Duality of multiparameter Hardy spaces H^p on spaces of homogeneous type

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Outline

Quasi-Metrics

A quasi-metric ρ on a set $\mathcal{X}$ is a function from $\mathcal{X} \times \mathcal{X}$ to $[0, \infty)$ such that the following three conditions hold:

1. $\rho(x, y) = 0$ if and only if $x = y$.
2. $\rho(x, y) = \rho(y, x)$ for all $x, y \in \mathcal{X}$
3. There exists a constant $A \geq 1$ such that for all $x, y, z \in \mathcal{X}$, we have
$$\rho(x, y) \leq A[\rho(x, z) + \rho(z, y)].$$

Thus, a quasi-metric is like a metric, except that the normal triangle inequality is replaced by a "triangle inequality" that holds up to a multiplicative constant.

Spaces of Homogeneous Type

Let $0 < \theta \leq 1$. A space of homogeneous type is a collection $(\mathcal{X}, \rho, \mu)_\theta$, where $\mathcal{X}$ is a set, ρ is a quasi-metric on $\mathcal{X}$, and μ is a nonnegative Borel regular measure on $\mathcal{X}$. We require that for all r such that $0 < r < \text{diam}(\mathcal{X})$, we have that $\mu(B(x, r)) \sim r$, where $B(x, r)$ is the ball of radius r centered at $x \in \mathcal{X}$. Lastly, we require that there is a constant $C_0 > 0$ such that for all $x, x', y \in \mathcal{X}$, we have

$$|\rho(x, y) - \rho(x', y)| \leq C_0 \rho(x, x')^\theta [\rho(x, y) + \rho(x', y)]^{1-\theta}.$$

Note that if $C_0 = 1$ and $\theta = 1$, this last condition is just the triangle inequality.

Slight Generalization

Instead of assuming that $\mu(B(x,r)) \sim r$, we could have assumed that $\mu(B(x,r)) \sim r^d$, where $d > 0$. However, in [MS79], Macias and Segovia proved that for $d > 0$, one can always find a quasi-metric $\overline{\rho}$ giving the same topology as ρ such that $\mu(B(x,r)) \sim r$, so throughout the paper the authors assume that $d = 1$.

Example

From the definition, it is clear that $\mathbb{R}$ is a space of homogeneous type. By the result of [MS79], it is clear that $\mathbb{R}^d$ is a space of homogeneous type for any natural number d.

Difficulties with spaces of Homogeneous Type

The main difficulty of working on spaces of homogeneous type is that there are no translations or dilations, and no Fourier transform. Thus, the challenge is to find methods to prove results without using these basic tools.

The paper in question relies heavily on Littlewood-Paley theory and a discrete Calderón reproducing formula for spaces of homogeneous type.

Dyadic Decomposition

There exists a collection $\{Q_a^k \subset \mathcal{X} : k \in \mathbb{Z}, a \in I_k\}$ of open subsets, called "cubes", where I_k is some index set, as well as constants $C_1, C_2 > 0$ such that

1. For each fixed k, we have $\mu(\mathcal{X} \setminus \cup_a Q_a^k) = 0$, and $Q_a^k \cap Q_b^k = \emptyset$ if $a \neq b$,
2. For any a, b, k, l with $l \geq k$, either $Q_b^l \subset Q_a^k$ or Q_b^l and Q_a^k are disjoint,
3. For each pair of k and a, there is a unique b such that $Q_a^k \subset Q_b^l$,
4. $\mathrm{diam}(Q_a^k) \leq C_1(1/2)^k$
5. each Q_a^k contains some ball of radius $C_2(1/2)^k$.

More on Dyadic Decomposition

Fix some large positive integer J. Let $k \in \mathbb{Z}$ and $\tau \in I_k$.

We denote by $N(k, \tau)$ the number of cubes Q_α^{k+J} that are contained in Q_τ^k, and we denote such a cube by $Q_\tau^{k,v}$, where $1 \leq v \leq N(k, \tau)$.

