RECTIFIABILITY OF MEASURES WITH BOUNDED RIESZ TRANSFORM OPERATOR: FROM SINGULAR OPERATORS TO GEOMETRIC MEASURE THEORY

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1. Lecture 3: Bounded Riesz operator: a reduction from nonhomogeneous sets to $(n-1)$-AD regular sets.

Let us recall our goals. They are reduced to proving

Theorem 1. *Let $\mathcal{H}^{n-1}(E) < \infty$. If R_n^{n-1} is bounded in $L^2(E, \mathcal{H}^{n-1})$, and E is $(n-1)$-AD regular, then E is $(n-1)$-uniformly rectifiable.*

Theorem 2. *Let $\mathcal{H}^{n-1}(E) < \infty$. If R_n^{n-1} is bounded in $L^2(E, \mathcal{H}^{n-1})$, then the set E is $(n-1)$-rectifiable.*

In this lecture we will reduce Theorem 2 to Theorem 3. In other words we will show why AD-regularity can be assumed without loss of generality. This is not very simple. The reduction will be based on Pajot's idea [Paj] and on elimination of *super nonhomogeneous* points of the measure $\mu := \mathcal{H}^{n-1}\lfloor E$. The dimension being $(n-1)$ will (unfortunately) play a crucial part in this latter elimination based on paper of Eiderman–Nazarov–Volberg [ENV].

1.1. The Main Lemma.

1.2. Statement of the Main Lemma. We say that a Borel measure μ in $\mathbb{R}^n$ has growth of degree d if there exists some constant c such that

$$\mu(B(x,r)) \le c\, r^d \qquad \text{for all } x \in \mathbb{R}^d,\ r > 0.$$

We define the upper and lower d-dimensional densities by

$$\theta^{d,*}(x,\mu) = \limsup_{r\to 0} r^{-d}\mu(B(x,r)) \text{ and } \theta^{d}_{*}(x,\mu) = \liminf_{r\to 0} r^{-d}\mu(B(x,r)),$$

respectively.

If μ and σ are Borel measures on $\mathbb{R}^n$, the notation $\mu \le \sigma$ means that $\mu(A) \le \sigma(A)$ for all Borel sets $A \subset \mathbb{R}^n$. Let R denote the operator with the kernel $R_n^d(x-y)$.

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Lemma 1 (Main Lemma). *Let μ be a compactly supported finite Borel measure in $\mathbb{R}^n$ with growth of degree d such that $\theta^d_*(x,\mu) > 0$ for μ-a.e. $x \in \mathbb{R}^n$. Suppose that R_μ is bounded in $L^2(\mu)$. Then there are finite Borel measures μ_k, $k \geq 1$, such that*

(a) $\mu \leq \sum_{k\geq 1} \mu_k$
(b) *μ_k is AD-regular for each $k \geq 1$ (with the AD-regularity constant depending on k), and*
(c) *for each $k \geq 1$, R_{μ_k} is bounded in $L^2(\mu_k)$.*

1.3. **Proof of Theorem 2 using the Main Lemma 1.** Let $E \subset \mathbb{R}^n$ be a bounded set with $\mathcal{H}^{n-1}(E) < \infty$. Set $\mu = \mathcal{H}^{n-1}\lfloor E$, and suppose that R_μ is bounded in $L^2(\mu)$.

Let E_0 be the subset of those $x \in E$ for which $\theta^{n-1}_*(x,\mu) = 0$. We can call such points *super nonhomogeneous points.* We want to get rid of them. We set

$$\mu_0 = \mu\lfloor E_0.$$

Then,

$$\theta^{n-1}_*(x,\mu_0) \leqslant \theta^{n-1,*}(x,\mu) = 0 \text{ for } \mu_0\text{-a.e. } x \in \mathbb{R}^n,$$

and, moreover, R_{μ_0} is bounded in $L^2(\mu_0)$. Then, by the main theorem of [ENV] (applied to the co-dimension 1 case) we deduce that $\mu_0 = 0$. That is,

$$\theta^{n-1}_*(x,\mu) > 0 \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n.$$

So the measure μ satisfies the assumptions of Main Lemma 1, and thus we may consider measures μ_k as in the statement of the Main Lemma.

