H^1 AND DYADIC H^1

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ABSTRACT. In this paper we give a simple proof of the fact that the average over all dyadic lattices of the dyadic H^1-norm of a function gives an equivalent H^1-norm. The proof we present works for both one-parameter and multi-parameter Hardy spaces.

The results of such type are known; cf. [2] for the one-parameter case. Also, by duality, such results are equivalent to the "BMO from dyadic BMO" statements proved in [4] (one parameter case) and [8] (two-parameter case).

While the paper generalizes these results to the multi-parameter setting, this is not its main goal. The purpose of the paper is to present an approach leading to a simple proof, which works in both one-parameter and multi-parameter cases.

The main idea of treating square function as a Calderón–Zygmund operator is a common-place in harmonic analysis; the main observation, on which the paper is based, is that one can treat the random dyadic square function this way. After that, all is proved by using the standard and well-known results about Calderón–Zygmund operators in the Hilbert-space-valued setting.

As an added bonus, we get a simple proof of the (equivalent by duality) inclusions BMO $\subset$ BMO$_\mathrm{d}$, $H_\mathrm{d}^1 \subset H^1$ in the multi-parameter case. Note, that unlike the one-parameter case, the inclusions in the general situation are far from trivial.

CONTENTS

NOTATION

$\mathcal{H} \otimes \mathcal{K}$ tensor product of Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$; we consider its endowed with the "canonical" norm, which makes $\mathcal{H} \otimes \mathcal{K}$ a Hilbert space. If one of the spaces is a function space, for example if $\mathcal{H} = L^2$, then $\mathcal{H} \otimes \mathcal{K}$ can be interpreted as L^2 with values in $\mathcal{K}$.

$L^p(X; \mathcal{H})$ L^p-space of functions on X with values in $\mathcal{H}$; X is usually $\mathbb{R}^N$ with the Lebesgue measure. Symbols X and/or $\mathcal{H}$ can be omitted if they are clear from the context.

dx Lebesgue measure in $\mathbb{R}^N$ if $x \in \mathbb{R}^N$.

$\mathcal{D}$ Dyadic Lattice in $\mathbb{R}^N$, see below.

$H^1, H_{\mathcal{D}}^1$ real variable Hardy space on $\mathbb{R}^N$ and its dyadic counterpart, associated to a dyadic lattice $\mathcal{D}$. We will also consider Hilbert-space-valued spaces, and the notation like $H^1(X; \mathcal{H})$ will be used.

$\ell(Q), |Q|$ sidelength and volume of a cube $Q \subset \mathbb{R}^N$.

$\mathbb{E}\xi$ expectation of a random variable ξ, $\mathbb{E}\xi = \int_\Omega \xi(\omega) dP(\omega)$; sometimes, to distinguish the random variable in a formula, the notation $\mathbb{E}\xi(\omega)$ or $\mathbb{E}_\omega \xi(\omega)$ will be used instead of $\mathbb{E}\xi$.

Cubes and dyadic lattices. Throughout the paper we will speak a lot about dyadic cubes and dyadic lattices, so let us first fix some terminology. A *cube* in $\mathbb{R}^N$ is an object obtained from the *standard cube* $[0,1)^N$ by dilations and shifts.

For a cube Q we denote by $\ell(Q)$ its *size*, i. e. the length of its side. Given a cube Q one can split it by dividing each side in halves into 2^N cubes Q_k of size $\ell(Q)/2$: we will call such cubes Q_k the *children of* Q.

For a cube Q and $\lambda > 0$ we denote by λQ the cube Q dilated λ times with respect to its center.

Now, let us define the *standard dyadic lattice* $\mathcal{D}_0$: for each $k \in \mathbb{Z}$ let us consider the cube $[0, 2^k)^N$ and all its shifts by elements of $\mathbb{R}^N$ with coordinates of form $j \cdot 2^k$, $j \in \mathbb{Z}$. The collection of all such cubes (union over all k) is called the *standard dyadic lattice*.

