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Integral Equations and Operator Theory

A Corona Theorem for Multipliers on Dirichlet Space

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Abstract. We prove a corona theorem for infinitely many functions from the multiplier algebra on Dirichlet space.

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In this paper we wish to extend the corona theorem on the multiplier algebra of Dirichlet space to infinitely many functions. For a finite number of functions, the corresponding theorem is due to Tolokonnikov [T]. (Another source for this is Nikolskii [N].) For infinitely many functions in $H^\infty(D)$, the corona theorem is due to Rosenblum [R] and to Tolokonnikov [T]. Our methods are in principle close to those of Rosenblum [R]. All of these efforts were made possible by Wolff's beautiful proof of Carleson's original corona theorem. (See [G].)

Our proof is based on three parts. First, since the reproducing kernel of Dirichlet space has one positive square, the commutant lifting theorem comes into play. This reduces the general $M(\mathcal{D}^2(D))$-corona problem to the $\mathcal{D}^2(D)$-corona problem and we may employ Hilbert space methods. Next, we have a series of tedious lemmas that basically say that multipliers on Dirichlet space can be naturally extended to multipliers on (boundary values of) Harmonic Dirichlet space. Third, a linear algebra result allows us to explicitly write down proposed solutions for the $\mathcal{D}(D)$-corona problem in the smooth case. This is our key innovation. Finally, simple estimates and a compactness argument allow us to remove the smoothness condition.

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We will establish our notation. $\mathcal{D}^2(D)$ or just $\mathcal{D}$ will denote the Dirichlet space on the unit disk, D. That is

$$\mathcal{D} = \{\, f : D \to \mathbb{C} \,|f \text{ is analytic on } D \text{ and for } f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

$$\|f\|_{\mathcal{D}}^2 = \sum_{n=0}^{\infty} (n+1)|a_n|^2 < \infty\}.$$

For a nice account of many interesting properties of Dirichlet space see the survey article of Wu [W].

We will use two other equivalent norms for smooth functions in $\mathcal{D}$. Namely,

$$\|f\|_{\mathcal{D}}^2 = \int_{-\pi}^{\pi} |f|^2 \, d\sigma + \int_D |f'(z)|^2 \, dA(z), \quad \text{where } dA(z) = \frac{dm_2(z)}{\pi}$$

and

$$\|f\|_{\mathcal{D}}^2 = \int_{-\pi}^{\pi} |f|^2 \, d\sigma + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} \, d\sigma d\sigma. \tag{1}$$

Also, we will consider $\mathcal{D}^2_{l^2}(D)$, or $\overset{\infty}{\underset{1}{\oplus}} \mathcal{D}$, which can be considered as l^2-valued Dirichlet space. The norms in this case are as above, but with absolute values replaced by l^2-norms in the appropriate spots. In addition, we will need Harmonic Dirichlet space, $H\mathcal{D}$ (restricted to the boundary of D), and its vector analog. But again we will only use this norm for smooth functions in this space. So if f is smooth on ∂D, the boundary of the unit disk, then

$$\|f\|_{H\mathcal{D}}^2 = \int_{-\pi}^{\pi} |f|^2 \, d\sigma + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} \, d\sigma d\sigma.$$

Of course, this is formally the same as (1), but the functions in $H\mathcal{D}$ may have nonvanishing negative fourier coefficients.

The algebra of operators which we consider is the multiplier algebra on Dirichlet space, $M(\mathcal{D}) = \{\, \phi \in \mathcal{D} : \phi f \in \mathcal{D} \quad \forall f \in \mathcal{D}\}$, and the multiplier algebra on Harmonic Dirichlet space, $M(H\mathcal{D})$, defined similarly (but only on ∂D). We will use M_ϕ to denote the operator multiplication by ϕ for $\phi \in M(\mathcal{D})$ (and for $\phi \in M(H\mathcal{D})$).

Given $\{f_j\}_{j=1}^{\infty} \subset M(\mathcal{D})$, we let $F(z) = (f_1(z), f_2(z), \dots)$. We use M_F^R to denote the (row) operator from $\overset{\infty}{\underset{1}{\oplus}} \mathcal{D}$ to $\mathcal{D}$ defined by

$$M_F^R(\{h_j\}_{j=1}^{\infty}) = \sum_{j=1}^{\infty} f_j h_j \quad \text{for } \{h_j\}_{j=1}^{\infty} \in \overset{\infty}{\underset{1}{\oplus}} \mathcal{D}.$$

Similarly, M_F^C will denote the (column) operator from $\mathcal{D}$ to $\overset{\infty}{\underset{1}{\oplus}} \mathcal{D}$ defined by

$$M_F^C(h) = \sum_{j=1}^{\infty} f_j h \quad \text{for } h \in \mathcal{D}.$$

The corona theorem for $H^\infty(D)$ is due to Carleson [C]. The infinite version, given by Rosenblum [R] and Tolokonnikov [T], can be formulated as follows:

Theorem C. *Let* $\{f_j\}_{j=1}^\infty \subset H^\infty(D)$ *with* $0 < \epsilon^2 \leq \sum_{j=1}^\infty |f_j(z)|^2 \leq 1$ *for all* $z \in D$. *Then there exist* $\{g_j\}_{j=1}^\infty \subset H^\infty(D)$ *such that* $\sum f_j g_j = 1$ *and* $\sup_{z\in D}\{\sum_{j=1}^\infty |g_j(z)|^2\} \leq \frac{9}{\epsilon^2} \ln \frac{1}{\epsilon^2}$ *for* $\epsilon^2 < \frac{1}{e}$.

(This bound is due to Uchiyama and can be found in, for example, Trent [Tr2].)

We note that $\mathcal{D}$ is a reproducing kernel (r.k.) Hilbert space with r.k. $k_w(z) = \frac{1}{\overline{w}z} \log \frac{1}{1-\overline{w}z}$ for $z, w \in D$. This means that if $f \in \mathcal{D}$ and $w \in D$ then $\langle f, k_w \rangle_{\mathcal{D}} = f(w)$. It has been shown, in for example [A], that

$$1 - \frac{1}{k_w(z)} = \sum_{n=1}^\infty c_n z^n \overline{w}^n, \quad \text{where } c_n > 0 \text{ for all } n.$$

This property of the reproducing kernel referred to as "one-positive square" or "Nevanlinna–Pick" (N-P) has been much studied recently in, for example, [A], [M], [AM], [BT], [BLTT], [BTV], [CM], and [GRS]. This property will be exploited for "$\frac{1}{2}$" of the corona theorem in Dirichlet space. The useful relationship between multiplier spaces and reproducing kernels is that for $\phi \in M(\mathcal{D})$ and $z \in D$

$$M_\phi^* k_z = \overline{\phi(z)} k_z. \tag{2}$$

It immediately follows from this that $\|\phi\|_\infty \leq \|M_\phi\|$, so $M(\mathcal{D}) \subset H^\infty(D)$.

