RECTIFIABILITY OF MEASURES WITH BOUNDED RIESZ TRANSFORMS: FROM SINGULAR OPERATORS TO GEOMETRIC MEASURE THEORY

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1. Lecture 1: Connectivity?

In what follows $\mathcal{H}^d$ denotes the Hausdorff measure (it is a Borel but not a sigma finite measure in $\mathbb{R}^n, 0 < d < n$). We will usually consider only compact sets in $\mathbb{R}^n$, and we are not very interested in $d = n$ case. The Hausdorff dimension $\dim E$ of such a set is the infimum of s such that $H^s(E) = 0$, (=the supremum of those s for which $H^s(E) = \infty$). Our main interest will be in compact sets E such that $0 < H^d(E) < \infty$. Obviously in this case $d = \dim E$.

Let us ask the following simple question: for which compact set E we can find another set Γ, which 1) contains E, 2) dim $\Gamma = 1$, 3) Γ is connected?

The necessary condition is of course dim $E \leqslant 1$. This is also sufficient, and Γ is easy to construct starting with $1/k$ nets of E and connecting points of each finer net to the closest point of the previous net.

However, here is a much more interesting and difficult question. It is again hinged on *connectivity* of the ambient set.

Question 1.1 For which compact set E we can find another set Γ, which 1) contains E, 2) $\mathcal{H}^1(\Gamma) < \infty$, 3) Γ is connected?

Again the obvious necessary condition is $\mathcal{H}^1(E) < \infty$ (and, thus, dim $\Gamma \leqslant 1$), but now it is very far from the sufficient condition. Question 1.1 can be (and is) called *Analytic Traveling Salesman Problem*. The necessary and sufficient condition for this to happen was found by Peter Jones in [PJ] if $n = 2$ (that is if $E \subset \mathbb{R}^2$). He also showed that his condition is sufficient in any $\mathbb{R}^n$. Kate Okikioulu proved the necessity of Peter Jones' condition for $n > 2$ in [KO].

The connected sets of finite length (that is of finite $\mathcal{H}^1$) can be rather nasty, but there is an interesting sub-class of such sets having the so-called 1-AD regularity (AD stands for Ahlfors–David).

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Definition 1.2 Let E be a d-dimensional set in $\mathbb{R}^n$. It is called d-AD regular if with some $0 < c < C < \infty$

$$cr^d \leqslant \mathcal{H}^d(E \cap B(x,r)) \leqslant Cr^d, \ \ \forall x \in E, \forall r \in (0, \operatorname{diam} E)\,.$$

Connected 1-AD regular sets turn out to be the right class sets with *bounded geometry*, and they appear the right class to answer a plethora of interesting questions.

1.1. **Some Answers.** For the set E to be a subset of a connected 1-AD regular set, it is necessary to satisfy the right hand side inequality in Definition 1.2. But again this is very far from being sufficient. The necessary and sufficient condition was found again by Peter Jones [PJ] (again the necessity for $n > 2$ is due to [KO]).

Later Mattila–Melnikov–Verdera [MMV] proved the following remarkable result:

Theorem 1. *If the set $E \subset \mathbb{R}^2$ is 1-AD regular (so it satisfies both inequalities in Definition 1.2, not only the right one) then such an E is contained in a connected 1-AD regular set if and only if the Cauchy integral operator is bounded in $L^2(E, \mathcal{H}^1)$.*

The same is true for 1-AD regular sets $E \subset \mathbb{R}^n, n > 2$, if the Cauchy operator is replaced by vector Riesz operator $R^1 := R^1_n$.

This a variant of analytic traveling salesman problem for 1-AD regular sets. The same is true for 1-AD regular sets $E \subset \mathbb{R}^n, n > 2$, if the Cauchy operator is replaced by vector Riesz operator $R^1 := R^1_n$.

We will need this operator very soon, so let us tell what it is.

Definition 1.4. The operator with the kernel

$$R^s(x,y) := \left(\frac{x_1 - y_1}{|x-y|^{s+1}}, \dots, (\frac{x_n - y_n}{|x-y|^{s+1}} \right)$$

is called the Riesz operator (we should write R^s_n, but we skip n every time it is clear what is n).