A *dyadic lattice* $\mathcal{D}$ is just a shift of the standard dyadic lattice $\mathcal{D}_0$. A collection of all cubes from a dyadic lattice $\mathcal{D}$ of a fixed size 2^k is called a *layer* of the lattice.

Random dyadic lattice. Our random lattice will contain the dyadic cubes of standard size 2^k ($k \in \mathbb{Z}$), but will be "randomly shifted" with respect to the standard dyadic lattice $\mathcal{D}_0$. The simplest idea would be to pick up a random variable x uniformly distributed over $\mathbb{R}^N$ and to define the random lattice as $x + \mathcal{D}_0$. This idea works for the torus $\mathbb{T}^N$, but unfortunately, there exists no such random variable x in $\mathbb{R}^N$, so we have to act in a little bit more sophisticated way.

Let us construct a random lattice of dyadic intervals on the real line $\mathbb{R}$, and then define a random lattice in $\mathbb{R}^N$ as the product of the lattices of intervals.

Let Ω_1 be some probability space and let $x(\omega)$ be a random variable uniformly distributed over the interval $[0, 1)$.

Let $\xi_j(\omega)$ be random variables satisfying $\mathbb{P}\{\xi_j = +1\} = \mathbb{P}\{\xi_j = -1\} = 1/2$. Assume also that $x(\omega), \xi_j(\omega), j \in \mathbb{N}$ are independent. Define the random lattice $\mathcal{D}(\omega)$ as follows:

(1) We require that $I_0(\omega) := [x(\omega) - 1, x(\omega)] \in \mathcal{D}(\omega)$; this gives us all intervals in $\mathcal{D}(\omega)$ of length 2^k, $k \leq 0$.

(2) To determine the rest of the intervals, it is enough to know dyadic intervals $I_k(\omega) \supset I_0(\omega)$, of length 2^k, $k \geq 0$. The intervals $I_k(\omega)$ are determined inductively: if $I_{k-1}(\omega) \in \mathcal{D}(\omega)$ is already known (and thus all intervals of length 2^{k-1} in $\mathcal{D}(\omega)$), then
  - $I_k(\omega)$ is the union of $I_{k-1}(\omega)$ and its right neighbor if $\xi_k(\omega) = +1$, and
  - $I_k(\omega)$ is the union of $I_{k-1}(\omega)$ and its left neighbor if $\xi_k(\omega) = -1$.

To get a random dyadic lattice in $\mathbb{R}^N$ we just take N independent random dyadic lattices $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_N$ in $\mathbb{R}$ and consider all cubes $Q = I_1 \times I_2 \times \ldots \times I_N$, $I_k \in \mathcal{D}_k$.

0. Introduction and main results

0.1. **One parameter case.** Let $H^1 = H^1(\mathbb{R}^N)$ be the usual *real variable* Hardy space on $\mathbb{R}^N$, and let $H_{\mathcal{D}}^1$ be its dyadic counterpart, defined as follows.

Consider a dyadic lattice $\mathcal{D}$ in $\mathbb{R}^N$. Let $E_k = E_k^{\mathcal{D}}$ be the averaging operator over cubes $Q \in \mathcal{D}$ of size 2^k, $E_k f(x) = |Q|^{-1} \int_Q f$, where Q is the cube in $\mathcal{D}$ of size 2^k containing x. If $Q \in \mathcal{D}$ is a cube of size 2^k define E_Q by $E_Q f = (|Q|^{-1} \int_Q f) \mathbf{1}_Q = \mathbf{1}_Q (E_k f)$.

Define the martingale differences $\Delta_k = \Delta_k^{\mathcal{D}} := E_{k-1}^{\mathcal{D}} - E_k^{\mathcal{D}}$, and again for a cube $Q \in \mathcal{D}$ of size 2^k define Δ_Q by $\Delta_Q f = \mathbf{1}_Q (E_k f)$.