Approximations to the Identity I

A sequence $\{S_k\}_{k\in\mathbb{Z}}$ is said to be an approximation to the identity of order ϵ, where $0 < \epsilon \leq \theta$, if there is some constant C such that the kernel $S_k(x, y)$ of S_k satisfies

1. $$|S_k(x,y)| \leq C \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x,y))^{1+\epsilon}},$$
2. For $\rho(x, x') \leq (1/2A)(2^{-k} + \rho(x, y))$, we have
$$|S_k(x,y) - S_k(x',y)| \leq C \left(\frac{\rho(x,x')}{2^{-k} + \rho(x,y)}\right)^{\epsilon} \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x,y))^{1+\epsilon}}$$
3. For $\rho(y, y') \leq (1/2A)(2^{-k} + \rho(x, y))$, we have
$$|S_k(x,y) - S_k(x,y')| \leq C \left(\frac{\rho(y,y')}{2^{-k} + \rho(x,y)}\right)^{\epsilon} \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x,y))^{1+\epsilon}}$$

Approximations to the Identity II

4. For $\rho(x, x'), \rho(y, y') \leq (1/2A)(2^{-k} + \rho(x, y))$ we have

$$|S_k(x,y) - S_k(x,y') - S_k(x',y) + S_k(x',y')| \leq$$
$$C\left(\frac{\rho(x,x')}{2^{-k} + \rho(x,y)}\right)^{\epsilon}\left(\frac{\rho(y,y')}{2^{-k} + \rho(x,y)}\right)^{\epsilon}\frac{2^{-k\epsilon}}{(2^{-k} + \rho(x,y))^{1+\epsilon}}$$

5. $$\int_{\mathcal{X}} S_k(x,y)\, d\mu(y) = \int_{\mathcal{X}} S_k(x,y)\, d\mu(x) = 1$$

Analogy

As k increases, the kernels $S_k(x, y)$ become more concentrated around the diagonal $x = y$. The approximation to the identity $S_k(x, y)$ is analogous to the Poisson kernel $P_{2^{-k}}(x - y)$ on the upper half plane.

One-parameter Test Functions I

Fix $\beta, \gamma, r > 0$. A function f defined on $\mathcal{X}$ is said to be a test function of type (β, γ) centered at $x_0 \in \mathcal{X}$ with width r if there is a constant $C > 0$ such that f satisfies the following conditions:

1. $$|f(x)| \le C \frac{r^\gamma}{(r + \rho(x, x_0))^{1+\gamma}}$$
2. If $\rho(x, x') \le \frac{1}{2A}(r + \rho(x, x_0))$ then
$$|f(x) - f(x')| \le C \left(\frac{\rho(x, x')}{r + \rho(x, x_0)} \right)^\beta \frac{r^\gamma}{(r + \rho(x, x_0))^{1+\gamma}}$$
3. $$\int_{\mathcal{X}} f(x)\, d\mu(x) = 0.$$

One-parameter Test Functions II

If f is such a test function, we say $f \in \mathcal{G}(x_0, r, \beta, \gamma)$ and we define its norm to be the infimum over the set of all C for which conditions 1 and 2 hold.

Two-Parameter Test Functions I

For $i = 1, 2$, fix $\gamma_i, \beta_i, r_i > 0$ and let $(x_0, y_0) \in \mathcal{X} \times \mathcal{X}$. We say that a function f defined on $\mathcal{X} \times \mathcal{X}$ is a test function of type $(\beta_1, \beta_2, \gamma_1, \gamma_2)$ centered at (x_0, y_0) and with widths (r_1, r_2) if there is a constant $C > 0$ such that

1. $$\|f(\cdot, y)\|_{\mathcal{G}(x_0, r_1, \beta_1, \gamma_1)} \leq C \frac{r_2^{\gamma_2}}{(r_2 + \rho(y, y_0))^{1+\gamma_2}},$$
2. For $\rho(y, y') \leq \frac{1}{2A}(r_2 + \rho(y, y_0))$ we have
$$\|f(\cdot, y) - f(\cdot, y')\|_{\mathcal{G}(x_0, r_1, \beta_1, \gamma_1)} \leq C \left(\frac{\rho(y, y')}{r_2 + \rho(y, y_0)} \right)^{\beta_2} \frac{r_2^{\gamma_2}}{(r_2 + \rho(y, y_0))^{1+\gamma_2}},$$
3. Condition 1 should hold with the roles of x and y reversed.
4. Condition 2 should hold with the roles of x and y reversed.

Two-Parameter Test Functions II

We denote this space of test functions by $\mathcal{G}(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ and define the norm in this space to be the smallest C such that the above definition holds.