Notice that for $s = 1, n = 2$ the components of R^1_2 are the real and imaginary parts of Cauchy kernel

$$C(z,w) := \frac{1}{z-w}\,.$$

Notice that the left hand side inequality of Definition 1.2 is much less natural than the right hand side one. In fact, the boundedness of R^1_n in $L^2(E, \mathcal{H}^1)$ implies (rather easily, see [D]) the right hand side inequality. But of course it does not imply the left hand side one.

Therefore, here is a natural

Question 1.5 Given $E \subset \mathbb{R}^n, \mathcal{H}^1(E) < \infty$, such that R_n^1 is bounded in $L^2(E, \mathcal{H}^1)$, what kind of geometry must E have?

It is easy to see that the answer should change, the operator can be bounded, but E might not be 1-uniformly rectifiable. However, a weaker type of connectedness is preserved!

Definition 1.6. A set E in $\mathbb{R}^n$ is called d-rectifiable if there is a countable number of C^1-manifolds of dimension d: $\Gamma_1, \dots, \Gamma_k, \dots$ such that $\mathcal{H}^d(E \setminus \cup_{k=1}^{\infty} \Gamma_k) = 0$.

Rectifiability is also a sort of connectedness condition. Somehow separate parts of E should "feel" that many of them are in fact parts of the same smooth manifold. Here is a theorem establishing such form of connectedness of E if Riesz operator R_n^1 is bounded on E. It has been proved by Guy David and Jean-Christoph Léger.

Theorem 2. *Let $\mathcal{H}^1(E) < \infty$ and R_n^1 is bounded in $L^2(E, \mathcal{H}^1)$. Then there is a piece E_1 of E, $\mathcal{H}^1(E_1) > 0$ such that there exists a C^1 manifold Γ_1 with the property $E_1 \subset \Gamma_1$.*

Using the classical structure theorem of Besicovitch that every set $E, \mathcal{H}^1(E) < \infty$ can be decomposed to 1-rectifiable set, a set of length zero, and a set which has zero length intersection with any C^1 manifold (such sets are called *purely unrectifiable*), one can easily deduce another theorem of David and Léger.

Theorem 3. *Let $\mathcal{H}^1(E) < \infty$ and R_n^1 is bounded in $L^2(E, \mathcal{H}^1)$. Then E is 1-rectifiable.*

Unlike Theorem 1, which is a Traveling Salesman Problem (he should visit all towns spending the finite amount of gasoline), Theorem 2 tells us the story of a *traveling bandit*: he should rob as much $\mathcal{H}^1$ piece of measure as he can with a limited amount of gasoline.

Theorems 1 and 2 are very remarkable. They show that the boundedness of some singular operators force the points of the set on which the operator is bounded to "feel" that they can be connected with a finite cost.

1.2. **Other than Dimension** 1. If one starts to consider $d \neq 1$ many questions appear. Of course we can again start asking the intrinsic conditions of a set $E, \mathcal{H}^d(E) < \infty$ in $\mathbb{R}^n$ to be a subset of a connected set Γ such that $\mathcal{H}^d(\Gamma) < \infty$. Notice that $d = 1$ case have a very special relation with connectivity. So maybe the notion of connectivity should be modified for $d \neq 1$.

We saw that in the $d = 1$ case (in what concerns the aspect of singular integral operators) brought the notion of rectifiability. Rectifiability is not only something related to connectedness. It is also related to smoothness. It is a very low regularity replacement of smoothness.

Let us formulate a very low regularity concept of *uniform smoothness.*

Definition 1.7. The set $E \subset \mathbb{R}^n, \mathcal{H}^d(E) < \infty$ is called d-uniformly rectifiable if it is 1) d-AD regular, 2) for every $x \in E$ and every $r \in (0, \operatorname{diam} E)$, there exists a Lipschitz map $g = g_{x,r}$ from d-dimensional ball $B_d(0,r)$ into $\mathbb{R}^n$, $g(0) = x$, such that its Lipschitz constant does not depend on x, r and such that

$$\mathcal{H}^d(E \cap g(B_d(0,r))) \geqslant cr^d$$

with certain $c > 0$ independent of x, r.

Notice that d is of course an integer here. Notice also easily that any d-uniformly rectifiable set is d-rectifiable.

The case $d = 1$ is very special because of the fact that any compact E which is 1-AD regular and connected is 1-uniformly rectifiable. This is not true anymore for integer $d \geqslant 2$, $n > d$. One can easily understand this effect by reading, for example, the paper of Hofmann-Martel–Mayboroda [HMM].