Similarly, if $\phi_{ij} \in M(\mathcal{D})$ and $M_{[\phi_{ij}]_{j=1}^\infty} \in B(\overset{\infty}{\underset{1}{\oplus}} \mathcal{D})$, then for $\underline{x} \in l^2$ and $z \in D$, we have

$$M_{[\phi_{ij}]}^*(\underline{x} k_z) = [\phi_{ij}(z)]^* \underline{x} k_z.$$

It again follows from this that

$$\sup_{z\in D} \|[\phi_{ij}(z)]\|_{B(l^2)} \leq \|M_{[\phi_{ij}]}\|_{B(\overset{\infty}{\underset{1}{\oplus}} \mathcal{D})}$$

and

$$M(\overset{\infty}{\underset{1}{\oplus}} \mathcal{D}) \subset H^\infty_{B(l^2)}(D).$$

For part of the pointwise hypothesis of Theorem C (that $\sum_{j=1}^\infty |f_j(z)|^2 \leq 1$), we note that if T_F^R and T_F^C are defined on $\overset{\infty}{\underset{1}{\oplus}} H^2(D)$ and $H^2(D)$ (Hardy spaces) in analogy to that of M_F^R and M_F^C, we have

$$\|T_F^R\| = \|T_F^C\| = \sup_{z\in D} (\sum_{j=1}^\infty |f_j(z)|^2)^{\frac{1}{2}}.$$

Thus one part of the pointwise hypothesis of Theorem C gives the boundedness of the operators T_F^R and T_F^C. We will need a similar hypothesis for our version on Dirichlet space. But since $M(\mathcal{D}) \subsetneqq H^\infty(D)$, (e.g., $\sum_{n=1}^\infty \frac{z^{n^3}}{n^2}$ is in $H^\infty(D)$ but is not in $\mathcal{D}$), a pointwise upper bound hypothesis will not suffice. Moreover, it is not hard to check that $I - 1 \otimes 1 = \sum_{n=1}^\infty c_n M_z^n M_z^{*n}$ and $\sum_{n=1}^\infty c_n = 1$, where $1 - \frac{1}{k_w(z)} = \sum_{n=1}^\infty c_n z^n \overline{w}^n$. Thus, for $f_j = \sqrt{c_j} M_{z^j}$ with $j \geq 1$ and $f_0 = 1$, we have $M_F^R = (M_{f_0}, M_{f_1}, \dots) \in B(\overset{\infty}{\underset{1}{\oplus}} \mathcal{D}, \mathcal{D})$ and $\|M_F^R\| = 1$.

Now

$$\|M_F^C(1)\|_{\mathcal{D}}^2 = \|\sum_{n=1}^\infty \sqrt{c_n} z^n\|_{\mathcal{D}}^2 = \sum_{n=1}^\infty (n+1)c_n \geq \sum_{n=1}^\infty n\, c_n.$$

But

$$2\sum_{n=1}^\infty n c_n u^{2n-1} = \frac{d}{du}\,(-(k_u(u))^{-1}) \approx (1-u)^{-1}(\ln(\frac{1}{(1-u)}))^{-2} \to \infty \text{ as } u \nearrow 1.$$

Thus $M_F^C \notin B(\mathcal{D}, \overset{\infty}{\underset{1}{\oplus}} \mathcal{D})$. However, as the next lemma shows, we always have $\|M_F^R\| \leq \sqrt{18}\, \|M_F^C\|$.

Lemma 1. *Let $M_F^C \in B(\mathcal{D}, \overset{\infty}{\underset{1}{\oplus}} \mathcal{D})$. Then $M_F^R \in B(\overset{\infty}{\underset{1}{\oplus}} \mathcal{D}, \mathcal{D})$ and $\|M_F^R\| \leq \sqrt{18}\, \|M_F^C\|$.*

Proof. First note that from our earlier discussion, $\|M_F^C\| \leq 1$ implies that $F(z)F(z)^* \leq 1$ for all z in D. Let $\{u_k\}_{k=1}^\infty \in \overset{\infty}{\underset{1}{\oplus}} \mathcal{D}$. Then

$$\begin{aligned}
\|M_F^R(\{u_k\}_{k=1}^\infty)\|^2 &= \int_{\partial D} |\sum_{k=1}^\infty f_k u_k|^2\, d\sigma + \int_D |\sum_{k=1}^\infty (f_k u_k)'|^2\, dA \\
&\leq \|\underline{u}\|_\sigma^2 + \int_D |\sum_{k=1}^\infty f_k u_k' + \sum_{k=1}^\infty f_k' u_k|^2\, dA \\
&\leq \|\underline{u}\|_\sigma^2 + 2\int_D \sum_{k=1}^\infty |u_k'|^2\, dA + 2\int_D |\sum_{k=1}^\infty f_k' u_k|^2\, dA \\
&\leq 2\,\|\underline{u}\|_{\mathcal{D}}^2 + 2\int_D \sum_{k=1}^\infty \sum_{j=1}^\infty f_k' u_k \overline{f}_j' \overline{u}_j\, dA \\
&\leq 2\,\|\underline{u}\|_{\mathcal{D}}^2 + 4\int_D \sum_{k=1}^\infty \sum_{j=1}^\infty |f_k' u_j|^2\, dA \\
&\leq 2\,\|\underline{u}\|_{\mathcal{D}}^2 + 4^2 \sum_{j=1}^\infty \|M_F^C(u_j)\|_{\mathcal{D}}^2 \\
&\leq 18\,\|\underline{u}\|_{\mathcal{D}}^2.
\end{aligned}$$

So

$$\|M_F^R\| \leq \sqrt{18}\,\|M_F^C\|. \qquad \square$$

Thus our replacement for "$\sum_{j=1}^{\infty}|f_j(z)|^2 \leq 1$" for $z \in D$ will be $\|M_F^C\| \leq 1$. We plan to prove

Theorem 1. *Let* $\{f_j\}_{j=1}^{\infty} \subset M(\mathcal{D})$. *Assume that*

$$\|M_F^C\| \leq 1 \ \text{ and } \ 0 < \epsilon^2 \leq \sum_{j=1}^{\infty}|f_j(z)|^2 \quad \text{for all } z \in D.$$

Then

$$\text{there exists } \{g_j\}_{j=1}^{\infty} \subset M(\mathcal{D})$$

so that

$$(1) \quad \sum_{j=1}^{\infty} f_j g_j = 1$$

and

$$(2) \quad \|M_G^C\| \leq \frac{1,500}{\epsilon^3}.$$

Notice that we are concluding that the strong bound, $\|M_G^C\| < \infty$, follows from the strong bound, $\|M_F^C\| < \infty$, together with the lower bound hypothesis on $F(z)F(z)^*$. Whether the weaker hypothesis that $\|M_F^R\| < \infty$ together with the lower bound hypothesis on $F(z)F(z)^*$ leads to $\|M_G^C\| < \infty$ is not known to us.

