RECTIFIABILITY OF MEASURES WITH BOUNDED RIESZ TRANSFORMS: FROM SINGULAR OPERATORS TO GEOMETRIC MEASURE THEORY

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1. Lecture 2: The description of removable sets of Lipschitz harmonic functions

1.1. **Critical Dimension.** Let $L(E)$ denote the set of harmonic functions in $\mathbb{R}^n \setminus E$ (E is compact as usual), such that they are bounded at infinity and Lipschitz in $\mathbb{R}^n$. Of course then they are bounded in $\mathbb{R}^n$. Constant functions belong to $L(E)$. We want to describe sets E such that $L(E)$ contains **only** constant functions.

Definition 2.1. Such sets are called *removable* for Lipschitz harmonic functions.

Two remarks are in order:

Remarks. 1) Removability is a local notion. What we mean by that is that if we consider a big ball containing E and denote by $L(E, B)$ the set of harmonic functions in $B \setminus E$ (E is compact as usual), such that they are Lipschitz in B, then removability of E is equivalent to

$$L(E, B) = L(\emptyset, B)\,,$$

which means that $L(E, B)$ consists only of functions harmonic (and Lipschitz) in the whole ball B.

2) Instead of the class $L(E)$ we could have considered a slightly bigger class of functions harmonic in $\mathbb{R}^n \setminus E$, Lipschitz in $\mathbb{R}^n \setminus E$ and continuous in $\mathbb{R}^n$. Everything below will be true. But if E does not have interior (and in many interesting cases it will not have interior) this is the same class.

First let us understand what is the critical dimension of removability/non-removability threshold.

Obviously, if $\dim E = s = n-1+\varepsilon$, there are plenty of non-constant functions in $L(E)$. Here is why. By Frostman's lemma [Carl] E carries a strictly positive measure σ, such that

$$\sigma(B(x, r)) \leqslant r^s,\ \forall x \in E.$$

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Consider the Newtonian potential (here we write it for $n \geqslant 3$)

$$U^\sigma(x) := \int \frac{1}{|x-y|^{n-2}} d\sigma(y).$$

It is a harmonic function outside the support of σ, so in $\mathbb{R}^n \setminus E$. Its gradient

$$\nabla U^\sigma(x) = R_n^{n-1}(\sigma)(x) := \left(\frac{x_1}{|x|^n}, \dots, \frac{x_n}{|x|^n}\right) * \sigma$$

will be a bounded in the whole $\mathbb{R}^n$ vector function. In fact,

$$\left|\int_0^{diam\, E} \frac{x_i - y_i}{|x-y|^n} d\sigma(y)\right| \leqslant \int_0^{diam\, E} \frac{1}{|x-y|^{n-1}} d\sigma(y) \leqslant$$

$$C\int_0^{diam\, E} \frac{\sigma(B(x,r))}{r^n} dr \leqslant C\int_0^{diam\, E} \frac{r^{n-1+\varepsilon}}{r^n} dr < C(n,\varepsilon)\,.$$

This shows that U^σ is a Lipschitz function. The fact that it is not constant follows from an elementary calculation of the flux. Let S denote a sphere in $\mathbb{R}^n \setminus E$ that has the whole E inside it, let ν be the outer unit normal to S. Then it is easy to see that (with strictly positive absolute constant $c(n)$)

$$\int_S \frac{\partial U^\sigma}{\partial \nu} d\mathcal{H}^{n-1} = -c(n)\sigma(E) \neq 0.$$

On the other hand, if dim $E < n-1$, or if the dimension is equal to $n-1$, but $\mathcal{H}^{n-1}(E) = 0$, then $L(E)$ contains only constant functions. Let us see this.

Our E can be put inside the union of finitely many closed balls B_i of radius r_i such that

$$\sum r_i^{n-1} < \varepsilon.$$

Let S_i be the boundaries of these balls. By $\Phi_x(y) = \frac{c(n)}{|x-y|^{n-2}}$ we denote the fundamental solution of Laplacian in $\mathbb{R}^n$ (here $n \geqslant 3$). Let Ω be a component of infinity of $\mathbb{R}^n \setminus \cup_i B_i$, and let S be boundary of Ω. Let ν be unit normal to S directed inside Ω. Choose a point x far from E. Then if ε is small it will be in Ω.