Another big difference is that any 1-uniformly rectifiable set $E \subset \mathbb{R}^n$, is a subset of connected 1-AD regular (and, thus, connected 1-uniformly rectifiable) set Γ. This follows from the the combination of results of David–Semmes [DS] and Theorem 1, see [MMV]. The result of Peter Jones [PJ] is used in the course of the proof. This claim again is not true anymore for integer $d \geqslant 2$, $n > d$. As we already mentioned several times the case $d = 1$ is really special because of the role of $d = 1$ for the concept of connectivity.

1.3. **Rectifiability versus Connectivity.** We have two roads that can be traveled: 1) Try to understand a purely geometric question when $E \subset \mathbb{R}^n$, $\mathcal{H}^s(E) < \infty$, $s \geqslant 1$, can be put into *connected* set Γ, $\mathcal{H}^s(\Gamma) < \infty$? 2) Try to understand a purely analytic problem when the Riesz transform operator R^s_n is bounded in $L^2(E, \mathcal{H}^s)$, namely, whether this boundedness imposes a low regularity smoothness on E.

We already discussed that for $s = 1$ these questions are very much related, and in the case of 1-AD regularity of E have the same family of sets as an answer: 1-uniformly rectifiable sets.

For $s > 1$ question 1) turns out to be not very interesting at all. It is easy to see that any compact set E, $\mathcal{H}^s(E) < \infty$ can be put in a connected set Γ, $\mathcal{H}^s(\Gamma) < \infty$. Again Γ is easy to construct starting with $1/k$ nets of E and connecting points of each finer net to the closest point of the previous net.

However, the answer to question 2) is suspected to be very interesting and beautiful. For general s, n question 2) is called David–Semmes problem, it is asked repeatedly in [DS].

David–Semmes' Conjecture 1. Let d be an integer. If R_n^d is bounded in $L^2(E, \mathcal{H}^d)$, and E is d-AD regular, then E is d-uniformly rectifiable.

David–Semmes' Conjecture 2. Let d be an integer. If R_n^d is bounded in $L^2(E, \mathcal{H}^d)$, then the set E is d-rectifiable.

David–Semmes' Conjecture 3. If R_n^s is bounded in $L^2(E, \mathcal{H}^s)$, then s is some integer d (and then if Conjecture 1 has positive answer, E has to be d-rectifiable).

We saw that for $d = 1$ Conjecture 1 has a positive answer by the work [MMV], moreover, by [DS] the converse is true. Again for the case $d = 1$ the second Conjecture is also true. This follows from David–Léger's Theorem 3 proved in [Le].

Till recently Conjecture 3 was attacked only for $s \in (0, 1)$. This is due to Laura Prat [Pr]. And also, till recently all attempts to approach Conjectures 1, 2 for $d \in [2, n) \cap \mathbb{Z}$ failed, as well as the attempts to prove Conjecture 3 for $s \in (1, n) \cap (\mathbb{R} \setminus \mathbb{Z})$.

1.4. **Recent Progress.** Recently a progress was made, and the main goal of the present lectures is to present this progress.

In the paper [HMM] it was proved that if E is a priori connected, and in fact is a boundary of so-called one sided NTA domain (a domain having certain geometric conditions: a corkscrew condition and a Harnack chain condition (it is a space analog of being a quasidisc on the plane–sort of)), if E is $(n-1)$-AD regular and if R_n^{n-1} is bounded in $L^2(E, \mathcal{H}^{n-1})$, then E is $(n-1)$-uniformly rectifiable. The method used a harmonic measure approach, and use this extra connectivity assumption and certain regularity (one sided NTA) for $\mathbb{R}^n \setminus E$.

Notice that the case $d = n - 1$, a co-dimension 1 case, has a special interest because of the relations with boundary value problems for the Laplacian in very low regularity domains, see [HMM].

This co-dimension 1 case was completely settled in papers of Nazarov–Tolsa–Volberg [NToV1], [NToV2]. Both Conjectures 1 and 2 have a positive answer for $d = n - 1$ case.