By the result of [NToV1] $\operatorname{supp}\mu_k$ is $(n-1)$-rectifiable. Therefore,

$$F = \bigcup_{k\geq 1} \operatorname{supp}\mu_k$$

is also $(n-1)$-rectifiable. Since

$$\mathcal{H}^{n-1}(E \setminus F) = \mu(\mathbb{R}^n \setminus F) \leq \sum_k \mu_k(\mathbb{R}^n \setminus F) = 0,$$

we infer that E is $(n-1)$-rectifiable too. Theorem 2 is proved up to the proof of Lemma 1. Below we never again use that $d = n-1$. The only two usages of this fact happened above.

1.4. **Preliminaries in the proof of the Main Lemma 1.** For the proof of the Main Lemma 1 we will need the following proposition.

Lemma 2. *Let μ and σ be Borel measures with growth of degree d in $\mathbb{R}^n$ such that R_μ is bounded in $L^2(\mu)$ and R_σ is bounded in $L^2(\sigma)$. Then, $R_{\mu+\sigma}$ is bounded in $L^2(\mu+\sigma)$.*

Proof. The boundedness of R_μ in $L^2(\mu)$ implies the boundedness of R from the space of real measures $M(\mathbb{R}^n)$ into $L^{1,\infty}(\mu)$. In other words, the following inequality holds for any $\nu \in M(\mathbb{R}^d)$ uniformly on $\varepsilon > 0$:

$$\mu\big\{x \in \mathbb{R}^n : |R_\varepsilon \nu(x)| > \lambda\big\} \leq c\,\frac{\|\nu\|}{\lambda} \qquad \text{for all } \lambda > 0.$$

For the proof, see Theorem 9.1 of [NTrV1]. Analogously, the same bound holds with μ replaced by σ. As a consequence, we infer that for all $\lambda > 0$,

$$(\mu + \sigma)\big\{x \in \mathbb{R}^n : |R_\varepsilon \nu(x)| > \lambda\big\} \leq c\,\frac{\|\nu\|}{\lambda}.$$

That is, R is bounded from $M(\mathbb{R}^n)$ into $L^{1,\infty}(\mu+\sigma)$. In particular, $R_{\mu+\sigma}$ is of weak type $(1,1)$ with respect to $\mu+\sigma$. This implies that $R_{\mu+\sigma}$ is bounded in $L^2(\mu+\sigma)$. For the proof, based on interpolation, see Theorem 10.1 of [NTrV1] (an alternative argument based on a good lambda inequality can be also found in Chapter 2 of the book [T-b]). □

Let us remark that the preceding lemma and its proof remain valid for more general Calderón-Zygmund operators. However, we will need it only for the Riesz transforms.

In the proof of the Main Lemma 1 it will be convenient to work with an ε-regularized version $\widetilde{R}_{\mu,\varepsilon}$ of the Riesz transform R_μ. We set

$$\widetilde{R}_{\mu,\varepsilon} f(x) = \int \frac{x-y}{\max(|x-y|,\varepsilon)^{d+1}}\, f(y)\, d\mu(y).$$

It is easy to check that

$$|\widetilde{R}_{\mu,\varepsilon} f(x) - R_{\mu,\varepsilon} f(x)| \leq c\, M_\mu f(x) \qquad \text{for all } x \in \mathbb{R}^n,$$

where c is independent of ε and M_μ is the centered maximal Hardy-Littlewood operator with respect to μ:

$$M_\mu f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f|\, d\mu.$$

Since M_μ is bounded in $L^2(\mu)$, it turns out that R_μ is bounded in $L^2(\mu)$ if and only if the operators $\widetilde{R}_{\mu,\varepsilon}$ are bounded in $L^2(\mu)$ uniformly on $\varepsilon > 0$. The advantage of $\widetilde{R}_{\mu,\varepsilon}$ over $R_{\mu,\varepsilon}$ is that the kernel

$$K_\varepsilon(x) = \frac{x}{\max(|x|,\varepsilon)^{d+1}}$$

is continuous and satisfies the smoothness condition

$$|\nabla K_\varepsilon(x)| \leq \frac{c}{|x|^{d+1}}, \quad |x| \neq \varepsilon$$

(with c independent of ε), which implies that $K_\varepsilon(x-y)$ is a Calderón-Zygmund kernel (with constants independent of ε), unlike the kernel of $R_{\mu,\varepsilon}$.

We follow the idea of Pajot [Paj], where some measures μ_k satisfying (a) and (b) of Lemma 1 are constructed. For the reader's convenience, we will repeat the arguments of the construction, and subsequently we will show that the statement (c) holds. Some extra calculation will be though needed for that.

It is known that the $L^2(\mu)$ boundedness of R^d_μ and the fact that μ has no point masses implies that μ has growth of degree d. See [Da4, Proposition 1.4, p.56].