Let $u \in L(E)$ and $u(\infty) = 0$. We write

$$|u(x)| \leqslant \left|\int_S u(y)\frac{\partial \Phi_x}{\partial \nu_y} d\mathcal{H}^{n-1}\right| + \left|\int_S \Phi_x(y)\frac{\partial u}{\partial \nu_y} d\mathcal{H}^{n-1}\right| \leqslant (C_1 + C_2)\mathcal{H}^{n-1}(S) \leqslant C\varepsilon.$$

Here C_1, C_2 are L^∞ bounds for $u|\mathbb{R}^n$ and $|\nabla u||\mathbb{R}^n$ correspondingly. As ε can be chose arbitrary close to zero, we see that $\nabla u = 0$ identically.

So we see that the critical dimension for removability for Lipschitz harmonic functions is $n-1$. Moreover there are only two cases left.

1.2. **Two Cases.** Here they are: 1) $0 < \mathcal{H}^{n-1}(E) < \infty$; 2) $\dim E = n-1$, $\mathcal{H}^{n-1}(E) = \infty$.

For $n = 2$ both cases are treated, the first by David–Mattila in [DM] and Nazarov–Treil–Volberg in [NTV] (see also the exposition in [Vo]). The second case is treated by Tolsa in [T1] (see also the exposition in [Vo], [T-b]).

However, for $n \geqslant 3$ we can teat only the first case. The reason is the same "cruelly missing" tool of Menger's curvature introduced into the subject by Melnikov. This tool is immediately missing if $n-1 = d > 1$.

By the word "treat" we mean a find a splitting of E as in 1) (or 2)) into removable and nonremovable sets depending on finer geometric indicators than dimension and than Hausdorff measure.

Even though we cannot treat the case 2) for $n \geqslant 3$, we can prove a highly nontrivial statement about it.

Theorem 1. *If $L(E) \setminus \{Constants\} \neq \emptyset$ (so we are in the cases 1) or 2) above), then E supports a strictly positive measure σ such that the Newtonian potential $U^\sigma \in L(E)$.*

Remark. Notice that nobody told us that function $u \in L(E) \setminus \{Constants\}$ is a Newtonian potential of a positive measure. Moreover, in general, this is false, there are many u in $L(E) \setminus \{Constants\}$ (if this class is nonempty) that are absolutely **not** Newtonian potentials of positive measures. However, Theorem 1 says, that if a non constant function in $L(E)$ exists, then a very special non constant function in $L(E)$ must also exist. For the case $n = 2$ this led Tolsa [T1] to the geometric description of sets in case 2). But Menger's curvature was used in [T1] (both in the proof of the Theorem 1 for $n = 2$ and in its geometric corollaries), and this is why we cannot continue from Theorem 1. For $n \geqslant 3$ Theorem 1 was proved in [Vo] by getting rid of Menger's curvature considerations.

Here is the main result we want to prove now.

Theorem 2. *Let $\mathcal{H}^{n-1}(E) < \infty$ and $\Gamma_{n-1,n}(E) = 0$. Then set E is purely $(n-1)$-unrectifiable, meaning that its intersection with any C^1 $(n-1)$-dimensional submanifold in $\mathbb{R}^n$ must have zero surface measure. The converse is also true (but the converse is a much easier statement).*

Remark. Purely d-unrectifiable sets E, $\mathcal{H}^d(E) < \infty$ have many equivalent characterizations, and the notion is very robust. For example, C^1 d-dimensional submanifolds can be replaced by Lipschitz images of $\mathbb{R}^d$, and also can be replaced by graphs of Lipschitz functions $\mathbb{R}^d \to \mathbb{R}^{n-d}$. The standard reference is the book of Federer [Fe].

Let us reformulate the main statement of Theorem 2.

Theorem 3. *Let E be a compact set in $\mathbb{R}^n$, and $0 < \mathcal{H}^{n-1}(E) < \infty$. Suppose $L(E) \setminus \{Constants\} \neq \emptyset$. Then E contains a piece E' of a C^1 $(n-1)$-dimensional manifold such that*

$$\mathcal{H}^{n-1}(E') > 0\,.$$

Below is the proof.

1.3. **The Newtonian Potential of Signed Measure in $L(E)$.** Given $L(E) \setminus \{Constants\} \neq \emptyset$ we will build now a nontrivial signed measure τ such that $U^\tau \in L(E)$.

This is easy. Let $u \in L(E) \setminus \{Constants\}$. Consider $u_\varepsilon := u * \phi_e$, where $\phi_e := \frac{1}{\varepsilon^n}\phi\big(\frac{x}{\varepsilon}\big)$, and $\phi \in C_0^\infty(B)$, where B is the unit ball centered at the origin.