Define the dyadic square function $S = S_{\mathcal{D}}$ by

$$(0.1) \qquad (S_{\mathcal{D}} f)(x) = \left(\sum_{k \in \mathbb{Z}} |\Delta_k^{\mathcal{D}} f(x)|^2 \right)^{1/2} = \left(\sum_{Q \in \mathcal{D}, Q \ni x} |\Delta_Q f(x)|^2 \right)^{1/2}.$$

One can also consider a slightly different square function $\widetilde{S} f = \widetilde{S}_{\mathcal{D}} f$

$$(0.2) \qquad (\widetilde{S}_{\mathcal{D}} f)(x) = \left(\sum_{k \in \mathbb{Z}} \left(E_k^{\mathcal{D}} |\Delta_k^{\mathcal{D}} f|^2 \right)(x) \right)^{1/2} = \left(\sum_{Q \in \mathcal{D}, Q \ni x} \left(E_Q^{\mathcal{D}} |\Delta_Q^{\mathcal{D}} f|^2 \right)(x) \right)^{1/2};$$

in other words, each term in the sum on the right equals the average of $|\Delta_Q f|^2$ if $x \in Q$ and 0 otherwise.

It is not hard to show and will be explained later that $C^{-1} \|Sf\|_1 \leq \|\widetilde{S}f\|_1 \leq C\|Sf\|_1$, where the constant C depends only on the dimension N.

Definition. A function $f \in L_{\mathrm{loc}}^1$ is in the dyadic Hardy space $H_{\mathcal{D}}^1$ (with respect to the dyadic lattice $\mathcal{D}$) if $\|f\|_{H_{\mathcal{D}}^1} := \|S_{\mathcal{D}} f\|_1 < \infty$

One can use the square function $\widetilde{S}_{\mathcal{D}} f$ here and get an equivalent norm.

And now one of the main results of the paper. Let $\mathcal{D}(\omega)$, $\omega \in \Omega$ be the random dyadic lattice, as described above, and let us recall that $\mathbb{E} = \mathbb{E}_\omega$ denotes the expectation (average with respect to ω)

Theorem 0.1. *A function $f \in L_{\mathrm{loc}}^1(\mathbb{R}^n)$ belongs to H^1 if and only if*

$$\int_{\mathbb{R}^N} \left[\mathbb{E}(|S_{\mathcal{D}(\omega)} f(x)|^2) \right]^{1/2} dx < \infty.$$

Moreover, the latter quantity gives an equivalent norm on H^1.

The same result is true with the square function $\widetilde{S}_{\mathcal{D}(\omega)}$ in place of $S_{\mathcal{D}(\omega)}$.

0.2. **Multi-parameter case.** The above results can be generalized to the case of multi-parameter Hardy spaces. Let $X_k = \mathbb{R}^{N_k}$, $k = 1, 2, \dots, n$ and let $H^1(X_1 \times X_2 \times \dots \times X_n)$ be the n-parameter Hardy space, see Section 2.1 below for the definition.

Define the dyadic Hardy space as follows. Let $\mathcal{D}_k$ be a dyadic lattice on X_k, $k = 1, 2, \dots, n$ and let $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2 \times \dots \times \mathcal{D}_n$ be the product dyadic lattice on $X = X_1 \times X_2 \times \dots \times X_n$; the elements on $\mathcal{D}$ are the "rectangles" (parallelepipeds) $R = Q_1 \times Q_2 \times \dots \times Q_n$, $Q_k \in \mathcal{D}_k$.

For a multiindex $\mathbf{k} = (k_1, k_2, \dots k_n)$ define on $X = X_1 \times X_2 \times \dots \times X_n$ the average $E_{\mathbf{k}} := E_{k_1}^1 E_{k_2}^2 \dots E_{k_n}^n$ and the martingale differences $\Delta_{\mathbf{k}} := \Delta_{k_1}^1 \Delta_{k_2}^2 \dots \Delta_{k_n}^n$, where $E_{k_j}^j$ and $\Delta_{k_j}^j$ are the "one variable" averages E_{k_j} martingale difference Δ_{k_j} as defined above by in Section 0.1, taken in the variable $x_j \in X_j$.

