm{Lip}} \leqslant \|R^H_{\delta\ell}\|_{\mathrm{Lip}}\|\Psi\varphi\|_{L^1(\mu)}$.

The short range term $\int \Psi\varphi[R^{H,\delta\ell}(\varphi\mu-\nu)]\,d\mu$ essentially reduces to estimate:

$$\frac{1}{2}\Big|\iint_{|x-y|\leqslant\delta\ell} R^H(x-y)(\Psi(x)-\Psi(y))\varphi(x)\varphi(y)\,d\mu(x)\,d\mu(y)\Big| \leqslant$$
$$\leqslant \frac{1}{2}\|\Psi\|_{\mathrm{Lip}(\mathrm{supp}\,\varphi)}\|\varphi\|^2_{L^\infty}\iint_{x,y\in\mathrm{supp}\varphi,|x-y|<\delta\ell}\frac{d\mu(x)\,d\mu(y)}{|x-y|^{d-1}} \leqslant$$
$$\leqslant C\delta\ell^{d+3}\|\Psi\|_{\mathrm{Lip}(\mathrm{supp}\,\varphi)}\|\varphi\|^2_{\mathrm{Lip}}\,.$$

Geometric Flattening Lemma.

We are heading to the proof that the boundedness of R_μ in $L^2(\mu)$ implies flatness of abundant family of cells. The first step is the following analysis-to-geometry Lemma. Fix some continuous function $\psi_0 : [0,+\infty) \to [0,1]$ such that $\psi_0 = 1$ on $[0,1]$ and $\psi_0 = 0$ on $[2,+\infty)$. For $z \in \mathbb{R}^{d+1}$, $0 < r < R$, define

$$\psi_{z,r,R}(x) = \psi_0\left(\frac{|x-z|}{R}\right) - \psi_0\left(\frac{|x-z|}{r}\right).$$

Lemma 3 (Geometric Flattening Lemma)**.** *Fix five positive parameters $A, \alpha, \beta, \widetilde{c}, \widetilde{C} > 0$. There exists $\rho > 0$ depending only on these parameters and the dimension d such that the following implication holds.*

Suppose that μ is a $\widetilde{C}$-good measure on a ball $B(x,R)$ centered at a point $x \in \mathrm{supp}\,\mu$ that is AD regular in $B(x,R)$ with lower regularity constant $\widetilde{c}$. Suppose also that

$$|[R(\psi_{z,\delta R,\Delta R}\mu)](z)| \leqslant \beta$$

for all $\rho < \delta < \Delta < \frac{1}{2}$ and all $z \in B(x,(1-2\Delta)R)$ such that $\mathrm{dist}(z,\mathrm{supp}\,\mu) < \frac{\delta}{4}R$.

Then there exist a scale $\ell > \rho R$, a point $z \in B(x, R-(A+\alpha)\ell)$, and a linear hyperplane H such that μ is geometrically (H, A, α)-flat at z on the scale ℓ.

Replacing μ by $R^{-d}\mu(x+R\cdot)$ if necessary, we may assume without loss of generality that $x = 0, R = 1$.

The absence of geometric flatness and also **the boundedness** of $[R(\psi_{z,\delta,\Delta}\mu)](z)$ are inherited by weak limits. More precisely, let ν_k be a sequence of $\widetilde{C}$-good measures on $B(0,1)$ and AD-regular there with lower regularity constant $\widetilde{c}$. Assume that ν is another measure on $B(0,1)$ and $\nu_k \to \nu$ weakly in $B(0,1)$.

Lemma 4.

- *If for some $A' > A$ and $0 < \alpha' < \alpha$, the measure ν is geometrically (H, A', α')-flat on the scale $\ell > 0$ at some point $z \in B(0, 1-(A'+\alpha)\ell)$, then for all sufficiently large k, the measure ν_k is geometrically (H, A, α)-flat at z on the scale ℓ.*
- *If for some $0 < \delta < \Delta < \frac{1}{2}$ and some $z \in B(0, 1-2\Delta)$ with $\mathrm{dist}(z, \mathrm{supp}\nu) < \frac{\delta}{4}$, we have $|[R(\psi_{z,\delta,\Delta}\nu)](z)| > \beta$, then for all sufficiently large k, we also have $\mathrm{dist}(z, \mathrm{supp}\nu_k) < \frac{\delta}{4}$ and $|[R(\psi_{z,\delta,\Delta}\nu_k)](z)| > \beta$.*

So suppose that with fixed 5 constants as above and with smaller and smaller ρ_k we still have μ_k's with **the absence** of geometric flatness and at the same time with **the boundedness** of $R[(\psi_{z,\delta,\Delta}\nu)](z)$, $0 < \rho_k < \delta < \Delta < 1/2$, for all $z \in B(0, 1-2\Delta)$, $\mathrm{dist}(z, \mathrm{supp}\,\mu_k) < \frac{\delta}{4}$ by the same β. Then we can come to a weak limit, and get that this limit μ negates the following **Alternative**.