To prove Theorem 1 we will establish Theorems A and B.

Theorem A. *Let* $\{f_j\}_{j=1}^{\infty} \subset M(\mathcal{D})$. *Assume that* $\|M_F^C\| \leq 1$ *and* $0 < \epsilon^2 \leq \sum_{j=1}^{\infty}|f_j(z)|^2$ *for all* $z \in D$. *Then*

$$\left(\frac{1,500}{\epsilon^3}\right)^{-2} I \leq M_F^R (M_F^R)^* \leq I.$$

Theorem B. *Assume that* $\delta^2 \leq M_F^R(M_F^R)^* \leq I$. *Then there exists* $\{g_j\}_{j=1}^{\infty} \subset M(\mathcal{D})$ *so that*

$$(1) \quad \sum_{j=1}^{\infty} f_j g_j = 1$$

$$(2) \quad \|M_G^C\| \leq \frac{1}{\delta}.$$

We refer to Theorem A as a $\mathcal{D}$-corona theorem, since it involves unknown functions from $\mathcal{D}$, not $M(\mathcal{D})$. When $\mathcal{D}$ is replaced by $H^2(D)$, Hardy space, this is the main result of Rosenblum [R] and Tolokonnikov [T]. Also, it has recently been shown to hold when $\mathcal{D}$ is replaced by $H^2(D^2)$. See Trent [Tr2] and for previous related work see Li [L] and Lin [Li]. Theorem B with $H^{\infty}(D)$ replacing $M(\mathcal{D})$ is referred to as a Toeplitz corona theorem. Theorem B has an interesting history. See, for example, [A], [CM], [SNF], [S], [KMT], [H], [BT], and [BTV].

Of course, Theorems A and B with $\delta = \left(\frac{1{,}500}{\epsilon^3}\right)^{-1}$ complete the proof of Theorem 1.

To prove Theorem B, we use the commutant lifting theorem for multipliers on Dirichlet space from [CM]. But we state it in our context using the version from [BTV]. The reader should note that the second part of the corona theorem for multiplier algebras on reproducing Hilbert space with complete N-P kernels holds as in the following argument.

Theorem 2 (CLT). *Let $\mathcal{M}_*, \mathcal{N}_*$ be invariant subspaces for M_z^* on $\overset{M}{\underset{1}{\oplus}} \mathcal{D}$ and $\overset{N}{\underset{1}{\oplus}} \mathcal{D}$, respectively, where $1 \leq M, N \leq \infty$. Assume that $X^* \in B(\mathcal{M}_*, \mathcal{N}_*)$ satisfies $X^* M_z^* \mid_{\mathcal{M}_*} = M_z^* \mid_{\mathcal{N}_*} X^*$. Then there exists a $Y^* \in B(\overset{M}{\underset{1}{\oplus}} \mathcal{D}, \overset{N}{\underset{1}{\oplus}} \mathcal{D})$ so that*

$$\begin{aligned} &\text{(i)} \quad Y^* \mid_{\mathcal{M}_*} = X^* \\ &\text{(ii)} \quad Y^* M_z^* = M_z^* Y^* \\ &\text{(iii)} \quad \|Y^*\| = \|X^*\|. \end{aligned}$$

Note that if we view Y as an M by N matrix, then the entries of Y are in $M(\mathcal{D})$ by (ii). *This follows since analytic polynomials are dense in $\mathcal{D}$.*

Proof of Theorem B. Define $M = \infty$, $N = 1$, $\mathcal{M}_* = range\ M_F^{R^*}$, $\mathcal{N}_* = \mathcal{D}$. Let

$$X^* = [M_F^R (M_F^R)^*]^{-1} (M_F^R).$$

Then

$$\|X^*\| \leq \frac{1}{\delta}$$

and

$$\begin{aligned} X^* M_z^* ((M_F^R)^* u) &= X^* (M_F^R)^* M_z^* u = M_z^* u \\ &= M_z^* (X^* (M_F^R)^* u). \end{aligned}$$

So

$$X^* M_z^* \mid_{\mathcal{M}_*} = M_z^* X^*.$$

Thus, by CLT there exists a $Y^* \in B(\overset{\infty}{\underset{1}{\oplus}} \mathcal{D}, \mathcal{D})$ satisfying (i), (ii), and (iii). By (ii), Y has entries in $M(\mathcal{D})$, say g_j, so $Y = M_G^C$. By (i), $Y^*(M_F^R)^* = I$, so $M_F^R M_G^C = I$. Finally, (iii) gives us that $\|M_G^C\| = \|Y\| = \|X\| \leq \frac{1}{\delta}$. □

Now we proceed with our proof of Theorem A. First we note that it is a simple operator theoretic fact that Theorem A is equivalent to Theorem A′.

Theorem A′. *Let $\{f_j\}_{j=1}^{\infty} \subset M(\mathcal{D})$. Assume that $\|M_F^C\| \leq 1$ and $0 < \epsilon^2 \leq \sum_{j=1}^{\infty} |f_j(z)|^2$ for all $z \in D$. Then for every $h \in \mathcal{D}$, there exists $\underline{u}_h \in \overset{\infty}{\underset{1}{\oplus}} \mathcal{D}$ with*

$$\text{(i)} \quad M_F^R(\underline{u}_h) = h$$

and

$$\text{(ii)} \quad \|\underline{u}_h\|_{\mathcal{D}} \leq \frac{1,500}{\epsilon^3}\|h\|_{\mathcal{D}}.$$

Actually, it suffices to prove Theorem A′ for any dense set of functions in $\mathcal{D}$. We take functions of $\mathcal{D}$ smooth across ∂D. The general plan is as follows. Assume F is analytic on $D_{1+\epsilon}(0)$. Given $h \in \mathcal{D}$ analytic on $D_{1+\epsilon}(0)$, write the most general solution of the pointwise problem on $\overline{D}$. That is

$$\underline{v}_h(z) = F(z)^*(F(z)F(z)^*)^{-1}h - Q(z)\underline{k}(z),$$

where *range* $Q(z) =$ *kernel* $F(z)$, $Q(z)$ is analytic, and $\underline{k}(z) \in l^2$ for $z \in \overline{D}$. In fact, we will show that for each $z \in D$,

$$(F(z)F(z)^*)I - F(z)^*F(z) = Q(z)Q(z)^*.$$

We must find $\underline{k}(z)$ so that $\underline{v}_h \in \overset{\infty}{\underset{1}{\oplus}}\mathcal{D}$. Thus we want

$$\overline{\partial}_z\underline{v}_h = 0 \quad \text{in } D.$$

So take $\underline{u}_h = F^*(FF^*)^{-1}h - Q\widehat{\frac{Q^*F'^*h}{(FF^*)^2}}$, where $\widehat{k}$ is the Cauchy transformation of k on D. That is, for k smooth on $\overline{D}$ and $z \in D$,

$$\widehat{\underline{k}}(z) \overset{def}{=} \int_D \frac{\underline{k}(w)}{z-w}\, dA(w)$$

and we have $\overline{\partial}_z\widehat{\underline{k}} = \underline{k}$ in D.