Let $D := \operatorname{diam} E$. Consider the subset $F \subset \operatorname{supp}\mu$ of those $x \in \mathbb{R}^n$ for which $\theta^d_*(x,\mu) > 0$, so that $\mu(\mathbb{R}^n \setminus F) = 0$. For positive integers p, s, we denote

$$F_p = \left\{ x \in F : \text{ for } 0 < r \le D,\, \mu(B(x,r)) \ge \tfrac{1}{p}\, r^d \right\},$$
$$F_{p,s} = \left\{ x \in F_p : \text{ for } 0 < r \le D,\, \mu(F_p \cap B(x,r)) \ge \tfrac{1}{ps}\, r^d \right\},$$

where $D = \operatorname{diam}(\operatorname{supp}\mu)$. From the definitions of F and F_p, it is clear that

$$F = \bigcup_{p\ge 1} F_p.$$

Also, $\theta^n_*(x,\mu) = \theta^n_*(x,\mu\lfloor F_p)$ for μ-a.e. $x \in F_p$ by the Lebesgue differentiation theorem, and thus

$$\mu\Big(F_p \setminus \bigcup_{s\ge 1} F_{p,s}\Big) = 0.$$

So we have

$$\mu \le \sum_{p,s\ge 1} \mu\lfloor F_{p,s}.$$

The strategy of the construction consists in adding a measure $\sigma_{p,s}$ to each $\mu\lfloor F_{p,s}$ so that the resulting measure is AD-regular, for each p, s.

It is easy to check that all the sets F_p and $F_{p,s}$ are compact. Fix p, s and denote

$$d(x) = \frac{1}{10}\,\operatorname{dist}(x, F_{p,s}).$$

Notice that $d(y) > 0$ if $x \notin F_{p,s}$, as $F_{p,s}$ is closed. Now we cover $F_p \setminus F_{p,s}$ by a family of balls of the form $B(x, d(x))$, with $x \in F_p \setminus F_{p,s}$, using Besicovitch's covering theorem. So there exists a family of points

$H_{p,s} \subset F_p \setminus F_{p,s}$, at most countable, such that

$$F_p \setminus F_{p,s} \subset \bigcup_{x \in H_{p,s}} B(x, d(x)),$$

and

$$\sum_{x \in H_{p,s}} \chi_{B(x,d(x))} \leq C_d \,.$$

Moreover, we can split $H_{p,s} = \bigcup_{i=1}^{N_d} H_{p,s}^i$ so that for each i, the balls from $\{B(x,d(x))\}_{x \in H_{p,s}^i}$ are pairwise disjoint (see Theorem 2.7 in p. 30 of [Ma]). Here C_n, N_n are some constants depending on n only.

To define $\sigma_{p,s}$, for each $x \in H_{p,s}$ we consider an arbitrary d-plane Π_x containing x and set $P_x = \Pi_x \cap B(x, \frac{1}{2}d(x))$. Then we define

$$\sigma_{p,s} = \mathcal{H}^d \lfloor \Pi_{p,s} + \sum_{x \in H_{p,s}} \mathcal{H}^d \lfloor P_x$$

where $\Pi_{p,s}$ is an arbitrary d-plane in $\mathbb{R}^n$ intersecting $F_{p,s}$. We set

$$\mu_{p,s} = \sigma_{p,s} + \mu \lfloor F_{p,s}.$$

We also denote

$$\sigma_{p,s}^i = \sum_{x \in H_{p,s}^i} \mathcal{H}^d \lfloor P_x,$$

so that $\sigma_{p,s} = \mathcal{H}^d \lfloor \Pi_{p,s} + \sum_{i=1}^{N_d} \sigma_{p,s}^i$. We will show now that $\mu_{p,s}$ is AD-regular.

1.5. Lower AD-regularity of $\mu_{p,s}$.

$$\mu_{p,s}(B(x,r)) \geqslant \delta(p,s) r^d, \; x \in \operatorname{supp} \mu_{p,s} \,.$$

We refer the reader to [Paj], [NToV2] for this purely geometric proof. It is interesting to notice that we would be able to prove this lower regularity *without* going to the seemingly "unnatural" construction of second order splitting $F_{p,s}$. In fact we could have considered only F_p, corresponding σ_p, μ_p and the lower AD-regularity of the latter measure would follow.