For a "rectangle" $R = Q_1 \times Q_2 \times \ldots \times Q_n$, $Q_k \in \mathcal{D}_k$ define $\Delta_R := \Delta_{Q_1}^1 \Delta_{Q_2}^2 \ldots \Delta_{Q_n}^n$, where again $\Delta_{Q_j}^j$ is the operator Δ_{Q_j} defined by (0.2) taken in the variable x_j.

Define the multi-parameter square function $S = S_{\mathcal{D}}$ by

$$(0.3) \qquad (S_{\mathcal{D}}f)(x) = \left(\sum_{\mathbf{k} \in \mathbb{Z}^n} |\Delta_{\mathbf{k}}f(x)|^2 \right)^{1/2} = \left(\sum_{R \in \mathcal{D}, R \ni x} |\Delta_R f(x)|^2 \right)^{1/2}.$$

One can also define the square function $\widetilde{S} = \widetilde{S}_{\mathcal{D}}$

$$(0.4) \qquad \widetilde{S}f(x) := \left(\sum_{k \in \mathbb{Z}} \left(E_k |\Delta_k f|^2 \right)(x) \right)^{1/2} = \left(\sum_{R \in \mathcal{D}, R \ni x} \left(E_R |\Delta_R f|^2 \right)(x) \right)^{1/2}.$$

The definitions look very similar to (0.1), (0.2), only here the sums in the right hand side is taken over all "rectangles" R, while in (0.1), (0.2) they are taken over all cubes.

We use the same notation for the one-parameter and multi-parameter square function, but since we will treat these cases in different sections, we hope to avoid the confusion.

Definition. Let $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2 \times \ldots \times \mathcal{D}_n$ be a product dyadic lattice on $X = X_1 \times X_2 \times \ldots \times X_n$. We say that a function $f \in L^1_{\mathrm{loc}}(X)$ belongs to the dyadic Hardy space $H^1_{\mathcal{D}}$ if $\|f\|_{H^1_{\mathcal{D}}} := \|S_{\mathcal{D}}f\|_1 < \infty$

Let now $\mathcal{D}_1(\omega), \mathcal{D}_2(\omega), \ldots, \mathcal{D}_n(\omega)$ be the independent random dyadic lattices on the spaces $X_1, X_2, \ldots, X_n$ respectively, and let $\mathcal{D}(\omega) = \mathcal{D}_1(\omega) \times \mathcal{D}_2(\omega) \times \ldots \times \mathcal{D}_n(\omega)$ be the multi-parameter random dyadic lattice.

Theorem 0.2. *A function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to $H^1(X_1 \times X_2 \times \cdots \times X_n)$ if and only if*

$$\int_{\mathbb{R}^N} \left[\mathbb{E}(|S_{\mathcal{D}(\omega)}f(x)|^2) \right]^{1/2} dx < \infty;$$

here $S_{\mathcal{D}(\omega)}$ is the multi-parameter random dyadic lattice defined above. Moreover, the latter quantity gives an equivalent norm on H^1.

The same result is true with the square function $\widetilde{S}_{\mathcal{D}(\omega)}$ in place of $S_{\mathcal{D}(\omega)}$.

0.3. **Some remarks.**

Remark 0.3. It is a simple exercise to show that in both one-parameter and multi-parameter cases

$$\mathbb{E}(|S_{\mathcal{D}(\omega)}f(x)|^2) = \lim_{Q \to \mathbb{R}^N} \frac{1}{|Q|} \int_Q |S_{\mathcal{D}}f(x+u)|^2 du$$

where $\mathcal{D}$ is a fixed dyadic lattice and the limits is taken over cubes (or parallelipipeds) $Q \subset \mathbb{R}^N$ ($Q \subset X$ in the multi-parameter case) centered at 0 whose sidelengths tend to ∞.