Clearly, $F\underline{u}_h = h$ and $\underline{u}_h$ is analytic. Thus we are done (in the smooth case of F) if

$$\|\underline{u}_h\|_{\mathcal{D}} \leq \frac{1,500}{\epsilon^3}\|h\|_{\mathcal{D}}.$$

Let $\underline{k} = \frac{Q^*F'^*h}{(FF^*)^2}$. Our procedure is to show that

$$\|F^*(FF^*)^{-1}h\|_{H\mathcal{D}} \leq C_1\|h\|_{\mathcal{D}},$$
$$\|Q\widehat{\underline{k}}\|_{H\mathcal{D}} \leq C_2\|\widehat{\underline{k}}\|_{H\mathcal{D}},$$

and the main estimate,

$$\|\widehat{\underline{k}}\|_{H\mathcal{D}} \leq C_3\|h\|_{\mathcal{D}}.$$

The next three lemmas involve extending multipliers on $\mathcal{D}$ to multipliers on $H\mathcal{D}$, where we are just considering $H\mathcal{D}$ on ∂D.

Lemma 2. (a) *Let* $\alpha \in M(\mathcal{D})$*, then* $\alpha \in M(H\mathcal{D})$ *and* $\|\alpha\|_{B(H\mathcal{D})} \leq \sqrt{20}\,\|\alpha\|_{B(\mathcal{D})}$.
(b) *Let* $\{f_i\}_{i=1}^{\infty} \subset M(\mathcal{D})$*. Then* $\|M_F^C\|_{B(H\mathcal{D},\overset{\infty}{\underset{1}{\oplus}} H\mathcal{D})} \leq \sqrt{20}\,\|M_F^C\|_{B(\mathcal{D},\overset{\infty}{\underset{1}{\oplus}}\mathcal{D})}$.

Proof. We will only show (a), since (b) follows by summing the results of (a). Let $\alpha \in M(\mathcal{D})$. As we have observed, $\|\alpha\|_\infty \leq \|\alpha\|_{B(\mathcal{D})} \overset{def}{=} C$. Let p, q_0 be analytic polynomials with $q_0(0) = 0$. We need only estimate $\|\alpha(p+\overline{q}_0)\|_{H\mathcal{D}}$, since $\{p+\overline{q}_0\}$ is dense in $H\mathcal{D}$.

$$
\begin{aligned}
\|\alpha(p+\overline{q}_0)\|_{H\mathcal{D}}^2 &= \int_{-\pi}^{\pi} |\alpha(p+\overline{q}_0)|^2\, d\sigma \\
&\quad + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|(\alpha(p+\overline{q}_0))(e^{it}) - (\alpha(p+\overline{q}_0))(e^{i\theta})|^2}{|e^{it}-e^{i\theta}|^2}\, d\sigma(t)d\sigma(\theta) \\
&\leq 2\int_{-\pi}^{\pi} |\alpha p|^2\, d\sigma + 2\int_{-\pi}^{\pi} |\alpha\overline{q}_0|^2\, d\sigma \\
&\quad + 2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|(\alpha p)(e^{it}) - (\alpha p)(e^{i\theta})|^2}{|e^{it}-e^{i\theta}|^2}\, d\sigma d\sigma \\
&\quad + 2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|(\alpha\overline{q}_0)(e^{it}) - (\alpha\overline{q}_0)(e^{i\theta})|^2}{|e^{it}-e^{i\theta}|^2}\, d\sigma d\sigma \\
&\leq 2C^2\|p\|_{\mathcal{D}}^2 + 2C^2\int_{-\pi}^{\pi} |\overline{q}_0|^2\, d\sigma \\
&\quad + 4\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|\alpha(e^{it}) - \alpha(e^{i\theta})|^2}{|e^{it}-e^{i\theta}|^2} |q_0(e^{it})|^2\, d\sigma d\sigma \\
&\quad + 4\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\alpha(e^{it})|^2 \frac{|q_0(e^{it}) - q_0(e^{i\theta})|^2}{|e^{it}-e^{i\theta}|^2}\, d\sigma d\sigma \\
&\leq 2C^2\|p\|_{\mathcal{D}}^2 + 2C^2\int_{-\pi}^{\pi} |\overline{q}_0|^2\, d\sigma + 16C^2\|q_0\|_{\mathcal{D}}^2 \\
&\quad + 4C^2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|q_0(e^{it}) - q_0(e^{i\theta})|^2}{|e^{it}-e^{i\theta}|^2}\, d\sigma d\sigma \\
&\leq 2C^2\|p\|_{\mathcal{D}}^2 + (4C^2 + 16C^2)\|q_0\|_{\mathcal{D}}^2 \\
&\leq 20C^2(\|p\|_{\mathcal{D}}^2 + \|q_0\|_{\mathcal{D}}^2) \\
&= 20C^2\|p+\overline{q}_0\|_{H\mathcal{D}}^2.
\end{aligned}
$$

Here we are using the fact that $p \perp \overline{q}_0$ in $H\mathcal{D}$ and $\|\overline{q}_0\|_{H\mathcal{D}}^2 = \|q_0\|_{\mathcal{D}}^2$. □