But not so for the seemingly easier upper AD-regularity. The beautiful idea of Pajot to split F_p to $F_{p,s}$ turns out to be critical for the proof of

1.6. Upper AD-regularity of $\mu_{p,s}$.

$$\mu_{p,s}(B(x,r)) \leqslant \Delta(p,s) r^d, \; x \in \operatorname{supp} \mu_{p,s} \,.$$

This is more difficult, the second splitting is essential in the proof, see [Paj], [NToV2].

1.7. **Boundedness of $R_{\mu_{p,s}}$ in $L^2(\mu_{p,s})$.** We set $r(x) = \frac{1}{2}\,d(x)$ for $x \in H^i_{p,s}$. So $\sigma^i_{p,s}$ is a measure supported on the union of the closed balls

$$B_x := B(x, r(x)) = B(x, \frac{1}{2}\,d(x)), \qquad x \in H^i_{p,s}, \tag{1}$$

coinciding with $\mathcal{H}^d \lfloor P_x$ inside B_x. Recall (see [NToV2], this is just because $H^i_{p,s} \subset F_p$) that

$$\mu(B_x) \geq \frac{1}{p}\, r(x)^d \tag{2}$$

Taking into account that $R_{\mu\lfloor F_{p,s}}$ is bounded in $L^2(\mu\lfloor F_{p,s})$, and that $R_{\mathcal{H}^d\lfloor \Pi_{p,s}}$ is bounded in $L^2(\mathcal{H}^d\lfloor \Pi_{p,s})$, it is enough to show that $R_{\sigma^i_{p,s}}$ is bounded in $L^2(\sigma^i_{p,s})$ for each $i = 1, \dots, N_n$. Then the repeated application of Lemma 2 yields the result.

To simplify notation, for fixed p, s, i, we denote $\sigma = \sigma^i_{p,s}$, $H = H^i_{p,s}$.

Now we define

$$\nu = \sum_{x \in H} c_x\, \mu\lfloor B_x,$$

with $c_x = \mathcal{H}^d(P_x)/\mu(B_x)$. Observe that the constants c_x, $x \in H$, are uniformly bounded by some constant depending on p, because of (2), and thus R_ν is bounded in $L^2(\nu)$. Further, $\nu(B_x) = \sigma(B_x)$ for each $x \in H$. Recall also that, by construction both σ and ν are supported on the union of the balls $B_x, x \in H$, and the double balls $2B_x$ are pairwise disjoint.

It is clear that, in a sense, ν can be considered as an approximation of σ (and conversely).

To prove the boundedness of R_σ in $L^2(\sigma)$, we will prove that $\widetilde{R}_{\sigma,\varepsilon}$ is bounded in $L^2(\sigma)$ uniformly on $\varepsilon > 0$ by comparing it to $\widetilde{R}_{\nu,\varepsilon}$.

1.8. **Boundedness of Local Operators related to σ.** First we need to introduce some local and non local operators: Given $z \in \bigcup_{x \in H} B_x$, we denote by $B(z)$ the ball $B_x, x \in H$, that contains z. Then we write, for $z \in \bigcup_{x, x \in H} B_x$,

$$R^{loc}_{\nu,\varepsilon} f(z) = \widetilde{R}_{\nu,\varepsilon}(f\chi_{B(z)})(z), \qquad R^{nl}_{\nu,\varepsilon} f(z) = \widetilde{R}_{\nu,\varepsilon}(f\chi_{\mathbb{R}^d \setminus B(z)})(z).$$

We define analogously $R^{loc}_{\sigma,\varepsilon} f$ and $R^{nl}_{\sigma,\varepsilon} f$. It is straightforward to check that $R^{loc}_{\nu,\varepsilon}$ is bounded in $L^2(\nu)$, and that $R^{loc}_{\sigma,\varepsilon}$ is bounded in $L^2(\sigma)$, both uniformly on ε (in other words, R^{loc}_ν is bounded in $L^2(\nu)$ and R^{loc}_σ is bounded in $L^2(\sigma)$). Indeed,

$$\|R^{loc}_{\sigma,\varepsilon} f\|^2_{L^2(\sigma)} = \sum_{x\in H} \|\chi_{B_x} \widetilde{R}_{\sigma,\varepsilon}(f\chi_{B_x})\|^2_{L^2(\sigma)} \le c \sum_{x\in H} \|f\chi_{B_x}\|^2_{L^2(\sigma)} = c\|f\|^2_{L^2(\sigma)},$$

by the boundedness of the d-Riesz transforms on d-planes. Using the boundedness of R_ν in $L^2(\nu)$, one derives the $L^2(\nu)$ boundedness of $R^{loc}_{\nu,\varepsilon}$ analogously.

