RECTIFIABILITY OF MEASURES WITH BOUNDED RIESZ TRANSFORM OPERATOR: FROM SINGULAR OPERATORS TO GEOMETRIC MEASURE THEORY

ALEXANDER VOLBERG

1. Lecture 4: Replacement of curvature: Riesz energy and its estimates

We are interested in the following singular Riesz transforms:

$$R^s\phi(x) = \int_{\mathbb{R}^n} R^s(x-y)f(y)\,d\mu(y)$$

understood as a Calderón-Zygmund operator. Here $x, y \in \mathbb{R}^n$, $s \in (0, n]$,

$$R^s(x) = \frac{x}{|x|^{s+1}}\,, R^s(x) = (R^s_1, \dots, R^s_n)\,, R^s_j(x) = \frac{x_j}{|x|^{s+1}}\,.$$

The measure μ is an Ahlfors–David (AD) regular measure in $\mathbb{R}^n$ meaning that

$$c\,r^s \leqslant \mu(B(x,r)) \leqslant C\,r^s$$

for all x in support of μ and all $r \leqslant \operatorname{diam} E$, where $E := \operatorname{supp}\mu$.

Conjecture 1. If the operator R_s (this is actually n operators) is bounded in $L^2(\mu)$ then

1) s is integer;
2) if $s = m$ is already integer, then support E of μ is m-rectifiable.

Definition 1. A set E in $\mathbb{R}^n$ is called m-rectifiable, if there are $\{\Gamma_k\}_{k=1}^\infty$ Lipschitz images of R^m, so that $\mathcal{H}^m(E \setminus \bigcup_{k=1}^\infty \Gamma_k) = 0$.

This notion is equivalent to m-rectifiability given in the previous lectures, [Fe].

The conjecture belongs to David and Semmes. For a special case $s = n-1$ it has received a lot of attention. In particular, because of its relations with regularity of solutions of Laplace equation in domains with very low regularity. For a long time it is remained open even

Date: March 13, 2014.

for the case $n = 2$ (and $1 < s < 2$). For $n = 2, s = 1$ it was done by Mattila–Melnikov–Verdera [MMV] in case of homogeneous sets (1-Ahlfors regular sets), [DM], [NTV] plus [Le] for the nonhomogeneous situation. For $s = 1$ Menger's curvature tool was available. It is "cruelly missing" for $s > 1$.

We present here a case of arbitrary n and $s = n-1$. That is the case of co-dimension 1. The case $0 < s \leqslant 1$ can be treated using Menger's curvature. This has been done by Laura Prat, Xavier Tolsa. In the case $s = n$ there is nothing to do. The case $n-1 < s < n$ was solved by Eiderman–Nazarov–Volberg recently [ENV].

Definition 2. Let $\mathcal{H}^s(E) < \infty$. We call a point x super s-nonhomogeneous (irregular) point of E if

$$\theta^s_*(\mathcal{H}^d\lfloor E, x) = \liminf_{r\to 0} \frac{1}{r^s}\mathcal{H}^s(E\cap B(x,r)) = 0.$$

The following geometric statement for $s = n-1$ is also quite easily discernible (but not conspicuously formulated) in [ENV]:

Theorem 1. *Let $\mu := \mathcal{H}^{n-1}\lfloor E$, where $0 < \mathcal{H}^{n-1}(E) < \infty$. Let $R^{n-1}_\mu : L^2(\mu) \to L^2(\mu)$ be a bounded operator. Then the μ measure of the set of super $(n-1)$-nonhomogeneous points of E is zero, that is $\theta^s_*(\mu, x) > 0$ for μ-a.e. x.*

New tools of Riesz energy were used in [ENV]. We start with these tools here.

We omit the use of the index $s = n-1$ in what appears below; that is we write

$$R := R^{n-1}(x) = (R_1, \dots, R_n)\,, R_j := R^d_j(x) = \frac{x_j}{|x|^n}.$$

Given a hyperplane H on which $x_n = \text{const}$ we consider $R^H := R^s(x) = (R^s_1, \dots, R^s_{n-1})$ and notice that the operator R^{H*} acts on vector fields: let $\psi = (\psi_1, \dots, \psi_{n-1})$ be an $L^p(m_{n-1})$ vector function on H. Then $R^{H*}\psi = R_1(\psi_1\, dm_{n-1}) + \cdots + R_n(\psi_{n-1}\, m_{n-1})$, where m_{n-1} is Lebesgue measure on H.