Lemma 3. *Assume that* $\|M_F^C\| \leq 1$. *Then*

$$M_{(FF^*)} \in B(H\mathcal{D})$$

and

$$\|M_{FF^*}\| \leq 86.$$

Proof. We show that $M_{(FF^*)^{\frac{1}{2}}} \in B(H\mathcal{D})$ with $\|M_{(FF^*)^{\frac{1}{2}}}\| \le \sqrt{86}$. This will complete the proof. Let p and q_0 be analytic polynomials with $q_0(0) = 0$. We let u denote $p + \overline{q}_0$. Then

$$\begin{aligned}
\|M_{(FF^*)^{\frac{1}{2}}}(u)\|^2_{H\mathcal{D}} = &\int_{\partial D} |FF^*|\,|u|^2\,d\sigma \\
&+ \int_{\partial D}\int_{\partial D} \frac{|(FF^*)^{\frac{1}{2}}(e^{it})u(e^{it}) - (FF^*)^{\frac{1}{2}}(e^{i\theta})u(e^{i\theta})|^2}{|e^{it}-e^{i\theta}|^2}\,d\sigma d\sigma \\
\le \|u\|^2_\sigma + 2 &\underbrace{\int_{\partial D}\int_{\partial D} \frac{|(FF^*)^{\frac{1}{2}}(e^{it}) - (FF^*)^{\frac{1}{2}}(e^{i\theta})|^2 |u(e^{it})|^2}{|e^{it}-e^{i\theta}|^2} d\sigma d\sigma}_{a} \\
&+ 2\int_{\partial D}\int_{\partial D} (FF^*)(e^{it}) \frac{|u(e^{it}) - u(e^{i\theta})|^2}{|e^{it}-e^{i\theta}|^2}\,d\sigma d\sigma.
\end{aligned}$$

$$|(FF^*)^{\frac{1}{2}}(e^{it}) - (FF^*)^{\frac{1}{2}}(e^{i\theta})|^2 \le \sum_k |F_k(e^{it}) - F_k(e^{i\theta})|^2.$$

So

$$\begin{aligned}
(a) &\le 2\int_{\partial D}\int_{\partial D} \sum_k \frac{|F_k(e^{it}) - F_k(e^{i\theta})|^2}{|e^{it}-e^{i\theta}|^2} |u(e^{it})|^2\,d\sigma d\sigma \\
&\le 4\int_{\partial D}\int_{\partial D} \sum_k \frac{|F_k(e^{it})u(e^{it}) - F_k(e^{i\theta})u(e^{i\theta})|^2}{|e^{it}-e^{i\theta}|^2} |u(e^{it})|^2\,d\sigma d\sigma \\
&\quad + 4\int_{\partial D}\int_{\partial D} \sum_k |F_k(e^{i\theta}) \frac{u(e^{it}) - u(e^{i\theta})|^2}{|e^{it}-e^{i\theta}|^2}\,d\sigma d\sigma \\
&\le 4\,\|M_F^C(u)\|^2_{H\mathcal{D}} + 4\,\|u\|^2_{H\mathcal{D}} \\
&\le (4\cdot 20 + 4)\|k\|^2_{H\mathcal{D}}
\end{aligned}$$

by Lemma 2. Thus

$$\|M_{(FF^*)^{\frac{1}{2}}}(u)\|^2_{H\mathcal{D}} \le 2\,\|u\|^2_{H\mathcal{D}} + 84\,\|u\|^2_{H\mathcal{D}} = 86\,\|u\|^2_{H\mathcal{D}}. \qquad \square$$

Lemma 4. *Let $H \in Mult(H\mathcal{D})$ with $1 \ge |H(e^{it})| \ge \epsilon > 0$ for σ-a.e. $t \in [-\pi, \pi]$. Then $\frac{1}{H} \in Mult(H\mathcal{D})$ and $\|\frac{1}{H}\|_{B(H\mathcal{D})} \le \frac{\sqrt{10}}{\epsilon^2}\|H\|_{B(\mathcal{D})}$.*

Proof. Let $r \in H\mathcal{D}$ be a rational polynomial on ∂D and let $\|H\|_{B(H\mathcal{D})} = C$. Then

$$\begin{aligned}
\|\frac{1}{H}r\|^2_{H\mathcal{D}} &= \int_{-\pi}^{\pi} |\frac{r}{H}|^2\, d\sigma + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|(\frac{r}{H})(e^{it}) - (\frac{r}{H})(e^{i\theta})|^2}{|e^{it}-e^{i\theta}|^2}\, d\sigma d\sigma \\
&\leq \frac{1}{\epsilon^2}\int_{-\pi}^{\pi} |r|^2\, d\sigma + \frac{1}{\epsilon^4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|H(e^{i\theta})r(e^{it}) - H(e^{it})r(e^{i\theta})|^2}{|e^{it}-e^{i\theta}|^2}\, d\sigma d\sigma \\
&\leq \frac{1}{\epsilon^2}\int_{-\pi}^{\pi} |r|^2\, d\sigma + \frac{2}{\epsilon^4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |H(e^{i\theta})|^2\, |\frac{r(e^{it}) - r(e^{i\theta})}{e^{it}-e^{i\theta}}|^2\, d\sigma d\sigma \\
&\quad + \frac{2}{\epsilon^4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|H(e^{i\theta}) - H(e^{it})|^2}{|e^{it}-e^{i\theta}|}\, |r(e^{it})|^2\, d\sigma d\sigma \\
&\leq \frac{2C^2}{\epsilon^4}\|r\|^2_{H\mathcal{D}} + \frac{2}{\epsilon^4}(4C^2\|r\|^2_{H\mathcal{D}}) \\
&= \frac{10C^2}{\epsilon^4}\|r\|^2_{H\mathcal{D}}.
\end{aligned}$$

□

Lemma 5. *Let $\{f_j\}_{j=1}^{\infty} \subset M(\mathcal{D})$. Assume that $\|M_F^C\| \leq 1$ and $0 < \epsilon^2 \leq F(z)F(z)^*$ for all $z \in D$. Then for $h \in \mathcal{D}$, we have*

$$\|\frac{F^*}{FF^*}h\|^2_{H\mathcal{D}} \leq \frac{10 \cdot 86^2 \cdot 20}{\epsilon^4}\|h\|^2_{\mathcal{D}}.$$

Proof. Let r be a rational polynomial on ∂D. Then by Lemma 2,

$$\begin{aligned}
\|F^*r\|^2_{H\mathcal{D}} &= \|M_F^C(\overline{r})\|^2_{H\mathcal{D}} \leq \|M_F^C\|_{B(H\mathcal{D})}\|\overline{r}\|^2_{H\mathcal{D}} \\
&\leq 20\,\|M_F^C\|\,\|r\|^2_{H\mathcal{D}} = 20\,\|r\|^2_{H\mathcal{D}}.
\end{aligned}$$

Lemmas 3 and 4 tell us that

$$\|M_{(FF^*)^{-1}}\|_{B(H\mathcal{D})} \leq \frac{\sqrt{10}}{\epsilon^2}\|M_{FF^*}\|_{B(H\mathcal{D})} \leq \frac{\sqrt{10}\cdot 86}{\epsilon^2}.$$

Finally for $h \in \mathcal{D}$

$$\|(FF^*)^{-1}F^*h\|^2_{H\mathcal{D}} \leq \frac{10\cdot 86^2}{\epsilon^4}\cdot 20\,\|h\|^2_{H\mathcal{D}} = \frac{10\cdot 86^2\cdot 20}{\epsilon^4}\|h\|^2_{\mathcal{D}}.$$ □

The next lemma is our linear algebra result which enables us to write down the most general pointwise solution of $F\underline{u}_h(z) = h(z)$. A more general version of this lemma can be found in Trent [Tr1].