1.9. **Boundedness of Non-Local Operators related to σ.** We must show that R^{nl}_σ is bounded in $L^2(\sigma)$. Observe first that, since $R^{nl}_{\nu,\varepsilon} = \widetilde{R}_{\nu,\varepsilon} - R^{loc}_{\nu,\varepsilon}$, and both $\widetilde{R}_{\nu,\varepsilon}$ and $R^{loc}_{\nu,\varepsilon}$ are bounded in $L^2(\nu)$, it turns out that $R^{nl}_{\nu,\varepsilon}$ is bounded in $L^2(\nu)$ (all uniformly on $\varepsilon > 0$).

We will prove below that, for all $f \in L^2(\nu)$ and $g \in L^2(\sigma)$ satisfying

$$\int_{B_x} f\,d\nu = \int_{B_x} g\,d\sigma \qquad \text{for all } x \in H, \tag{3}$$

we have

$$I(f,g) := \int |R^{nl}_{\nu,\varepsilon} f - R^{nl}_{\sigma,\varepsilon} g|^2\, d(\nu+\sigma) \le c\,(\|f\|^2_{L^2(\nu)} + \|g\|^2_{L^2(\sigma)}), \tag{4}$$

uniformly on ε. Let us see how the boundedness of R^{nl}_σ in $L^2(\sigma)$ follows from this estimate. As a preliminary step, we show that $R^{nl}_\sigma : L^2(\sigma) \to L^2(\nu)$ is bounded. To this end, given $g \in L^2(\sigma)$, we consider a function $f \in L^2(\nu)$ satisfying (3) that is constant on each ball B_j. It is straightforward to check that

$$\|f\|_{L^2(\nu)} \le \|g\|_{L^2(\sigma)}.$$

Then from the $L^2(\nu)$ boundedness of R^{nl}_ν and (4), we obtain

$$\|R^{nl}_{\sigma,\varepsilon} g\|_{L^2(\nu)} \le \|R^{nl}_{\nu,\varepsilon} f\|_{L^2(\nu)} + I(f,g)^{1/2} \le c\|f\|_{L^2(\nu)} + c\,\|g\|_{L^2(\sigma)} \le c\,\|g\|_{L^2(\sigma)},$$

which proves that $R^{nl}_\sigma : L^2(\sigma) \to L^2(\nu)$ is bounded.

Notice that R^{nl}_ε is antisymmetric. Indeed, its kernel is

$$\left[1 - \sum_{x\in H} \chi_{B_x}(z)\chi_{B_x}(y)\right] \frac{z-y}{\max(|z-y|,\varepsilon)^{d+1}}\,.$$

Then, by duality, we deduce that $R^{nl}_\nu : L^2(\nu) \to L^2(\sigma)$ is bounded. To prove now the $L^2(\sigma)$ boundedness of R^{nl}_σ, we consider an arbitrary function $g \in L^2(\sigma)$, and we construct $f \in L^2(\nu)$ satisfying (3) which is constant in each ball B_x. Again, we have $\|f\|_{L^2(\nu)} \le \|g\|_{L^2(\sigma)}$.

Using the boundedness of $R^{nl}_\nu : L^2(\nu) \to L^2(\sigma)$ together with (4), we obtain

$$\|R^{nl}_{\sigma,\varepsilon} g\|_{L^2(\sigma)} \le \|R^{nl}_{\nu,\varepsilon} f\|_{L^2(\sigma)} + I(f,g)^{1/2} \le c\|f\|_{L^2(\nu)} + c\,\|g\|_{L^2(\sigma)} \le c\,\|g\|_{L^2(\sigma)},$$

as wished.

It remains to prove that (4) holds for $f \in L^2(\nu)$ and $g \in L^2(\sigma)$ satisfying (3). For $z \in \bigcup_{x\in H} B_x$, we have

$$|R^{nl}_{\nu,\varepsilon} f(z) - R^{nl}_{\sigma,\varepsilon} g(z)| \le \sum_{x\in H: z\notin B_x} \left| \int_{B_x} K_\varepsilon(z-y)(f(y)\,d\nu(y) - g(y)\,d\sigma(y)) \right|,$$

where $K_\varepsilon(z)$ is the kernel of the ε-regularized n-Riesz transform. By standard estimates, using (3), the fact that the balls $2B_x$, $x \in H$, are pairwise disjoint, and the smoothness of K_ε, it follows that