Remark 0.4. Note that for every dyadic lattice $\mathcal{D}$ one has $\|f\|_{H^1} \leq C\|S_{\mathcal{D}}f\|_1$, thus by Tonelli theorem and Hölder inequality we have for $p \leq 2$

$$\|f\|_{H^1} \leq C \int_{\mathbb{R}^N} \left[\mathbb{E}(|S_{\mathcal{D}(\omega)}f(x)|^p) \right]^{1/p} dx \leq C \int_{\mathbb{R}^N} \left[\mathbb{E}(|S_{\mathcal{D}(\omega)}f(x)|^2) \right]^{1/2} dx.$$

In the one-parameter case the estimate $\|f\|_{H^1} \leq C\|S_{\mathcal{D}}f\|_1$ is trivial and well known. Indeed, H^1-BMO duality ($(H^1)^* = \mathrm{BMO}$, $(H^1_{\mathcal{D}})^* = \mathrm{BMO}_{\mathcal{D}}$, see the definitions below in Section 3.1) and the trivial inclusion $\mathrm{BMO} \subset \mathrm{BMO}_{\mathcal{D}}$ imply the $H^1_{\mathcal{D}} \subset H^1$ with the desired estimates of the norms.

Since BMO $\neq$ BMO$_\mathcal{D}$, one can conclude that th inclusion $H_\mathcal{D}^1 \subset H^1$ is proper.

In the multi-parameter case the estimate $\|f\|_{H^1} \leq C\|S_\mathcal{D}f\|_1$is also known to specialists, but it is less trivial. In fact, the only place that the author is aware of, where this is proved is the Ph.D. thesis of J. Pipher; this proof is far from trivial and the calculations are quite tedious. Below in Section 2.3 we present a different, quite simple prove of this fact. This proof is based on one-parameter (Hilbert-space-valued) H^1-BMO theory.

Remark 0.5. Applying Hölder inequality to Theorem 0.1 (to Theorem 0.2 in the multi-parameter case) we get that for $p \leq 2$

$$(0.5) \qquad \int_{\mathbb{R}^N} \left[\mathbb{E}(|S_{\mathcal{D}(\omega)}f(x)|^p)\right]^{1/p} dx \leq \int_{\mathbb{R}^N} \left[\mathbb{E}(|S_{\mathcal{D}(\omega)}f(x)|^2)\right]^{1/2} dx \leq C\|f\|_{H^1}$$

If $p = 1$ Tonelli Theorem and (0.5) implies that

$$(0.6) \qquad \mathbb{E}\|S_{\mathcal{D}(\omega)}f\|_1 \leq C\|f\|_{H^1}.$$

This was proved in one-parameter case in [2]. Note, that by duality (0.6) is equivalent to "BMO from dyadic BMO" statements: if f_ω $\omega \in \Omega$ is a measurable family of functions, $f_\omega \in \text{BMO}_{\mathcal{D}(\omega)}$, $\|f_\omega\|_{\text{BMO}_{\mathcal{D}(\omega)}} \leq 1$, then for f defined by $f(x) = \mathbb{E}_\omega f_\omega(x)$ one has $f \in \text{BMO}$, $\|f\|_{\text{BMO}} \leq C$.

This "BMO from dyadic BMO" result was proved directly in [4] (one-parameter case) and in the recent paper [8] (the two-parameter case).

1. Proof of Theorem 0.1

1.1. "Vectorization" of the square function.
There is a standard way of making the nonlinear operator $S_\mathcal{D}$ into a linear one by treating $S_\mathcal{D}f$ as a vector-valued function.