Riesz Energy. We wish to give the estimate from below for the expression

$$\mathcal{E}(f, E) := \int_H (Rf)^2(x) f(x)\, dm_{n-1}(x)\,,$$

where $E \subset H$ and $0 \leqslant f \leqslant 1$ function is supported on E. We want to give the estimate from below of $\mathcal{E}(f, E)$ in terms of $|E| := \mathcal{H}^{n-1}(E) =$

$m_{n-1}(E) < \infty$ and

$$\text{mass} := \text{mass}(f\, dm_{n-1}) = \text{mass}(f\, d\mathcal{H}^{n-1}) := \int_H f\, dm_{n-1}\,.$$

Theorem 2. *$\mathcal{E}(f, E) \geqslant c_n \frac{(\text{mass})^5}{|E|^4}$, where $c_n > 0$.*

To do that we want first the following vector field ψ on H:

- $\int_H |\psi|\, dm_{n-1} \leqslant C_1 < \infty$;
- $\int_H |\psi|^2\, dm_{n-1} \leqslant C_2 < \infty$;
- $R^{H*}\psi(x) = 1$, m_{n-1} a. e. on E.

To do this put $\psi_0 = 0$, $\phi_0 = \chi_E$,

(1) $$\psi_{k+1} - \psi_k = \chi_{\{R\phi_k > A^{-k}\}} R\phi_k\,,$$

and

$$\chi_E R^*(\psi_{k+1} - \psi_k) = \phi_k - \phi_{k+1}\,,$$

where $A := 2 + \varepsilon$ will be chosen momentarily. Here we use R for R^H temporarily.

Then

$$\phi_{k+1} = \chi_E R^*(\chi_{\{R\phi_k \leqslant A^{-k}\}} R\phi_k)$$

By induction (using that $m_{n-1}(E) < \infty$) $\|\phi_{k+1}\|_2 \leqslant C\|\phi_{k+1}\|_4 \leqslant \Big(\int |R\phi_k|^2 A^{-2k}\Big)^{1/4} \leqslant C2^{-k/2}A^{-k/2} \leqslant 2^{-k-1}$.

Automatically ψ_k converges in $L^2(H, m_{n-1})$, see (1). But also in $L^1(H, m_{n-1})$. In fact,

$$\int_H |\psi_{k+1} - \psi_k|\, dm_d \leqslant C\|\phi_k\|_2 |\{R\phi_k > A^{-k}\}|^{1/2} \leqslant C\, 2^{-k}(2^{-2k}A^{2k})^{1/2} =$$

$= C\, 4^{-k}A^k \leqslant C\, q^k$, and $q < 1$ if $A < 4$. Hence, $\psi := \lim_k \psi_k$ is in $L^1(H) \cap L^2(H)$. As $\psi_0 = 0, \phi_0 = \chi_E$, we use (1) again:

$$\chi_E R^*\psi_N = \chi_E - \phi_N\,.$$

Taking the limit in $N \to \infty$ we get $R^*\psi = 1$ on E.

Now we start the estimate of the Riesz energy from below. Suppose that $\mathcal{E}(f, E) < \lambda \cdot \text{mass}$ with a very small λ. Let $H = \{x_n = 0\}$. Consider a new measure

$$d\nu := f(x) dm_d \times \delta^{-1}\chi_{[0,\delta]} dx_n\,.$$

Lemma 1. *$\int |R^H \nu|^2\, d\nu \to \mathcal{E}(f, E)$ when $\delta \to 0$.*

In fact, notice that given intervals I containing 0 and of length δ we have for almost every $x \in E \subset H$:

$$\lim_{\delta\to 0}\sup_{I}\Big|\frac{1}{|I|}\int_I (R^H f)(x,x_n)\,dx_n - (R^H f)(x,0)\Big| = 0\,.$$

Moreover this convergence is dominated by an $L^2(H)$ majorant.

Therefore,

$$\lim_{\delta\to 0}\sup_{x_n\in[0,\delta]}\Big|\int_E |(R^H f)(x,x_n)|^2\,dm_{n-1} - \int_E |(R^H f)(x,0)|^2\,dm_{n-1}\Big| = 0\,,$$

which immediately means that

$$\int |R^H\nu|^2\,d\nu \to \mathcal{E}(f,E)\,.$$

This proves the lemma. Hence we can assume that

$$\mathcal{E}(\nu) := \int |R^H\nu|^2\,d\nu < \lambda\cdot \mathrm{mass}(\nu)\,. \tag{2}$$