$$\begin{aligned}
&\left| \int_{B_x} K_\varepsilon(z-y)(f(y)\,d\nu(y) - g(y)\,d\sigma(y)) \right| \\
&\qquad = \left| \int_{B_x} (K_\varepsilon(z-y) - K_\varepsilon(z-x))(f(y)\,d\nu(y) - g(y)\,d\sigma(y)) \right| \\
&\qquad \le c \int_{B_x} \frac{|x-y|}{|x-y|^{d+1}} (|f(y)|\,d\nu(y) + |g(y)|\,d\sigma(y)) \\
&\qquad \approx \frac{r(x)}{\operatorname{dist}(B(z), B_x)^{d+1}} \int_{B_x} (|f|\,d\nu + |g|\,d\sigma).
\end{aligned}$$

Recall that $B(z)$ stands for the ball $B_x, x \in H$, that contains z.

We consider the operators

$$T_\nu(f)(z) = \sum_{x\in H: z\notin B_x} \frac{r(x)}{\operatorname{dist}(B(z), B_x)^{d+1}} \int_{B_x} f\,d\nu\,,$$

and T_σ, which is defined in the same way with ν replaced by σ. Observe that

$$\begin{aligned}
I(f,g) &\le c\|T_\nu(|f|) + T_\sigma(|g|)\|^2_{L^2(\nu+\sigma)} \\
&\le 2c\|T_\nu(|f|)\|^2_{L^2(\nu+\sigma)} + 2c\|T_\sigma(|g|)\|^2_{L^2(\nu+\sigma)} \\
&= 4c\|T_\nu(|f|)\|^2_{L^2(\nu)} + 4c\|T_\sigma(|g|)\|^2_{L^2(\sigma)},
\end{aligned}$$

where, for the last equality, we took into account that both $T_\nu(|f|)$ and $T_\sigma(|g|)$ are constant on each ball B_x and that $\nu(B_x) = \sigma(B_x)$ for all $x \in H$.

To finish the proof of (4) it is enough to show that T_ν is bounded in $L^2(\nu)$ and T_σ in $L^2(\sigma)$. We only deal with T_σ, since the arguments for T_ν are analogous. We argue by duality again. So we consider non-negative functions $f, h \in L^2(\sigma)$. We have

$$\int T_\sigma(f)\, h\, d\nu \approx \int \left(\sum_{x \in H: z \notin B_x} \frac{r(x)}{\operatorname{dist}(z, B_x)^n} \int_{B_x} f\, d\sigma \right) h(z)\, d\sigma(z)$$
$$= \sum_{x \in H} r(x) \int_{B_x} f\, d\sigma \int_{\mathbb{R}^d \setminus B_x} \frac{1}{\operatorname{dist}(z, B_x)^{d+1}}\, h(z)\, d\sigma(z).$$

From the growth of degree d of σ and the fact that the balls $2B_x$ are disjoint, it follows easily that

$$\int_{\mathbb{R}^n \setminus B_x} \frac{1}{\operatorname{dist}(z, B_x)^{d+1}}\, h(z)\, d\sigma(z) \leq \frac{c}{r(x)}\, M_\sigma h(y),$$

for all $y \in B_x$, where M_σ stands for the (centered) maximal Hardy-Littlewood operator (with respect to σ). Then we deduce that

$$\int T_\sigma(f)\, h\, d\sigma \lesssim \sum_{x \in H} \int_{B_x} f(y)\, M_\sigma h(y)\, d\sigma(y) \lesssim \|f\|_{L^2(\sigma)} \|h\|_{L^2(\sigma)},$$

by the $L^2(\sigma)$ boundedness of M_σ. Thus T_σ is bounded in $L^2(\sigma)$.

So we completely proved that R^d_σ is bounded in $L^2(\sigma)$. Then we have that $R^d_{\mu_{p,s}}$ is bounded in $L^2(\mu_{p,s})$ (this is just a repeated application of Lemma 2).

But $\mu_{p,s}$ is a d-AD regular measure. So Lemma 1 (the Main Lemma) is proved. And, by the way, $d = n - 1$ was not used in its proof. Therefore, Theorem 2 got entirely reduced to Theorem 3.

In the rest of our lectures we will be mostly concerned with indicating the main points of the proof of Theorem 3:

Theorem 3. *Let $\mathcal{H}^{n-1}(E) < \infty$. If R^{n-1}_n is bounded in $L^2(E, \mathcal{H}^{n-1})$, and E is $(n-1)$-AD regular, then E is $(n-1)$-uniformly rectifiable.*

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