Conjecture 3 also was considered, and for the case $s \in (n-1, n)$ it has been shown that R_n^s is never bounded in $L^2(E, \mathcal{H}^s), 0 < \mathcal{H}^s(E) < \infty$. Thus Conjecture 3 is confirmed for two intervals of s: $s \in (0, 1)$ (Prat [Pr]) and $s \in (n - 1, n)$. This latter result was obtained by Eiderman–Nazarov–Volberg [ENV].

Combining these results with [MMV] and [DS], we see that on the plane all Conjectures 1, 2, 3 have positive answers.

1.5. **Rectifiability versus Removability. Free Boundary Questions.** Free boundary questions usually deal with (harmonic) functions who are solutions of certain natural variational problems (like a minimal energy function majorizing a certain obstacle function).

However, sometimes free boundary is understood in a more general way, as a question about an a priori arbitrary set such that in its complement a certain nontrivial harmonic function exists. It then becomes a question of description of removable sets of singularities of some class of functions. In many simpler situations such removability is described by saying that a certain (Sobolev) capacity of a set is zero.

Arguably, one of the much less standard and in fact very difficult problem was a famous problem of Painlevé:

Describe $E \subset \mathbb{R}^2$ such that any analytic function outside of E, which is bounded outside of E must be constant.

Such sets are called the sets of analytic capacity zero. The reader should not be cheated by the use of the same word "capacity" in this setting in comparison with more classical Sobolev capacities. Vitushkin introduced analytic capacity, but he met immediately huge difficulties to answer the most basic question: Is analytic capacity sub-additive

$$\gamma(E_1 \cup E_2) \leqslant \gamma(E_1) + \gamma(E_2)?$$

Such sub-additivity is obvious for all classical Sobolev capacities. But this is still strictly speaking unknown for analytic capacity of Vitushkin.

However, in his celebrated work Tolsa [T1] described the sets E such that $\gamma(E) = 0$ (answered Painlevé's question), he also proved semi-additivity of γ (answered Vitushkin's question up to an absolute constant):

$$\gamma(E_1 \cup E_2) \leqslant C(\gamma(E_1) + \gamma(E_2))!$$

Here constant C is universal.

The real difference between the classical Sobolev capacities and Vitushkin's analytic capacity hides in the fact, that they are hinged on *positive kernels* (of the type $\frac{1}{|x-y|^s}$, for $s = 1, n = 2$ it is just $\frac{1}{|z-w|}$), and analytic capacity is hinged on *singular kernel* $\frac{1}{z-w}$. This small difference brings immense difficulties.

Lipschitz Harmonic Functions, Removability. The analogous question is the following (our E's are always compact): What are the sets $E \subset \mathbb{R}^n$, $n \geqslant 2$, such that there exits a function u outside of E, having the following properties: 1) u is Lipschitz in $\mathbb{R}^n \setminus E$, 2) E is continuous on $\mathbb{R}^n$ (it is more of a technical condition), 3) u is bounded at infinity, 4) u is **not** harmonic on all $\mathbb{R}^n$. The last property can be replaced by 4') u is not a constant function (because bounded at infinity Lipschitz function is of course bounded in $\mathbb{R}^n$, and bounded in $\mathbb{R}^n$ harmonic functions are constants).

Here it is also instructive to compare the question with classical Sobolev capacity $C_{n-1,n}$, which seeks to find the sets **not** supporting strictly positive measures such that their convolution with the kernel $\frac{1}{|x-y|^{n-1}}$ is bounded in $\mathbb{R}^n$.

It turns out (see later) that removability question for Lipschitz harmonic functions as above seeks to characterize compact sets $E \subset \mathbb{R}^n$ such that they do **not** support strictly positive measures such that their convolution with the kernel $R_n^{n-1}(x-y) = (\frac{x_1-y_1}{|x-y|^n}, \dots, (\frac{x_n-y_n}{|x-y|^n})$ is bounded in $\mathbb{R}^n$. For such sets we write $\Gamma_n^{n-1}(E) = 0$.

So $C_{n-1,n}(E) = 0$ means that if μ is a nonnegative measure on E such that

$$\int \frac{1}{|x-y|^{n-1}} d\mu(y) \in L^\infty \Rightarrow \mu = 0,$$

while $\Gamma_{n-1,n}(E) = 0$ means that if μ is a signed measure on E such that

$$\int R_n^{n-1}(x-y) d\mu(y) \in L^\infty \Rightarrow \mu = 0.$$

Notice that singularity of the kernels is the same, it is $-(n-1)$. But the classes of sets have nothing to do with each other.