Lemma 6. *Let $\{c_j\}_{j=1}^{\infty} \in l^2$ and $C = (c_1, c_2, \dots) \in B(l^2, \mathbb{C})$. Then $\exists Q$ such that entries of Q are either 0 or $\pm c_j$ for some j and*

$$CC^*I - C^*C = QQ^*.$$

Proof. For $k = 1, 2, \dots,$ let

$$A_k = \begin{bmatrix} 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \\ c_{k+1} & c_{k+2} & c_{k+3} & \dots \\ -c_k & 0 & 0 & \dots \\ 0 & -c_k & 0 & \dots \\ 0 & 0 & -c_k & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where the first k rows of A_k have only 0 entries.
Then

$$A_k A_k^* = \begin{bmatrix} 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ 0 & \dots & 0 & \sum_{j=k+1}^{\infty} |c_j|^2 & -\overline{c}_k c_{k+2} & -\overline{c}_k c_{k+3} & \dots \\ 0 & \dots & 0 & -c_k \overline{c}_{k+2} & |c_k|^2 & 0 & \dots \\ 0 & \dots & 0 & -c_k \overline{c}_{k+3} & 0 & |c_k|^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Thus

$$\sum_{k=1}^{\infty} A_k A_k^* = \begin{bmatrix} \sum_{k\neq 1}^{\infty} |c_k|^2 & -\overline{c}_1 c_2 & -\overline{c}_1 c_3 & \dots \\ -\overline{c}_2 c_1 & \sum_{k\neq 2}^{\infty} |c_k|^2 & -\overline{c}_2 c_3 & \dots \\ -\overline{c}_3 c_1 & -\overline{c}_3 c_2 & \sum_{k\neq 3}^{\infty} |c_k|^2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$= CC^* I - C^* C.$$

So let

$$Q = [A_1, A_2, \dots] \in B(\overset{\infty}{\underset{1}{\oplus}} l^2, l^2). \qquad \square$$

Lemma 7. *Let $\{f_j\}_{j=1}^{\infty} \subset M(\mathcal{D})$. Assume that for each j, f_j is analytic on $D_{1+\epsilon}(0)$ and $\|M_F^C\|_{B(\mathcal{D})} \leq 1$. Associate $Q(z)$ to $F(z)$ for each $|z| = 1$. Then*

$$\|Q\|_{B(\overset{\infty}{\underset{1}{\oplus}} H\mathcal{D})} \leq \sqrt{86}.$$

Proof. Since $\|M_F^C\|_{B(\mathcal{D})} \leq 1$, we have $\|F(z)\|_{l^2}^2 \leq 1$. By Lemma 6, for $z \in \overline{D}$, $Q(z)Q(z)^* \leq (F(z)F(z)^*)\, I_{l^2}$, so $\|Q(z)\|_{B(l^2)} \leq 1$. First, note that if $r \in H\mathcal{D}$ is a rational polynomial in z, then

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\|F(e^{it})-F(e^{i\theta})\|_{l^2}^2}{|e^{it}-e^{i\theta}|^2}|r(e^{it})|^2 d\sigma d\sigma$$
$$\leq 2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\|(F\,r)(e^{it})-(F\,r)(e^{i\theta})\|^2}{|e^{it}-e^{i\theta}|^2}\,d\sigma d\sigma$$
$$+\,2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{|r(e^{it})-r(e^{i\theta})|^2}{|e^{it}-e^{i\theta}|^2}\,d\sigma d\sigma$$
$$\leq 2\,\|M_F^C\|^2_{B(H\mathcal{D},\overset{\infty}{\underset{1}{\oplus}}H\mathcal{D})}\|r\|^2_{H\mathcal{D}}+2\,\|r\|^2_{H\mathcal{D}}$$
$$\leq 42\,\|r\|^2_{H\mathcal{D}}$$

by Lemma 2.

Now for $\underline{r}\in\overset{\infty}{\underset{1}{\oplus}}H\mathcal{D}$

$$\|Q\,\underline{r}\|_{H\mathcal{D}}=\int_{-\pi}^{\pi}\|(Q\,\underline{r})(e^{it})\|_{l^2}^2\,d\sigma+\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\|(Q\,\underline{r})(e^{it})-(Q\,\underline{r})(e^{i\theta})\|_{l^2}^2}{|e^{it}-e^{i\theta}|^2}\,d\sigma d\sigma$$
$$\leq\int_{-\pi}^{\pi}\|(\underline{r})(e^{it})\|_{l^2}^2 d\sigma+2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\|Q(e^{it})-Q(e^{i\theta})\|_{B(l^2)}^2}{|e^{it}-e^{i\theta}|^2}\|\underline{r}(e^{it})\|_{l^2}^2 d\sigma d\sigma$$
$$+\,2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\|Q(e^{i\theta})\|_{B(l^2)}^2\frac{\|\underline{r}(e^{it})-\underline{r}(e^{i\theta})\|^2}{|e^{it}-e^{i\theta}|^2}\,d\sigma d\sigma.$$

But using Lemma 6 pointwise with $c_j=f_j(e^{it})-f_j(e^{i\theta})$, we get

$$(Q(e^{it})-Q(e^{i\theta}))(Q(e^{it})-Q(e^{i\theta}))^*\leq(F(e^{it})-F(e^{i\theta}))(F(e^{it})-F(e^{i\theta}))^* I_{l^2}$$

so

$$\|Q(e^{it})-Q(e^{i\theta})\|_{B(l^2)}^2\leq\|F(e^{it})-F(e^{i\theta})\|^2.$$

Combining the two estimates above, we get that

$$\|Q\,\underline{r}\|^2_{H\mathcal{D}}\leq 2\,\|\underline{r}\|^2_{H\mathcal{D}}+2\cdot 42\,\|\underline{r}\|^2_{H\mathcal{D}}$$
$$\leq 86\,\|\underline{r}\|^2_{H\mathcal{D}}.$$

□

We need one more lemma to handle Cauchy transforms.