The Space $\overset{\circ}{\mathcal{G}}$

It is not difficult to see that no matter which x_0 and y_0 we choose, we get the same space of functions with equivalent norm. We choose some fixed (x_0, y_0) and let $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2) = \mathcal{G}(x_0, y_0; 1, 1; \beta_1, \beta_2; \gamma_1, \gamma_2)$. If $0 < \beta_1, \beta_2, \gamma_1, \gamma_2 < \theta$, we define the space $\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ to be the completion of $\mathcal{G}(\theta, \theta; \theta, \theta)$ in $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$. Note that $\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ is a Banach space.

We define the Littlewood-Paley operators D_k as $S_k - S_{k-1}$, where S is an approximation of the identity. We can now define the *Littlewood-Paley-Stein* square function by

$$g(f)(x_1, x_2) = \left[\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |D_j D_k(f)(x_1, x_2)|^2 \right]^{1/2}.$$

It can be show that, for $f \in L^p$ for $1 < p < \infty$, we have $\|g(f)\|_p \approx \|f\|_p$.

Let $\{S_k\}$ be an approximation to the identity of order θ. Suppose that $\frac{1}{1+\theta} < p \leq 1$ and $\frac{1}{p} - 1 < \beta_i, \gamma_i < \theta$. Then we define the Hardy space $H^p(\mathcal{X} \times \mathcal{X})$ to be the set of all $f \in \left(\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2, \gamma_1, \gamma_2)\right)'$ such that $\|g(f)\|_{L^p(\mathcal{X}\times\mathcal{X})} < \infty$. We define

$$\|f\|_{H^p(\mathcal{X}\times\mathcal{X})} = \|g(f)\|_{L^p(\mathcal{X}\times\mathcal{X})}$$

The Carleson Measure Spaces $\mathrm{CMO}^{\mathrm{p}}$

Let β_i, γ_i, etc. be as before. The Carleson measure space $\mathrm{CMO}^{\mathrm{p}}(\mathcal{X} \times \mathcal{X})$ is the set of all $f \in (\mathring{\mathcal{G}}(\beta_1, \beta_2, \gamma_1, \gamma_2))'$ such that

$$\sup_{\Omega} \left(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \int_{\Omega} \sum_{Q_{\tau_1}^{k_1,v_1} \times Q_{\tau_2}^{k_2,v_2} \subset \Omega} |D_{k_1} D_{k_2}(f)(x,y)|^2 \right.$$
$$\left. \chi_{Q_{\tau_1}^{k_1,v_1}}(x) \chi_{Q_{\tau_2}^{k_2,v_2}}(y)\, d\mu(x) d\mu(y) \right)^{1/2}$$

is finite. The above expression defines the norm in $\mathrm{CMO}^{\mathrm{p}}$.

Some Intuition

The above definition can be though of as analogous to the single parameter case on $\mathbb{R}$, where $\phi \in \mathrm{BMO}$ if and only if $|\nabla u|^2 y \, dx \, dy$ is a Carleson measure, where u is the harmonic extension of ϕ to the upper half plane.

Classical Calderón Reproducing Formula

$$f(x) = \int_0^\infty (\psi_t * \psi_t * f)(x) \frac{dt}{t}$$

where $\psi_t(x) = t^{-n}\psi(x/t)$. Also, we assume that ψ is real, radial, and well behaved, and that $\int_0^\infty (\widehat{\psi}(tx))^2 \frac{dt}{t} = 1$ for all $x \neq 0$.

This is not difficult to prove by using the Fourier transform.

Discreet Calderón Reproducing Formula

Let R denote an arbitrary dyadic rectangle of the form $Q_{\tau_1}^{k_1,v_1} \times Q_{\tau_2}^{k_2,v_2}$. Let the notation be the same as above. For each R, choose a point $(x', y') \in R$. Then there are families of linear operators $\{\widetilde{D}_k\}$ and $\{\overline{D}_k\}$ such that

$$\begin{aligned} f(x,y) &= \sum_R \mu(R) \widetilde{D}_{k_1} \widetilde{D}_{k_2}(x,y,x',y') D_{k_1} D_{k_2}(f)(x',y') \\ &= \sum_R \mu(R) D_{k_1} D_{k_2}(x,y,x',y') \overline{D}_{k_1} \overline{D}_{k_2}(f)(x',y'), \end{aligned}$$

The Main Result

The dual space of H^p is $\mathrm{CMO^p}$ if $\dfrac{2}{2+\theta} = p_0 < p \leq 1$.