Definition 1. 8. The sets E with $\Gamma_{n-1,n}(E) = 0$ are called the sets of zero Lipschitz harmonic capacity.

Tolsa proved

Theorem 4. *On the plane the sets of zero analytic capacity and the sets of zero Lipschitz harmonic capacity are the same.*

This is highly non trivial, but easy to explain. Notice that on the plane we can form $f = \partial u$, and if u is Lipschitz, then f is bounded, and if u is harmonic, then f is analytic. This is the beginning of the proof of Theorem 4, but it is *only* the beginning.

On the plane there is another remarkable fact, relating sets with $\Gamma_{1,2}(E) = 0$ with rectifiability.

The following theorem was proved independently and in different ways by David–Mattila [DM] and by Nazarov–Treil–Volberg [NTV], [Vo].

Theorem 5. *Let $\mathcal{H}^1(E) < \infty$ and $\Gamma_{1,2}(E) = 0$. Then set E is purely 1-unrectifiable, meaning that its intersection with any C^1 curve on the plain, or its intersection with any rectifiable curve on the plane must have zero length.*

The converse is also true (but the converse is a much easier statement). Such sets sometimes are called "invisible". This theorem solved an old problem of Denjoy. We will need bits and pieces of the proof later.

In the next lecture we will prove the following higher dimensional analog of Theorem 5:

Theorem 6. *Let $\mathcal{H}^{n-1}(E) < \infty$ and $\Gamma_{n-1,n}(E) = 0$. Then set E is purely $n-1$-unrectifiable, meaning that its intersection with any C^1 $n-1$ dimensional submanifold in $\mathbb{R}^n$ must have zero surface measure. The converse is also true (but the converse is a much easier statement.*

This is again the jump from $d = 1$ to $d = n-1$ (of course for $n = 2$ there should not be any jump, and there is none). Tools specific to the case of $d = 1$ cannot be used. This specific tool consists of use of Menger's curvature, discovered by Melnikov. It is an amazing geometric tool. But we cannot use it because it is "cruelly missing" by the expression of Guy David if $d > 1$. It was formally shown, however, that the Menger curvature approach is rigidly restricted to $d \leqslant 1$ case by Farag [Fa].

In the proof of Theorem 6 the most essential part will be to refer to the solution of David–Semmes Conjectures 1 and 2 in the co-dimension 1 case. So the rest of the lectures will be devoted to a some sort of explanation of the following results.

Theorem 7. *Let $\mathcal{H}^{n-1}(E) < \infty$. If R_n^{n-1} is bounded in $L^2(E, \mathcal{H}^{n-1})$, and E is $n-1$-AD regular, then E is $n-1$-uniformly rectifiable.*

This is proved in [NToV1]. The converse is also true, see [DS].

Theorem 8. *Let $\mathcal{H}^{n-1}(E) < \infty$. If R_n^{n-1} is bounded in $L^2(E, \mathcal{H}^{n-1})$, then the set E is $n-1$-rectifiable.*

This is proved in [NToV2].

1.6. **Traveling Salesman Problem (TSP) again.** We want to return to the case $d = 1$ again (even though it will not be very important for us). In this case, if the set E is a subset of vertices of a plane graph, then as Peter Jones writes in [PJ] "to compute the length of the shortest Hamiltonian cycle passing through E is the same up to a constant multiple as asking for the infimum of $\mathcal{H}^1(F)$ where F is a curve, $E \subset F$. (Such a F is called a spanning tree in TSP theory.) For infinite sets E, we cannot hope in general to have E be a subset of a Jordan curve. What we should therefore look at is connected sets which contain E". In [PJ] the algorithm to find the minimal, up to an absolute constant, spanning tree is given. This algorithm, unlike the algorithms in [Law] use the Euclidean structure of the ambient space. The advantage of algorithm in [PJ] that it can give minimal up to a constant connected set containing E for infinite sets E, which cannot be done by algorithms of [Law] as they work for finite graphs with triangle inequality. See also [PS]. In principle, the algorithm in [PJ] gives constant depending on the dimension n of the ambient euclidean space. Raanan Schul won over this "curse of dimensionality" in [RS].

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