Now we will estimate the **Riesz Energy** $\mathcal{E}(\nu)$ from below. For that purpose introduce functional on functions $a \in L^\infty(\nu)$:

$$\mathcal{H}(a) := \lambda\|a\|_\infty \mathrm{mass}(\nu) + \int |R^H(a\,d\nu)|^2\,a d\nu;$$

under the assumptions $a \geqslant 0$, $\mathrm{mass}(a\,d\nu) = \mathrm{mass}(\nu)$. The minimum is attained. In fact, let $\{a_k\}$ be a minimizing sequence.

$$\lambda\|a_k\|_\infty \mathrm{mass} \leqslant \mathcal{H}(a_k) \leqslant \mathcal{H}(1) = \lambda\,\mathrm{mass} + \mathcal{E}(\nu) < 2\lambda\,\mathrm{mass}$$

by assumption (2). Therefore, $\|a_k\|_\infty \leqslant 2$. WLOG $a_k \to a \in L^\infty(\nu)$ weakly. So

- $\|a\|_\infty \leqslant \liminf_k \|a_k\|_\infty$.
- $R^H(a_k\,d\nu)$ are uniformly in any $L^p(\nu)$ ($p = 4$, say).
- For every compact subset $S \subset \mathrm{supp}\,\nu$ we can conclude that $R^H(a_k\,d\nu)(x)$ converge to $R^H(a\,d\nu)(x)$ uniformly for $x \in S$.

This last assertion follows from the observation that the set $\{R^H(x-\cdot)\}_{x\in S}$ is a continuous image of the compact set S into $L^1(\nu)$, and hence, it is compact in $L^1(\nu)$. Integrating it with $a_k(x)$ that converges weakly to a in $L^\infty(\nu)$, we obtain the uniform convergence on S. The existence of a minimizer a, $\|a\|_\infty \leqslant 2$ and $\mathcal{H}(a) \leqslant 2\lambda\,\mathrm{mass}$ is very important. Denote

$$\nu_a := a\,d\nu$$

and let U be a set, where $a > 0$. Denote

$$\nu_{a_t} := a(1 - t\chi_U)\nu\,.$$

$$\mathcal{H}(a_t) = \mathcal{H}(a) - t\Big[\int_U |R^H\nu_a|^2\,d\nu_a + 2\int_U R^{H*}\big[(R^H\nu_a)d\nu_a\big]\,d\nu_a\Big] + o(t^2)\,.$$

The mass of ν_{a_t} is $(\text{mass} - t\nu_a(U))$, therefore a_t is not admissible. To make it admissible consider $\frac{\text{mass}}{\text{mass}-t\nu_a(U)}a_t = \Big(1 - t\frac{\nu_a(U)}{\text{mass}}\Big)^{-1}a_t$.

Then

$$\mathcal{H}(a) \leqslant \mathcal{H}\Big(\frac{\text{mass}}{\text{mass} - t\nu_a(U)}a_t\Big) \leqslant \Big(1 - t\frac{\nu_a(U)}{\text{mass}}\Big)^{-3}\mathcal{H}(a_t) \leqslant \mathcal{H}(a) +$$

$$t\Big[3\frac{\nu_a(U)\mathcal{H}(a)}{\text{mass}} - \Big(\int_U |R^H\nu_a|^2\,d\nu_a + 2\int_U R^{H*}\big[(R^H\nu_a)d\nu_a\big]\,d\nu_a\Big)\Big] + o(t^2)$$

This immediately implies:

$$\int_U |R^H\nu_a|^2\,d\nu_a + 2\int_U R^{H*}\big[(R^H\nu_a)d\nu_a\big]\,d\nu_a \leqslant 3\nu_a(U)\frac{\mathcal{H}(a)}{\text{mass}}\,.$$

This holds for every U on which a is strictly positive. We use also $\mathcal{H}(a) \leqslant 2\lambda\,\text{mass}$. Then point-wisely

$$|R^H\nu_a|^2 + 2R^{H*}\big[(R^H\nu_a)d\nu_a\big] \leqslant 6\lambda$$

on $O := \{x \in \mathbb{R}^n : a > 0\}$. But all functions here are continuous (this is why we replaced $f\,dm_{n-1}$ by "mollified" ν). So this holds on $\operatorname{clos} O$.