(Recall that θ is a parameter measuring the regularity of the space of Homogeneous type. The higher θ, the more regular the space is

Sequence Spaces

The sequence space s^p is defined to be the space of all complex valued sequences $\{\lambda_{Q_{\tau_1}^{k_1,v_1} \times Q_{\tau_2}^{k_2,v_2}}\}$ such that

$$\|\lambda\|_{s^p} = \left\| \left[\sum_R \left(|\lambda_R| \widetilde{\chi_R}(\cdot) \right)^2 \right]^{1/2} \right\|_{L^p} < \infty.$$

Here $\widetilde{\chi_R} = \mu(R)^{-1/2} \chi_R$.

The sequence space c^p is the space of all sequences $\{t_{Q^{k_1,v_1}_{\tau_1} \times Q^{k_2,v_2}_{\tau_2}}\}$ such that

$$\|t\|_{c^p} = \sup_{\Omega} \left(\mu(\Omega)^{1-(2/p)} \int_{\Omega} \sum_{R \subset \Omega} \left(|t_R| \widetilde{\chi}(x,y)\right)^2 \, d\mu(x) d\mu(y) \right)^{1/2} < \infty.$$

Notice the Similarity

$$\|\lambda\|_{s^p} = \left\| \left[\sum_R \left(|\lambda_R| \widetilde{\chi_R}(\cdot) \right)^2 \right]^{1/2} \right\|_{L^p} < \infty.$$

$$\|f\|_{H^p} = \left\| \left[\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |D_j D_k(f)(x_1, x_2)|^2 \right]^{1/2} \right\|^{1/2} .$$

Notice the Similarity

$$\|t\|_{c^p} = \sup_{\Omega} \left(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \int_{\Omega} \sum_{R \subset \Omega} (|t_R| \widetilde{\chi}(x,y))^2 \; d\mu(x) d\mu(y) \right)^{1/2}.$$

$$\|f\|_{\mathrm{CMO^p}} =$$

$$\sup_{\Omega} \left(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \int_{\Omega} \sum_{R \subset \Omega} |D_{k_1} D_{k_2}(f)(x,y)|^2 \chi_R(x,y) \, d\mu(x) d\mu(y) \right)^{1/2}$$

Duality

The authors show that the dual of s^p is c^p. To do this, they use inequalities involving the Hardy-Littlewood maximal function and also use the duality properties of certain weighted ℓ^2 spaces related to both s^p and c^p.

Let all notation be the same as in definition 2.6. Let $\{P_k\}$ be another approximation to the identity of order θ, and let E_k denote the corresponding Littlewood-Paley operators. Suppose that $\frac{1}{1+\theta} < p \leq 1$ and $\frac{1}{p} - 1 < \beta_i, \gamma_i < \theta$. Then there is a constant $C > 0$ such that for all $f \in \left(\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2, \gamma_1, \gamma_2)\right)'$, we have

Min-Max for Hardy Spaces II

$$\Bigg\| \Bigg(\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \sum_{v_1=1}^{N(k_1,\tau_1)} \sum_{v_2=1}^{N(k_2,\tau_2)} \sup_{z \in Q_{\tau_1}^{k_1,v_1}, w \in Q_{\tau_2}^{k_2,v_2}} |D_{k_1} D_{k_2}(f)(z,w)|^2 \times \chi_{Q_{\tau_1}^{k_1,v_1}}(x) \chi_{Q_{\tau_2}^{k_2,v_2}}(y) \Bigg)^{1/2} \Bigg\|_{L^p}$$

$$\leq C \Bigg\| \Bigg(\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \sum_{v_1=1}^{N(k_1,\tau_1)} \sum_{v_2=1}^{N(k_2,\tau_2)} \inf_{z \in Q_{\tau_1}^{k_1,v_1}, w \in Q_{\tau_2}^{k_2,v_2}} |E_{k_1} E_{k_2}(f)(z,w)|^2 \times \chi_{Q_{\tau_1}^{k_1,v_1}}(x) \chi_{Q_{\tau_2}^{k_2,v_2}}(y) \Bigg)^{1/2} \Bigg\|_{L^p}$$

where the integration for the L^p norm is respect to $d\mu(x)d\mu(y)$.