All u_ε are uniformly Lipschitz and smooth (not uniformly). Consider functions Δu_e and measures $\tau_\varepsilon := \Delta u_e dx$ with these densities. Uniformly $\int_{\partial B(x,r)} |\frac{u_\varepsilon}{\partial \nu}| d\mathcal{H}^{n-1} \leqslant Cr^{n-1}$. Then it is easy to see that uniformly in ε, x, r we have

$$|\tau_\varepsilon(B(x,r))| \leqslant Cr^{n-1}.$$

But measure τ_ε is supported by the ε-neighborhood of E. And $\mathcal{H}^{n-1}(E) < \infty$.

Then the last display inequality implies that the total variation measures $|\tau_\varepsilon|$ satisfy

$$|\tau_\varepsilon|(\mathbb{R}^n) \leqslant 2L$$

for all sufficiently small ε. Let us consider a weak limit τ of τ_{ε_k}. This measure τ is of finite total variation, it lies on E, and it satisfies

$$|\tau|(B(x,r)) \leqslant Cr^{n-1}.$$

Obviously we just proved that function's u distributional Laplacian Δu is equal to signed measure τ satisfying the properties above. Moreover, it is quite easy to see now that

$$u(x) = u(\infty) + U^\tau(x),$$

where U^τ is Newtonian potential of τ.

We proved that if $L(E)$ contains a non constant function, it also contains a non constant Newtonian potential of a signed measure. Moreover, this measure is absolutely continuous with respect to $\mathcal{H}^{n-1}|E$, and the density is bounded (and signed).

Conclusion: There exists a bounded measurable function b on E, such that for $d\tau = b\, d\mathcal{H}^{n-1}|E$ the Newtonian potential

$$U^\tau(x) := \int_E \frac{1}{|x-y|^{n-2}} b(y) d\mathcal{H}^{n-1}(y)$$

has bounded gradient, which is not identically zero in $\mathbb{R}^n \setminus E$.

1.4. **One More Step:** $\tau(E) \neq 0$**.** Notice that $\tau(E)$ is the flux of u on any surface S surrounding E. This flux can vanish, even if ∇U^τ is not identically zero.

Let ϕ be an arbitrary function C^∞ with compact support. The distributional Laplacian of $\phi\, U^\tau$ is a measure that can be written as

$$\Delta(\phi\, U^\tau) = \phi\, d\tau + U^\tau \Delta(\phi) dx + (\nabla U^\tau \cdot \nabla \phi) dx\,.$$

The second and the third terms have bounded density with respect to Lebesgue measure in $\mathbb{R}^n$. Newtonian potentials of measures that have bounded density with respect to Lebesgue measure in $\mathbb{R}^n$ have obviously bounded gradients.

Now the Newtonian potential of the left hand side is also Lipschitz. In fact, the Newtonian potential of the left hand side is just $\phi\, U^\tau$ itself, and this function has a bounded gradient because U^τ has a bounded gradient.

Combining all this we conclude that $U^{\phi\, d\tau}$ has a bounded gradient for any ϕ. But for some ϕ then

$$\int_E \phi\, d\tau \neq 0.$$

Otherwise, τ would be a zero measure, which contradicts the fact that ∇U^τ is not identically zero.

Conclusion: There exists a bounded measurable b on E, such that for $d\tau = b\, d\mathcal{H}^{n-1}|E$ Newtonian potential

$$U^\tau(x) := \int_E \frac{1}{|x-y|^{n-2}} b(y) d\mathcal{H}^{n-1}(y)$$

has bounded gradient, and such that $\int_E b\, d\mathcal{H}^{n-1} \neq 0$.

1.5. **Bound on the Growth of Underlying Positive Measure.** The measure τ is **not** positive as a rule. However, it can be written as $d\tau = b\, d\mu$, just by denoting by μ the restriction of $\mathcal{H}^{n-1}|E$ onto $E^+ := \{x \in E : b(x) \neq 0\}$. Notice that then

$$\frac{1}{r^{n-1}} \mu(B(x,r)) = \frac{|\tau(B(x,r))|}{r^{n-1}} \Big/ \frac{|\tau(B(x,r))|}{\mu(B(x,r))}.$$

Now let x belong to the set of Lebesgue points of $b|E^+$. Then by the Lebesgue Differentiation Theorem there exists a nonzero (so strictly positive) limit of the fraction $\frac{|\tau(B(x,r))|}{\mu(B(x,r))}$ in the denominator above. On the other hand, we already saw that

$$|\tau(B(x,r))| \leqslant Cr^{n-1}.$$

We conclude that

$$\text{for } \mu \text{ a.e. } x \ \limsup_{r\to 0} \frac{1}{r^{n-1}}\mu(B(x,r)) < \infty\,.$$

We established (nonuniform) $(n-1)$-growth condition on positive measure $\mu = \mathcal{H}^{n-1}|E$.