Namely, let us consider the space $L^1(\ell^2)$ consisting of functions on $\mathbb{Z} \times \mathbb{R}^N$ such that

$$\|f\|_{L^1(\ell^2)} := \int_{\mathbb{R}^N} \left(\sum_{k \in \mathbb{Z}} |f(k,x)|^2\right)^{1/2} dx < \infty.$$

Define the vector-valued square function $\mathbf{S}_\mathcal{D}f$ by

$$\mathbf{S}_\mathcal{D}f(k,x) := \Delta_k^\mathcal{D} f(x), \qquad k \in \mathbb{Z}, \ x \in \mathbb{R}^N$$

($\mathbf{S}_\mathcal{D}f$ is a function on $\mathbb{Z} \times \mathbb{R}^N$). We will treat this function as a function of the argument $x \in \mathbb{R}^N$ with values in $\ell^2 = \ell^2(\mathbb{Z})$.

Clearly, $\left|\mathbf{S}_\mathcal{D}f(\cdot, x)\right|_{\ell^2} = S_\mathcal{D}f(x)$, so $f \in H_\mathcal{D}^1$ if and only if $\mathbf{S}f \in L^1(\ell^2)$. Moreover, $\|\mathbf{S}_\mathcal{D}f\|_{L^1(\ell^2)} = \|S_\mathcal{D}f\|_1 = \|f\|_{H_\mathcal{D}^1}$.

Let now $\mathcal{D}(\omega)$ be the random dyadic lattice, and let $\Omega, \mathbb{P}$ be the corresponding probability space. Consider the space $\mathcal{L} = L^1(\ell^2 \otimes L^2(\Omega, \mathbb{P}))$,

$$\|f\|_\mathcal{L} := \int_{\mathbb{R}^N} \left(\int_\Omega \sum_{k \in \mathbb{Z}} |f(k,\omega,x)|^2 d\mathbb{P}(\omega)\right)^{1/2} dx.$$

It is an L^1 space with values in the Hilbert space $\ell^2 \otimes L^2(\Omega, \mathbb{P})$; here again f is treated as function of the argument $x \in \mathbb{R}^N$ with values in $\ell^2 \otimes L^2(\Omega, \mathbb{P})$.

Define the vector-valued square function $\mathbf{S}$ with values in $\ell^2 \otimes L^2(\Omega, \mathbb{P})$ by

$$(1.1) \qquad \mathbf{S}f(k,\omega,x) = \mathbf{S}_{\mathcal{D}(\omega)}f(k,x), \qquad x \in \mathbb{R}^N, \ k \in \mathbb{Z}, \omega \in \Omega.$$

here and below we will use notation $\mathbf{S}(\cdot, \cdot, x) =: \mathbf{S}f(x) \in \ell^2 \otimes L^2(\Omega, \mathbb{P})$.

Clearly, $\|\mathbf{S}f(x)\|_{\ell^2 \times L^2(\Omega,\mathbb{P})} = \left[\mathbb{E}(|S_{\mathcal{D}(\omega)}f(x)|^2)\right]^{1/2}$, so $\|\mathbf{S}f\|_{\mathcal{L}} = \int_{\mathbb{R}^N} \left[\mathbb{E}(|S_{\mathcal{D}(\omega)}f(x)|^2)\right]^{1/2} dx$ (the norm in $\mathcal{L}$ was constructed so this would hold).

Similar "vectorization" can be performed to the square function $\widetilde{S}$. Namely, fix some ordering of the "children" of a dyadic cube (the same one for all cubes), and for each dyadic cube Q define operator from the functions constant on the "children" of Q to $\mathbb{C}^{2^N}$ by

$$U_Q \mathbf{1}_{Q_k} = |Q_k|^{1/2} e_k, \qquad k = 1, 2, \ldots, 2^{2^N},$$

where Q_k are "children" of Q and $\{e_k\}_{k=1}^{2^N}$ is the standard basis in $\mathbb{R}^{2^N}$. Let now $\mathcal{L} := L^1(\ell^2 \otimes L^2(\Omega,\mathbb{P}) \otimes \mathbb{C}^{2^N})$ with the norm

$$\|f\|_{\mathcal{L}} := \int_{\mathbb{R}^N} \left(\int_\Omega \sum_{k \in \mathbb{Z}} |f(k,\omega,x)|^2_{\mathbb{C}^{2^N}} d\mathbb{P}(\omega) \right)^{1/2} dx.$$