Min-Max for Hardy Spaces III

This lemma is very useful for several reasons.

- It lets one show that the definition of H^p given does not depend on the choice of approximation to the identity. To see this, note that the left hand side is greater than the H^p norm with operators D_k, and the right side is less than C times the H^p norm with operators E_k.
- Also, the lemma in question allows one to approximate the integration defining the L^p norm of the g function by a sort of "Riemann sum". The lemma states that it does not matter whether we take the smallest or largest possible "Riemann sum", we still get an equivalent norm. Of course, the sums in the lemma are infinite. As the coefficient k gets larger, we need to sum over smaller and smaller dyadic boxes $Q_\tau^{k,v}$. This lemma allows us to relate Hardy space functions to sequences, which is crucial in proving duality.

A very important theorem is the Min-Max comparison principle for $\mathrm{CMO^p}$. For ease of notation, let R denote an arbitrary dyadic rectangle $Q_{\tau_1}^{k_1,v_1} \times Q_{\tau_2}^{k_2,v_2}$. If $2/(2+\theta) < p \leq 1$, then there is some constant $C > 0$ such that

$$\sup_{\Omega} \left(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R \subset \Omega} \mu(R) \sup_{(x,y) \in R} |D_{k_1} D_{k_2}(f)(x,y)|^2 \right)^{1/2}$$

$$\leq C \sup_{\Omega} \left(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R \subset \Omega} \mu(R) \inf_{(x,y) \in R} |D_{k_1} D_{k_2}(f)(x,y)|^2 \right)^{1/2}$$

where $R = Q_{\tau_1}^{k_1,\nu_1} \times Q_{\tau_2}^{k_2,\nu_2}$.

Min-Max for CMO^p II

- In fact, one can use different approximations to the identity on the left and right side of the inequality, which allows us to see that the CMO spaces are well defined.
- As before, another reason this theorem is important is because it allows us to consider a sort of “Riemann sum” instead of integration in the definition of CMO. This allows us to relate the *CMO* spaces to sequences, which again is key to proving the duality.

Discreet Calderón Reproducing Formula

Let R denote an arbitrary dyadic rectangle of the form $Q_{\tau_1}^{k_1,v_1} \times Q_{\tau_2}^{k_2,v_2}$. Let the notation be the same as above. For each R, choose a point $(x', y') \in R$. Then there are families of linear operators $\{\widetilde{D}_k\}$ and $\{\overline{D}_k\}$ such that

$$\begin{aligned} f(x,y) &= \sum_R \mu(R) \widetilde{D}_{k_1} \widetilde{D}_{k_2}(x,y,x',y') D_{k_1} D_{k_2}(f)(x',y') \\ &= \sum_R \mu(R) D_{k_1} D_{k_2}(x,y,x',y') \overline{D}_{k_1} \overline{D}_{k_2}(f)(x',y'), \end{aligned}$$

Overview of Proof of Min-Max for CMO

1. Choose (x', y') to be the point where $|D_{k_1} D_{k_2}(f)(x, y)|$ is minimized in R.
2. Apply $D_{k_1} D_{k_2}$ to both sides of the Discrete Calderón Reproducing Formula.
3. We obtain an expression that can be used to relate the left side of the inequality in the Min-Max Theorem to something resembling the right side.
4. Change the order of summation and carefully consider the various geometric quantities involved to prove the Min-Max Theorem.

The Lifting Operator

Define the lifting operator

$$S_D(f) = \left\{ \mu(Q_{\tau_1}^{k_1,\nu_1})^{1/2} \mu(Q_{\tau_2}^{k_2,\nu_2})^{1/2} D_{k_1} D_{k_2}(f)(y_1, y_2) \right\},$$

which maps elements of $(\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$ into sequences. Here y_j is the center of $Q_{\tau_j}^{k_j,\nu_j}$. Also, $0 < \beta_j, \gamma_j < \theta$.