1.6. **The Bound on the Maximal Singular Operator.** Let us denote temporarily by letter T the operator with the kernel $R := R_n^{n-1}$. We want to estimate $T^*_\mu(b)$, where b,μ are those above. Here

$$T^*_\mu(b)(x) := \sup_{\varepsilon>0} |T^\varepsilon_\mu(b)(x)| := \sup_{\varepsilon>0}\left|\int_{y:|y-x>\varepsilon} R(x-y)b(y)d\mu(y)\right|\,.$$

Theorem 4. *For μ a. e. x_0 we have $T^*_\mu(b)(x_0)<\infty$.*

Proof. Take x_0 for which $(n-1)$ growth holds. Consider $B = B(x,\varepsilon)$ and write the expression

$$I := \frac{1}{|B|}\int_B \left(\int_{E\setminus B} R(x-y)b(y)d\mu(y)\right)dx.$$

Function under $\int_B$ sign is harmonic. Therefore, expression $I = T^\varepsilon_\mu(b)(x_0)$. On the other hand, recall that a bounded function $U^\tau(x), x\in\mathbb{R}^n\setminus E$, is exactly $\int_E R(x-y)b(y)d\mu(y)dx$. And I can be written as

$$I = \frac{1}{|B|}\int_B\left(\int_E R(x-y)b(y)d\mu(y)\right)dx + \frac{1}{|B|}\int_B\left(\int_B R(x-y)b(y)d\mu(y)\right)dx$$
$$=: II + III\,,$$

and both expressions II, III make perfect sense and they are bounded, because in II we integrate a bounded function U^τ over B, and III we can estimate it as follows:

$$|III| \leqslant \frac{C}{\varepsilon^n}\|b\|_\infty \int_B d\mu(y)\int_B |R(x-y)|dx \leqslant$$
$$\frac{C}{\varepsilon^n}\int_{B(x_0,\varepsilon)} d\mu(y)\int_{B(x_0,\varepsilon)}\frac{1}{|x-y|^{n-1}}dy \leqslant \frac{C}{\varepsilon^n}\varepsilon\mu(B(x_0,\varepsilon)) \leqslant C(x_0)\,.$$

The theorem is proved.

$\square$

1.7. **Nonhomogeneous Nonaccretive Tb Theorem.** T1 and Tb theorems are the main tools in the Singular Integral theory, especially in what concerns Calderón–Zygmund (CZ) operators (but the ideology goes much beyond CZ operators). They were proved by David–Journé for CZ operators with respect to Lebesgue measure in $\mathbb{R}^n$. Christ extend this into CZ operators with respect to doubling measures (homogeneous space setting) [Ch].

T1 theorem says that operator with CZ kernel (say, anti-symmetric kernel) is bounded in $L^2(dx)$ if it is bounded uniformly on χ_B, where B runs through all balls. Tb theorem says that operator with CZ kernel (say, anti-symmetric kernel) is bounded in $L^2(dx)$ if it is bounded uniformly on $b\chi_B$, where B runs through all balls, and b is an *accretive* function, namely,

$$\frac{1}{|B|}\left|\int b dx\right| \geqslant \delta > 0$$

independently of B ($b = 1$ is of course accretive).

As we said Christ [Ch] proved this kind of results if dx is replaced by $d\mu$, $|B|$ by $\mu(B)$, and measure μ is *doubling*: $\mu(B(x, 2r)) \leqslant C\mu(B(x, r))$.

Metric measure spaces with such measure (homogeneous spaces) often occur in important questions in Analysis. However, for our metric measure space (E, μ) (E is provided with the usual euclidean metric, $\mu := \mathcal{H}^{n-1}|E^+$) the doubling is beyond its reach completely. So we are dealing with a *nonhomogeneous space*.