However, $R^H\mu$ is harmonic outside of the support of μ for any μ. In our case $\mu = \nu_a$ and $\operatorname{supp}\nu_a = \operatorname{clos} O$. All functions above are subharmonic and continuous. **Maximum Principle** shows now that

$$|R^H\nu_a|^2 + 2R^{H*}\big[(R^H\nu_a)d\nu_a\big] \leqslant 6\lambda \tag{3}$$

is true **everywhere** in $\mathbb{R}^n$. In particular, it is true on H on which ψ lives. Integrate (3) with respect to $|\psi|\,dm_{n-1}\lfloor H$. We remember:

- $\int_H |\psi|\,dm_{n-1} \leqslant C_1 < \infty\,;$
- $\int_H |\psi|^2\,dm_{n-1} \leqslant C_2 < \infty\,;$
- $R^{H*}\psi(x) \geqslant 1\,,\ m_d$ a. e. on $E\,.$

From the very beginning we can think that E is the union of finitely many (but a very large number) of $n-1$ dimensional balls in H. Then we can mollify ψ to keep the first two claims and to have the third one holding in a small neighborhood $N(E)$ of E already in $\mathbb{R}^n$ (R^{H*} applied to a smooth vector function on H produces a continuous function in $\mathbb{R}^n$). So we will get

$$R^{H*}\psi(x) \geqslant 1/2\,,\ m_n \text{ a. e. on } N(E)\,. \tag{4}$$

We get

$$\int |R^H(\nu_a)|^2|\psi|\, dm_{n-1} \leqslant 6C_1\lambda + \Big|\int R^H(|\psi|\, dm_{n-1})\cdot R^H(\nu_a)d\nu_a\Big|$$

$$\leqslant 6C_1\lambda + \sqrt{2}\mathcal{H}(a)^{1/2}\Big(\int |R^H(|\psi|\, dm_{n-1})|^2\, d\nu\Big)^{1/2}.$$

The last integral can be taken "layer" by "layer" as $d\nu = dm_{n-1}\times \delta^{-1}dx_n$. On each layer we use that R^H is bounded in $L^2(m_{n-1})$. Hence, we continue

$$\int |R^H(\nu_a)|^2|\psi|\, dm_{n-1} \leqslant 6C_1\lambda + 2\lambda^{1/2}\text{mass}^{1/2}\Big(\int_H |\psi|^2\, dm_{n-1}\Big)^{1/2}.$$

Temporarily normalize by $|E|\leqslant 1 \Rightarrow$ mass $\leqslant 1$. Then integrate (4) with respect to $d\nu_a$ whose support lies in $N(E)$ if δ is sufficiently small. Cauchy inequality gives then $\frac12\text{mass} = \frac12\text{mass}(\nu_a)\leqslant \frac12|\int R^{H*}\psi\, d\nu_a| =$

$$\frac12\Big|\int R^H(\nu_a)\psi\, dm_{n-1}\Big| \leqslant \frac12\int |R^H(\nu_a)||\psi|\, dm_{n-1}\leqslant C(\lambda+\lambda^{1/2})^{1/2}\leqslant C\lambda^{1/4}.$$

Therefore, using the assumption $|E|\leqslant 1$ we finally get the estimate on λ from below $\lambda\geqslant c(\text{mass})^4 = c(\int f\, dm_{n-1})^4$. This gives us immediately the following estimate on the Riesz energy from below (see (2) with minimal λ):

$$\mathcal{E}(f,E)\geqslant c\Big(\int f\, dm_{n-1}\Big)^5.$$

To get rid of the assumption $|E|\leqslant 1$ we just use the scaling invariance to get

$$\mathcal{E}(f,E)\geqslant c\Big(\frac{\int f\, dm_{n-1}}{|E|}\Big)^4\int f\, dm_{n-1} = \frac{\text{mass}^5}{|E|^4}. \tag{5}$$

Theorem 2 is proved.

References

[Carl] L. Carleson Selected Problems on Exceptional Sets, Van Nostrand, 1967.

[Ch] M. Christ, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral, Colloq. Math. 60/61, 1990, pp. 601-628.

[D] G. David, Wavelets and singular integrals on curves and surfaces, Lecture Notes in Mathematics, vol. 1465, Springer-Verlag, 1991.

[D1] G. David, Unrectifiable 1-sets have vanishing analytic capacity, Revista Mat. Iberoamericana, 14(2), 1998, pp. 369-479.

[Da4] G. David, *Wavelets and singular integrals on curves and surfaces,* Lecture Notes in Math. 1465, Springer-Verlag, Berlin, 1991.

[DM] G. David and P. Mattila. Removable sets for Lipschitz harmonic functions in the plane. Rev. Mat. Iberoamericana 16(1) (2000), 137–215.

[DS] G. David, S. Semmes, Analysis of and on uniformly rectifiable sets, Mathematical Surveys and Monographs, Volume 38, 1993, AMS, Providence, RI.