The Projection Operator

For any sequence, we define the projection operator by

$$T_{\widetilde{D}}(s)(x_1, x_2) =$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_1=1}^{N(k_1,\tau_1)} \sum_{\nu_2=1}^{N(k_2,\tau_2)} s_{Q_{\tau_1}^{k_1,\nu_1} \times Q_{\tau_2}^{k_2,\nu_2}}$$

$$\times \mu(Q_{\tau_1}^{k_1,\nu_1})^{1/2} \mu(Q_{\tau_2}^{k_2,\nu_2})^{1/2} \widetilde{D}_{k_1} \widetilde{D}_{k_2}(x_1, x_2, y_1, y_2)$$

The $\widetilde{D}_k$ are the kernels of the operators associated with the D_k by the Calderón reproducing formula.

Relating H^p to the s^p

Let $\dfrac{1}{1+\theta} < p \leq 1$. For any $f \in H^p(\mathcal{X} \times \mathcal{X})$,

$$\|S_D(f)\|_{s^p} \leq C\|f\|_{H^p}.$$

Also, for any $s \in s^p$,

$$\|T_{\widetilde{D}}(s)\|_{H^p} \leq C\|s\|_{s^p}.$$

In addition, $T_{\widetilde{D}} \circ S_D$ equals the identity on H^p.

Relating $\mathrm{CMO^p}$ to c^p

Let $\dfrac{2}{2+\theta} < p \leq 1$. For any $f \in \mathrm{CMO^p}$,

$$\|S_D(f)\|_{s^p} \leq C\|f\|_{\mathrm{CMO^p}}.$$

Also, for any $t \in c^p$,

$$\|T_{\widetilde{D}}(t)\|_{\mathrm{CMO^p}} \leq C\|t\|_{c^p}.$$

In addition, $T_{\widetilde{D}} \circ S_D$ equals the identity on $\mathrm{CMO^p}$.

A Word About the Proofs

- The proofs of these two statements are similar to the proof of the Min-Max inequalities.
- Also, note that the adjoint of S_D is T_D and the adjoint of $T_{\widetilde{D}}$ is $S_{\widetilde{D}}$.
- The fact that $(H^p)' = \mathrm{CMO^p}$ follows fairly easily from the above statements about the lifting and projection operators, and from the fact that $(s^p)' = c^p$.

First, we need to show that

$$\left|\sum_R s_R \overline{t_R}\right| \leq \|s\|_{s^p} \|t\|_{c^p}.$$

Define the sets

- $\Omega_k = \left\{(x_1, x_2) : \left(\sum_R (|s_R| \widetilde{\chi_R}(x_1, x_2))^2\right)^{1/2} > 2^k\right\}$
- $B_k = \left\{R : \mu(\Omega_k \cap R) > \frac{1}{2}\mu(R),\ \mu(\Omega_{k+1} \cap R) \leq \frac{1}{2}\mu(R)\right\}$
- $\widetilde{\Omega_k} = \left\{(x, y) : \mathcal{M}_s(\chi_{\Omega_k}) > \frac{1}{2}\right\}$ where $\mathcal{M}_s$ is the strong maximal function.

Duality of s^p and c^p

$$\left|\sum_R s_R \overline{t_R}\right| \leq \left(\sum_k \left(\sum_{R\in B_k} |s_R|^2\right)^{p/2} \left(\sum_{R\in B_k} |t_R|^2\right)^{p/2}\right)^{1/p}$$

$$\leq \left(\sum_k \mu(\widetilde{\Omega_k})^{1-\frac{p}{2}} \left(\sum_{R\in B_k} |s_R|^2\right)^{p/2} \left(\frac{1}{\mu(\widetilde{\Omega_k})^{\frac{2}{p}-1}} \sum_{R\subset \widetilde{\Omega_k}} |t_R|^2\right)^{p/2}\right)^{1/p}$$

$$\leq \left(\sum_k \mu(\widetilde{\Omega_k})^{1-\frac{p}{2}} \left(\sum_{R\in B_k} |s_R|^2\right)^{p/2}\right)^{1/p} \|t\|_{c^p}$$

$$
\begin{aligned}
\frac{1}{2}\sum_{R\in B_k}|s_R|^2 &\le \sum_{R\in B_k}|s_R|^2\mu(R)^{-1}\frac{1}{2}\mu(R) \\
&\le \sum_{R\in B_k}|s_R|^2\mu(R)^{-1}\mu(\widetilde{\Omega}_k\setminus\Omega_{k+1}\cap R) \\
&\le \int_{\widetilde{\Omega}_k\setminus\Omega_{k+1}}\sum_{R\in B_k}(|s_R|\widetilde{\chi}_R(x))^2\,d\mu(x) \\
&\le 2^{2(k+1)}\mu(\widetilde{\Omega}_k\setminus\Omega_{k+1}) \\
&\le C2^{2k}\mu(\Omega_k)
\end{aligned}
$$