Moreover, function b from Theorem 4 is **not** accretive, and there is no hope that it can be "made" accretive. It is an arbitrary bounded nonzero function. Frankly there is a very weak trace of accretivity in b. Namely, in Section 1.4 we saw that

$$\frac{1}{\mu(Q_0)}\left|\int_{Q_0} b\,d\mu\right| = \frac{|\tau(E)|}{\mu(Q_0)} = \delta > 0\,,$$

where Q_0 is a large cube containing the whole compact E. So we do have a meager accretivity: we have *accretivity in one scale*.

Here is a nonhomogeneous nonaccretive Tb theorem from [NTV], [Vo] that played a very important role in multitude of recent achievements, including Tolsa's proofs of Painlevé and Vitushkin's conjecture [T1].

Theorem 5. *Let T be operator with kernel R (in fact, any CZ kernel of singularity $-(n-1)$, and $n > 1$ can be non-integer as well), let measure μ satisfies a nonuniform $(n-1)$ growth condition:*

$$\limsup_{r\to 0} \frac{1}{r^{n-1}}\mu(B(x, r)) < \infty \ \mu\ a.e.\ x \in E.$$

Let $b \in L^\infty(\mu)$ and b satisfies one-scale accretivity condition as above:

$$\frac{1}{\mu(E)}\left|\int_E b\,d\mu\right| = \delta > 0.$$

Finally let $|T^*_\mu(b)(x)|$ is finite μ almost everywhere.

Then there exists a measurable subset $E' \subset E$ such that

1) $\mu(E') \geqslant \frac{\delta}{2\|b\|_{L^\infty(\mu)}}\mu(E)$,

and

2) $T_\mu : L^2(E',\mu) \to L^2(E',\mu)$ is a bounded operator.

The proof is too complicated and long to be given here. It is based on dyadic techniques mixed with a probabilistic approach. This is a random way how the proof chooses a piece E' of E. Notice that this random geometric constructions (see [NTV], [Vo], [T-b]) got many extensions by Nazarov, Treil, Volberg, Hytönen, Martikainen, Lacey, Sawyer, Uriatre-Tuero and many others.

Notice also that usually "a piece" to be chosen is "a leftover" after "bad" pieces are deleted. This brings in a difficult task of controlling that something is actually left after all deleting! See the details in [NTV], [Vo], [T-b].

1.8. **Recapitualation.** We established the following. If $L(E)$ consists of not only constant functions, then it should contain a non-constant function of the type $U^\tau = \int_E \frac{1}{|x-y|^{n-2}}\,d\tau(y)$ (here $n \geqslant 3$ with obvious change for $n = 2$). Measure τ is $d\tau = b\,d\mu$, where $\mu := \mathcal{H}^{n-1}|E^+$, E^+ is just a certain piece of E of positive $\mathcal{H}^{n-1}$ measure, $b \in L^\infty(\mu)$, $\int_E b\,d\mu \neq 0$.

Then Theorem 5 steps into the main stage, and we got

$$E' \subset E, \mathcal{H}^{n-1}(E') > 0$$

such that

$$T_{\mathcal{H}^{n-1}|E'} : L^2(E',\mathcal{H}^{n-1}) \to L^2(E',\mathcal{H}^{n-1}) \text{ is bounded.} \tag{1}$$

This is not exactly what we promised. We promised to find $E'' \subset E, \mathcal{H}^{n-1}(E'') > 0$, such that it lies on a C^1 $(n-1)$-dimensional submanifold.

The reader, however, should be recalled that in Lecture 1 we formulated the following two theorems (proved in [NToV1] and [NToV2] correspondingly).

Theorem 6. *Let $\mathcal{H}^{n-1}(E) < \infty$. If R^{n-1}_n is bounded in $L^2(E,\mathcal{H}^{n-1})$, and E is $(n-1)$-AD regular, then E is $(n-1)$-uniformly rectifiable.*

Theorem 7. *Let $\mathcal{H}^{n-1}(E) < \infty$. If R_n^{n-1} is bounded in $L^2(E, \mathcal{H}^{n-1})$, then the set E is $(n-1)$-rectifiable.*

Notice that Theorem 7 now gives us a desired conclusion about the "hidden smoothness and connectivity" of E'. Therefore, the main Theorems of Lecture 2: Theorem 3 and Theorem 2 are already proved.

Remark. We recall one more time that for $n = 2$ Theorem 6 was proved by Mattila–Melnikov–Verdera in [MMV], and Theorem 7 was proved by David and Léger in [Le]. However, for $n > 2$ Menger's curvature approach of these papers does not work anymore.

The next lecture is devoted to the reduction of Theorem 7 to Theorem 6. And later in our lectures we go along the proof of Theorem 6 in almost all details.

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