Lemma 8. *Let $\underline{k}$ be smooth and l^2-valued on ∂D. Then*

$$\|\widehat{\underline{k}}\|^2_{H\mathcal{D}}\leq\|\underline{k}\|^2_A+\|\widehat{\underline{k}}\|^2_\sigma.$$

Proof.

$$\|\widehat{\underline{k}}\|^2_{H\mathcal{D}}=\|\widehat{\underline{k}}\|^2_\sigma+\underbrace{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\|\widehat{\underline{k}}(e^{it})-\widehat{\underline{k}}(e^{i\theta})\|_{l^2}^2}{|e^{it}-e^{i\theta}|^2}\,d\sigma d\sigma}_{a}$$

Since all entries in $\widehat{\underline{k}}(e^{it})$ involve only negative Fourier coefficients, we see that

$$(a) = \sup\{|\underbrace{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\langle \frac{\widehat{\underline{k}}(e^{it}) - \widehat{\underline{k}}(e^{i\theta})}{e^{it}-e^{i\theta}}, \frac{\overline{\underline{p}_0(e^{it})} - \overline{\underline{p}_0(e^{i\theta})}}{e^{it}-e^{i\theta}}\rangle\, d\sigma d\sigma}_{b}|^2 : \underline{p}_0 \text{ has}$$

analytic polynomial entries that vanish at 0 and $\|\underline{p}_0\|_{\mathcal{D}} \le 1\}$.

But

$$\begin{aligned}(b) &= -\int_D\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\langle \underline{k}(z)[\frac{1}{z-e^{it}} - \frac{1}{z-e^{i\theta}}](e^{it}-e^{i\theta})^{-1}, \frac{\overline{\underline{p}_0(e^{it})} - \overline{\underline{p}_0(e^{i\theta})}}{e^{it}-e^{i\theta}}\rangle d\sigma d\sigma dA \\
&= \int_D\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\langle \underline{k}(z)\frac{1}{(e^{it}-z)(e^{i\theta}-z)}, \overline{\left(\frac{\underline{p}_0(e^{it}) - \underline{p}_0(e^{i\theta})}{e^{it}-e^{i\theta}}\right)}\rangle e^{it}e^{i\theta} d\sigma(t)d\sigma(\theta)dA \\
&= \int_D \frac{1}{2\pi i}\int_{\partial D}\frac{1}{2\pi i}\int_{\partial D}\langle \underline{k}(z)\frac{1}{(u-z)(v-z)}, \overline{\left(\frac{\underline{p}_0(u)-\underline{p}_0(v)}{u-v}\right)}\rangle\, du\, dv\, dA(z) \\
&= \int_D \frac{1}{2\pi i}\int_{\partial D}\langle \underline{k}(z)\frac{1}{(v-z)}, \overline{\left(\frac{\underline{p}_0(z)-\underline{p}_0(v)}{(z-v)}\right)}\rangle\, dv\, dA(z) \\
&= \int_D \langle \underline{k}(z), \overline{\underline{p}_0'(z)}\rangle\, dA(z)\end{aligned}$$

by two applications of Cauchy's theorem.

Now

$$\begin{aligned}|\int_D \langle \underline{k}(z), \overline{\underline{p}_0'(z)}\rangle\, dA| &\le (\int_D \|\underline{k}\|^2\, dA)^{\frac{1}{2}}(\int_D \|\underline{p}_0'\|^2\, dA)^{\frac{1}{2}} \\
&\le \left(\int_D \|\underline{k}\|^2\, dA\right)^{\frac{1}{2}}\end{aligned}$$

So

$$(a) \le \|\underline{k}\|_A^2. \qquad \square$$

We are now ready to proceed with a proof of Theorem A′ in the smooth case.

Proof. Assume that $\{f_j\}_{j=1}^{\infty}$ are analytic in $|z| < 1+\delta$ for all j, $\|M_F^C\| \le 1$, and $0 < \epsilon^2 \le F(z)F(z)^*$ for all $z \in D$, where $F(z) = (f_1(z), f_2(z), \dots)$. Let h be analytic in $|z| < 1+\delta$. Define

$$\underline{u}_h = \frac{F^*h}{FF^*} - Q\left(\widehat{\frac{Q^*F'^*h}{(FF^*)^2}}\right)^z$$

pointwise on $\overline{D}$. From our construction, the entries of $Q(z)$ are contained in $\{0, \pm f_j\}_{j=1}^{\infty}$, so

$$\overline{\partial}\, \underline{u}_h(z) = \underline{0} \quad \text{for } z \in D.$$

Thus we need only show that

$$\|\underline{u}_h\|_{\mathcal{D}}^2 \le \left(\frac{1,500}{\epsilon^3}\right)^2 \|h\|_{\mathcal{D}}^2.$$

Combining Lemmas 5, 7, and 8, we get

$$\begin{aligned}\|\underline{u}_h\|_{\mathcal{D}} = \|\underline{u}_h\|_{H\mathcal{D}} &= \|\frac{F^*h}{FF^*} - Q\widehat{\frac{Q^*F'^*h}{(FF^*)^2}}^z\|_{H\mathcal{D}} \\ &\le \|\frac{F^*h}{FF^*}\| + \|Q\widehat{\frac{Q^*F'^*h}{(FF^*)^2}}^z\|_{H\mathcal{D}} \\ &\le \frac{86\cdot\sqrt{200}}{\epsilon^2}\|h\|_{\mathcal{D}} + \sqrt{86}\sqrt{\underbrace{\|\frac{Q^*F'^*h}{(FF^*)^2}\|_A^2}_{a} + \underbrace{\|\widehat{\frac{Q^*F'^*h}{(FF^*)^2}}^z\|_\sigma^2}_{b}}.\end{aligned}$$

But since $\|\frac{Q(z)}{\sqrt{FF^*}}\|_{B(l^2)} \le 1$,

$$(a) \le \frac{1}{\epsilon^6}\|M_F^C(\overline{h})\|_{H\mathcal{D}}^2 \le \frac{20}{\epsilon^6}\|h\|_{\mathcal{D}}^2.$$

To estimate (b), we need the corona estimates for the $H^\infty(D)$ corona theorem. Using the Wolff procedure (see Garnett [G]) of Paley-Littlewood estimates, we get that

$$\begin{aligned}(b) &\le \left(\frac{8}{\epsilon^2}\ln\frac{1}{\epsilon^2}\right)^2 \|h\|_\sigma^2 \\ &\le \left(\frac{8}{\epsilon^3}\right)^2 \|h\|_{\mathcal{D}}^2.\end{aligned}$$

See Trent [Tr2] for more details.

Combining these estimates we see that in the smooth case,

$$\|\underline{u}_h\| \le \frac{1,500}{\epsilon^3}\|h\|_{\mathcal{D}}.$$ □

We show that the same estimate holds for the general case. The following two lemmas hold for any N-P r.k. kernel on the ball or polydisk in $\mathbb{C}^n$.