Substitute back into the previous slide to see that

$$
\frac{1}{\|t\|_{c^p}}\left|\sum_R s_R\overline{t_R}\right| \le C\sum_k 2^{kp}\mu(\widetilde{\Omega}_k) \le C\sum_k 2^{kp}\mu(\Omega_k) \le C\,\|s\|_{s^p}
$$

Thus $\|L\| \le C\|t\|_{c^p}$.

Other Direction of Duality

- Now we have shown that $c^p \subset (s^p)'$, and we need to show the other direction.
- Define $s^i_R = 1$ when $R = R_i$ and $s^i_r = 0$ otherwise. Then $\|s^i_R\|_{s^p} = 1$.
- We can write $s = \{s_R\} = \sum_i s_{R_i} s^i_{R_i}$.
- For $L \in (s^p)'$, define a sequence t by $\overline{t_{R_i}} = L(s_i)$.
- Then $L(s) = \sum_R s_R \overline{t_R}$.
- Need to show that $\|t\|_{c^p} \leq \|L\|$.

$\ell^2(\overline{\mu})$ Spaces

- For fixed Ω, define $\overline{\mu}(R) = \frac{\mu(R)}{\mu(\Omega)}$.
- Define $\|s\|_{\ell^2(\overline{\mu})} = \left(\sum_{R \subset \Omega} |s_R|^2 \frac{\mu(R)}{\mu(\Omega)^{\frac{p}{2}-1}} \right)^{1/2}$
- Then $(\ell^2(\overline{\mu}))' = \ell^2(\overline{\mu})$.
- This will allow us to estimate norms of sequences in c^p by using $\ell^2(\overline{\mu})$ duality.

$\ell^2(\overline{\mu})$ Spaces I

$$\left(\frac{1}{\mu(\Omega)^{\frac{p}{2}-1}}\sum_{R\subset\Omega}|t_R|^2\right)^{1/2} = \|\mu(R)^{-1/2}|t_R|\|_{\ell^2(\overline{\mu})}$$

$$= \sup_{s:\|s\|_{\ell^2(\overline{\mu})}\leq 1}\left|\sum_{R\subset\Omega}(|t_R|\mu(R)^{-1/2})s_R\frac{\mu(R)}{\mu(\Omega)^{\frac{p}{2}-1}}\right|$$

$$\leq \sup_{s:\|s\|_{\ell^2(\overline{\mu})}\leq 1}\left|L\left(\chi_{R\subset\Omega}(R)\cdot\frac{\mu(R)^{1/2}|s_R|}{\mu(\Omega)^{\frac{p}{2}-1}}\right)\right|$$

$$\leq \sup_{s:\|s\|_{\ell^2(\overline{\mu})}\leq 1}\|L\|\left\|\chi_{R\subset\Omega}(R)\cdot\frac{\mu(R)^{1/2}|s_R|}{\mu(\Omega)^{\frac{p}{2}-1}}\right\|_{s^p}$$

But by Hölder's Inequality, this is less than or equal to

$$\sup_{s:\|s\|_{\ell^2(\overline{\mu})}\leq 1} \|L\| \left(\sum_{R\subset\Omega} |s_R|^2 \frac{\mu(R)}{\mu(\Omega)^{\frac{p}{2}-1}} \right)^{1/2}$$
$$\leq \sup_{s:\|s\|_{\ell^2(\overline{\mu})}\leq 1} \|L\| \|s\|_{\ell^2(\overline{\mu})} \leq \|L\|.$$

So $\|t\|_{c^p} \leq \|L\|$.