Alternative. If ν is any good measure on $B(0,1)$ that is AD regular there, then either for every $A, \alpha > 0$ there exist a scale $\ell > 0$, a point $z \in B(0, 1-(A+\alpha)\ell)$ and a linear hyperplane H such that ν is geometrically (H, A, α)-flat at z on the scale ℓ, or

$$\sup_{\substack{0<\delta<\Delta<\frac{1}{2} \\ z\in B(0,1-2\Delta), \mathrm{dist}(z,\mathrm{supp}\nu)<\frac{\delta}{4}}} |[R(\psi_{z,\delta,\Delta}\nu)](z)| = +\infty\,.$$

We are left to prove the Alternative.

Sketching the proof of the Alternative The negation of every of the two condition of the Alternative is inherited by all tangent measures of ν. Since ν is finite and AD regular in $B(0,1)$, its support is nowhere dense in $B(0,1)$. Take any point $z' \in B(0, \frac{1}{2}) \backslash \mathrm{supp}\,\nu$. Let z be a closest point to z' in suppν. Note that since $0 \in \operatorname{supp}\nu$, we have $|z-z'| \leqslant |z'|$, so $|z| \leqslant 2|z'| < 1$. Also, the ball $B = B(z', |z-z'|)$ doesn't contain any point of $\operatorname{supp}\nu$. Let n be the outer unit normal to ∂B at z. Consider the blow-ups $\nu_{z,\lambda}$ of ν at z. As $\lambda \to 0$, the supports of $\nu_{z,\lambda}$ lie in a smaller and smaller neighborhood of the half-space $S = \{x \in \mathbb{R}^{d+1} : \langle x, n\rangle \geqslant 0\}$ bounded by the linear hyperplane $H = \{x \in \mathbb{R}^{d+1} : \langle x, n\rangle = 0\}$. So, every tangent measure of ν at z must have its support in half-space S. Thus, starting with any measure ν that gives a counterexample to the alternative we are trying to prove, we can modify it so that it is supported on a half-space. But this is impossible: either support is then on the boundary of S (then geometric flatness "almost" follows) or if otherwise, then the integral $\int_{B(0,\Delta)} \frac{\langle x,n\rangle}{|x|^{d+1}}\, d\nu(x)$ blows up.

We proved the geometric flattening lemma.

The flattening lemma.

This is the major step in the argument: from **geometric flatness** and the absence of large oscillation of $R^H\mu$ on $\operatorname{supp}\mu$ near some fixed point z on scales $\asymp \ell$ to the flatness of μ at z on scale ℓ.

Lemma 5. *Fix four positive parameters $A, \alpha, \widetilde{c}, \widetilde{C}$. $\exists A', \alpha' > 0$ depending on $A, \alpha, \widetilde{c}, \widetilde{C}$ and d such that: if H is a linear hyperplane in $\mathbb{R}^{d+1}$, $z \in \mathbb{R}^{d+1}$, L is the affine hyperplane containing z and parallel to H, $\ell > 0$, and μ is a $\widetilde{C}$-good finite measure in $\mathbb{R}^{d+1}$ that is AD regular in $B(z, 5A'\ell)$ with the lower regularity constant $\widetilde{c}$. Assume that μ is geometrically $(H, 5A', \alpha')$-flat at z on the scale ℓ and, in addition, for every (vector-valued) Lipschitz function g with* supp$g \subset B(z, 5A'\ell)$, *$\|g\|_{\mathrm{Lip}} \leqslant \ell^{-1}$, and $\int g\, d\mu = 0$, one has*

$$|\langle R^H_\mu 1, g\rangle_\mu| \leqslant \alpha' \ell^d\,.$$

Then μ is (H, A, α)-flat at z on the scale ℓ.

The proof is rather involved, see [NToV1]. In the next lecture we will prove that the cells that are not flat are very rare (satisfy the Carleson measure condition that was mentioned at the beginning of this lecture).

Discussion.

The first step in proving the rectifiability of a measure is showing that its support is almost planar on many scales in the sense of the geometric $(H, 5A', \alpha')$-flatness in the assumptions of the Flattening Lemma. This step is not that hard and we will carry it next.

The second assumption involving the Riesz transform means, roughly speaking, that $R^H_\mu 1$ is almost constant on $\operatorname{supp}\mu \cap B(z, A'\ell)$ in the sense that its "wavelet coefficients" near z on the scale ℓ are small. There is no canonical smooth wavelet system in $L^2(\mu)$ when μ is an arbitrary measure but mean zero Lipschitz functions serve as a reasonable substitute. The boundedness of R^H_μ in $L^2(\mu)$ implies that $R^H_\mu 1 \in L^2(\mu)$ (because for finite measures μ, we have $1 \in L^2(\mu)$), so an appropriate version of the Bessel inequality can be used to show that large wavelet coefficients have to be rare and the balls satisfying the second assumption should also be viewed as typical.

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