Lemma 9. *Let $\{f_j\}_{j=1}^\infty \subset M(\mathcal{D})$ with $\|M_F^C\| = 1$. For $0 \le r \le 1$, let $F_r(z) = F(rz)$. Then $\|M_{F_r}^C\| \le \|M_F^C\|$ and thus $F_r \in M(\mathcal{D}, \overset{\infty}{\underset{1}{\oplus}}\mathcal{D})$.*

Proof. We claim that

$$I - M_{F_r}^C(M_{F_r}^C)^* \ge 0.$$

That is, for any $\{\underline{c}_j\}_{j=1}^n \subset l^2$ and $\{z_j\}_{j=1}^n \subset D$,

$$0 \le \sum\sum\langle (I - F(rz_k)F(rz_j)^*)\underline{c}_j, \underline{c}_k\rangle k_{z_j}(rz_k). \tag{3}$$

But

$$(3) = \sum_{j=1}^{n}\sum_{k=1}^{n}\langle (I - F(rz_k)F(rz_j)^*)\,\underline{c}_j, \underline{c}_k\rangle k_{rz_j}(rz_k)\cdot [\frac{k_{z_j}(z_k)}{k_{rz_j}(rz_k)}] \tag{4}$$

The expression (4) without the "boxed terms" is positive since $I - M_F^C M_F^{C*} \geq 0$. We need only note that the matrix whose ij-th entry is the boxed term is positive. Then Schur's lemma gives us that (4) is positive.

Now $k_w(z)$ is an N-P kernel, in fact

$$1 - \frac{1}{k_w(z)} = \sum_{n=1}^{\infty} c_n z\overline{w}^n \text{ and } c_n > 0 \text{ for all } n.$$

Thus

$$\begin{aligned}\frac{k_{z_j}(z_k)}{k_{z_j r}(z_k r)} &= (1 - \sum_{n=1}^{\infty} c_n r^{2n}\overline{z}_j^n z_k^n)k_{z_j}(z_k)\\ &= (1 - \sum_{n=1}^{\infty} c_n \overline{z}_j^n z_k^n + \sum_{n=1}^{\infty}(1 - r^{2n})c_n z_k^n \overline{z}_j^n)k_{z_j}(z_k)\\ &= 1 + \sum_{n=1}^{\infty} c_n(1 - r^{2n})z_j^n \overline{z}_k^n k_{z_j}(z_k).\end{aligned}$$

Thus, $[\frac{k_{z_j}(z_k)}{k_{rz_j}(rz_k)}]_{j,k=1}^n$ is positive and we are done. □

Lemma 10. *Let $\mathcal{F} \in M(\overset{\infty}{\underset{1}{\oplus}}\mathcal{D})$. Then s-$\lim_{r\to 1^-} M_{\mathcal{F}_r}^* = M_{\mathcal{F}}^*$.*

Proof. Assume that $\|M_{\mathcal{F}}\|_{B(\overset{\infty}{\underset{1}{\oplus}}\mathcal{D})} \leq 1$. By Lemma 8, $\|M_{\mathcal{F}_r}\|_{B(\overset{\infty}{\underset{1}{\oplus}}\mathcal{D})} \leq 1$ for all $0 \leq r \leq 1$. Thus we need only show that $\lim_{r\to 1^-}\|(M_{\mathcal{F}_r}^* - M_{\mathcal{F}}^*)\underline{x}\| = 0$ for $\underline{x}$ in a dense subset of $\overset{\infty}{\underset{1}{\oplus}}\mathcal{D}$. By considering finite sums of the form $\sum_{j=1}^N \underline{c}_j k_{z_j}$, with $\{\underline{c}_j\}_{j=1}^N \subset l^2$ and $\{z_j\}_{j=1}^N \subset D$, we need only show that for $\underline{e} \in l^2$ and $z \in D$, $\lim_{r\to 1^-}\|(M_{\mathcal{F}_r}^* - M_{\mathcal{F}}^*)\,\underline{e}k_z\|_{\mathcal{D}} = 0$.

Now

$$\begin{aligned}(M_{\mathcal{F}_r}^* - M_{\mathcal{F}}^*)(\underline{e}k_z) &= \mathcal{F}(rz)^*\underline{e}k_z - \mathcal{F}(z)^*\underline{e}k_z\\ &= \mathcal{F}(rz)^*\underline{e}k_{rz}\frac{k_z}{k_{rz}} - \mathcal{F}(z)^*\underline{e}k_z\\ &= M_{\mathcal{F}}^*(\underline{e}k_{rz})\frac{k_z}{k_{rz}} - M_{\mathcal{F}}^*(\underline{e}k_z).\end{aligned}$$

Thus

$$\|(M_{\mathcal{F}_r}^* - M_{\mathcal{F}}^*)(\underline{e}k_z)\| \leq \|k_{rz} - k_z\| + \|k_{rz}\|\left(\sup_{w\in D}|\frac{k_z(w)}{k_{rz}(w)} - 1|\right). \tag{5}$$

But

$$|\frac{k_z(w)}{k_{rz}(w)} - 1| = |\frac{k_z(w) - k_z(rw)}{k_{rz}(w)}| \leq \frac{|k_z(w) - k_z(rw)|}{k_z(1)}. \tag{6}$$

Since k_z is uniformly continuous on $\overline{D}$, we see that combining (5) and (6) completes the proof. □

Proof of Theorem 1. Let $\{f_j\}_{j=1}^{\infty} \subset M(\mathcal{D})$, $\|M_F^C\| \leq 1$ and $\epsilon^2 \leq F(z)F(z)^*$ for all $|z| < 1$. By Lemma 8 for $0 \leq r < 1$, we have $\|M_{F_r}^C\| \leq 1$ and $\epsilon^2 \leq F_r(z)F_r(z)^*$ for all $|z| < 1$. By the proof of Theorem A′ in the smooth case we have

$$\left(\frac{1,500}{\epsilon^3}\right)^{-2} I \leq M_{F_r}^R (M_{F_r}^R)^* \leq I \quad \text{for } 0 \leq r < 1.$$

By Theorem B, $\exists\, G_r \in M(\mathcal{D}, \overset{\infty}{\underset{1}{\oplus}} \mathcal{D})$ so that $M_{F_r}^R M_{G_r}^C = I$ and $\|M_{G_r}^C\| \leq \frac{1,500}{\epsilon^3}$.

By compactness, we may choose a net with $G_{r_\alpha}^* \overset{WOT}{\to} G^*$ as $r_\alpha \to 1^-$. Note that $G \in M(\mathcal{D}, \overset{\infty}{\underset{1}{\oplus}} \mathcal{D})$, since the multiplier algebra (as operators) is WOT closed. Since Lemma 9 says that $F_{r_\alpha}^* \overset{S}{\to} F^*$, we get

$$I = G_{r_\alpha}^* F_{r_\alpha}^* \overset{WOT}{\to} G^* F^* \quad \text{and} \quad FG = I.$$

with entries of G in $M(\mathcal{D})$ and $\|M_G^C\| \leq \frac{1,500}{\epsilon^3}$. □

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