Some Words About the Min-Max Inequality

The Min-Max Inequalities for H^p and for $\mathrm{CMO^p}$ are crucial to the paper.
Recall:
If $2/(2+\theta) < p \leq 1$, then there is some constant $C > 0$ such that

$$\sup_{\Omega} \left(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R \subset \Omega} \mu(R) \sup_{(x,y)\in R} |D_{k_1} D_{k_2}(f)(x,y)|^2 \right)^{1/2}$$

$$\leq C \sup_{\Omega} \left(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R \subset \Omega} \mu(R) \inf_{(x,y)\in R} |D_{k_1} D_{k_2}(f)(x,y)|^2 \right)^{1/2}$$

where $R = Q_{\tau_1}^{k_1,\nu_1} \times Q_{\tau_2}^{k_2,\nu_2}$.

Let R denote an arbitrary dyadic rectangle of the form $Q_{\tau_1}^{k_1,v_1} \times Q_{\tau_2}^{k_2,v_2}$. Let the notation be the same as above. For each R, choose a point $(x', y') \in R$. Then there are families of linear operators $\{\widetilde{D}_k\}$ and $\{\overline{D}_k\}$ such that

$$\begin{aligned} f(x,y) &= \sum_R \mu(R) \widetilde{D}_{k_1} \widetilde{D}_{k_2}(x,y,x',y') D_{k_1} D_{k_2}(f)(x',y') \\ &= \sum_R \mu(R) D_{k_1} D_{k_2}(x,y,x',y') \overline{D}_{k_1} \overline{D}_{k_2}(f)(x',y'), \end{aligned}$$

1. Choose (x', y') to be the point where $|D_{k_1} D_{k_2}(f)(x, y)|$ is minimized in R.
2. Apply $D_{k_1} D_{k_2}$ to both sides of the Discrete Calderón Reproducing Formula.
3. We obtain an expression that can be used to relate the left side of the inequality in the Min-Max Theorem to something resembling the right side.
4. Change the order of summation and carefully consider the various geometric quantities involved to prove the Min-Max Theorem.

Almost Orthogonality

- Almost orthogonality of $D_j \widetilde{D}_k$:

$$|D_j \widetilde{D}_k(x,y)| \le C 2^{-|k-j|\varepsilon} \frac{2^{-\min(k,j)\theta}}{(2^{-\min(k,j)} + \rho(x,y))^{1+\theta}}$$

  where $0 < \varepsilon < \theta$
- Basically, $|D_j \widetilde{D}_k(x,y)|$ is small if k and j are far apart, or if x and y are far apart.
- Now we apply $D_{k_1} D_{k_2}$ to both sides of the Calderon Reproducing Formula, where x' and y' are chosen such that $|D_{k_1} D_{k_2}(f)(x', y')|$ is minimized in R.
- Use the almost orthogonality to get an estimate.

Some Notation

- $R = Q_{\tau_1}^{k_1,\nu_1} \times Q_{\tau_2}^{k_2,\nu_2}$,
- $S_R = \sup_{(x,y)\in R} |D_{k_1} D_{k_2}(f)(x,y)|^2$
- $T_R = \inf_{(x,y)\in R} |D_{k_1} D_{k_2}(f)(x,y)|^2$
- $v(R,R')$ and $r(R,R')$ and $P(R,R')$ are geometric quantities depending on μ and the rectangles R and R'.

By using the estimate obtained earlier and analysis of the geometric quantities involved, we find that the left side of the Min-Max inequality, which is

$$\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R \subset \Omega} \mu(R) S_R$$

is bounded by

$$\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R \subset \Omega} \sum_{R'} v(R, R') r(R, R') P(R, R') T_{R'}.$$

To finish the proof, we want to show that this is bounded by

$$\sup_{\Omega'} \frac{1}{\mu(\Omega')^{\frac{2}{p}-1}} \sum_{R' \subset \Omega'} \mu(R) T_{R'}$$

where Ω' ranges over all open sets with finite measure.

Let $\Omega_0 = \bigcup_{R \subset \Omega} 3A^2 R$.

- For each R, we consider smallest $j, k \geq 0$ such that $3A^2 \cdot 2^j Q_1' \times 3A^2 \cdot 2^k Q_2'$ intersects $3A^2 R$. Here $R' = Q_1' \times Q_2'$.
- We divide the R' into different classes based on the values of j and k.
- We switch orders of summation.
- Divide the R' into even more classes based on what fraction of their area lies in Ω_0. We estimate the geometric quantities v, r, and P for the various classes.
- Eventually, after analyzing many different cases, we get the estimate we want.

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