[ENV] V. Eiderman, F. Nazarov, A. Volberg, The s-Riesz transform of an s-dimensional measure in $\mathbb{R}^2$ is unbounded for $1 < s < 2$, available from http://arxiv.org/abs/1109.2260.

[Fa] H. Farag, The Riesz kernels do not give rise to higher-dimensional analogues of the Menger-Melnikov curvature, Publ. Mat. 43 (1999), no. 1, 251–260.

[Fe] H. Federer, Geometric Measure Theory, Springer 1969.

[HMM] S. Hofmann, J. M. Martell, S. Mayboroda, Uniform rectifiability and harmonic measure III: Riesz transform bounds imply uniform rectifiability of boundaries of 1-sided NTA domains, http://arxiv.org/abs/1207.1527.

[PJ] P.W. Jones, Rectifiable sets and the traveling salesman problem, Invent. Math. 102 (1990), 1–15.

[Law] Lawler, E.L.: The Traveling Salesman Problem. New York: Wiley-lnterscience, 1985.

[Le] J. C. Léger, Menger curvature and rectifiability, Ann. of Math. 149 (1999), 831–869.

[KO] K. Okikiolu, Characterization of subsets of rectifiable sets in $\mathbb{R}^n$, J. London Math. Soc., (2) 46 (1992), pp. 336–348.

[Ma] P. Mattila. *Geometry of sets and measures in Euclidean spaces,* Cambridge Stud. Adv. Math. 44, Cambridge Univ. Press, Cambridge, 1995.

[MMV] P. Mattila, M. Melnikov, J. Verdera, The Cauchy integral, analytic capacity, and uniform rectifiability, Ann. of Math. (2) 144, 1996, pp. 127-136.

[MPa] P. Mattila and P.V. Paramonov. On geometric properties of harmonic Lip1-capacity, Pacific J. Math. 171:2 (1995), 469490.

[NToV1] F. Nazarov, X. Tolsa and A. Volberg, On the uniform rectifiability of AD-regular measures with bounded Riesz transform operator: the case of codimension 1, arXiv:1212.5229, to appear in Acta Math.

[NToV2] F. Nazarov, X. Tolsa and A. Volberg, The Riesz transform, rectifiability, and removability for Lipschitz harmonic functions, arXiv:1212.5431, to appear in Publ. Mat.

[NTV] F. Nazarov, S. Treil and A. Volberg, The Tb-theorem on non- homogeneous spaces that proves a conjecture of Vitushkin. Preprint of 2000 available at www.crm.cat/Paginas/Publications/02/Pr519.pdf or arXiv:1401.2479.

[NTrV1] F. Nazarov, S. Treil and A. Volberg, Weak type estimates and Cotlar inequalities for Calderón-Zygmund operators in nonhomogeneous spaces, Int. Math. Res. Notices **9** (1998), 463–487.

[Paj] H. Pajot, Théorème de recouvrement par des ensembles Ahlfors-réguliers et capacité analytique. (French) C. R. Acad. Sci. Paris Sér. I Math. 323 (1996), no. 2, 133–135.

[Pr] L. Prat, Potential theory of signed Riesz kernels: capacity and Hausdorff measure. Int. Math. Res. Not. 2004, no. 19, 937981.

[PS] Preparata, F.P., Shamos, M.I.: Computational Geometry. Berlin Heidelberg New York: Springer 1985.

[RS] Subsets of Rectifiable curves in Hilbert Space-The Analyst's TSP, arXiv:math/0602675, J. Anal. Math. 103 (2007), 331375.

[T1] X. Tolsa, Painlevé's problem and the semiadditivity of analytic capacity, Acta Math. 190:1 (2003), 105–149.

[T2] X. Tolsa, Uniform rectifiability, Calderón–Zygmund operators with odd kernels, and quasiorthogonality, Proc. London Math. Soc. 98(2), 2009, pp. 393-426.

[T3] X. Tolsa, Principal values for Riesz transform and rectifiability, J. Func. Anal. vol. 254(7), 2008, pp. 1811-1863.

[T-b] X. Tolsa, *Analytic capacity, the Cauchy transform, and non-homogeneous Calderón-Zygmund theory.* To appear (2012).

[Vo] A. Volberg, *Calderón-Zygmund capacities and operators on nonhomogeneous spaces.* CBMS Regional Conf. Ser. in Math. 100, Amer. Math. Soc., Providence, 2003.

Alexander Volberg, Department of Mathematics, Michigan State University, East Lansing, Michigan, USA