Define the square function $\widetilde{\mathbf{S}}$ by

$$\widetilde{\mathbf{S}}f(k,\omega,x) := U_Q \Delta_Q f \in \mathbb{C}^{2^N},$$

where $Q \in \mathcal{D}(\omega)$ is the cube of size 2^k containing x. From the construction it is clear that $\|\widetilde{\mathbf{S}}f\|_{\mathcal{L}} = \int_{\mathbb{R}^N} \left[\mathbb{E}(|\widetilde{S}_{\mathcal{D}(\omega)}f(x)|^2)\right]^{1/2} dx$.

Now the proof of Theorem 0.1 can be outlined in few sentences. First, it is nor hard to show that $\mathbf{S}$ (or $\widetilde{\mathbf{S}}$) is a Calderón–Zygmund operator, whose kernel takes values in the Hilbert space $\ell^2 \otimes L^2(\Omega,\mathbb{P})$. It is a well known fact that such Calderón–Zygmund operators map H^1 to L^1, so

$$\int_{\mathbb{R}^N} \left[\mathbb{E}(|S_{\mathcal{D}(\omega)}f(x)|^2)\right]^{1/2} dx = \|\mathbf{S}f\|_{\mathcal{L}} \le C\|f\|_{H^1}.$$

As we discussed above in Section 0.1, the opposite inequality $\|f\|_{H^1} \le C\|\mathbf{S}f\|_{\mathcal{L}}$ is trivial.

Of course, the classical theory of Calderón–Zygmund operators deals with the scalar-valued kernels. But, as it is well known to the specialists, all the facts that we need, are valid in the case of Hilbert space valued kernels too.

However, the blind trust is not expected from the reader: all relevant facts will be presented below.

1.2. $\mathbf{S}$ as a Calderón–Zygmund operator.
Let us recall that the classical Calderón–Zygmund kernel on $\mathbb{R}^N$ is the function $K(\cdot,\cdot)$ defined on $\mathbb{R}^N \times \mathbb{R}^N \setminus \{(x,x) : x \in \mathbb{R}^N\}$ satisfying

(1) $|K(x,y)| \le C|x-y|^{-N}$;
(2) There exists $\delta > 0$ such that

$$|K(x,y) - K(x_0,y)|, \ \left|K(y,x) - K(y,x_0)\right| \le C\frac{|x-x_0|^\delta}{|y-x_0|^{N+\delta}},$$

whenever $|y-x_0| \ge 2|x-x_0|$

One can also consider operator-valued kernels, $K(x,y) \in B(X,Y)$ for arbitrary Banach spaces X and Y. In this case $|\cdot|$ in the left hand side should be replaced by the norm in $B(\mathcal{X},\mathcal{Y})$.

1.2.1. *Vector-valued Calderón–Zygmund operators.* A Calderón–Zygmund operator with (operator-valued) Calderón–Zygmund kernel K is a bounded operator $T : L^2(\mathbb{R}^N; \mathcal{X}) \to L^2(\mathbb{R}^N, \mathcal{Y})$ such that for all compactly supported $f \in L^2(\mathbb{R}^N; \mathcal{X})$ and $g \in L^2(\mathbb{R}^N; \mathcal{Y}^*)$ with separated supports

$$\langle Tf, g \rangle = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \langle K(x,y) f(y), g(y) \rangle \, dy dx.$$

Such operators with operator-valued kernels were considered, for example, in [9], and it was proved there (see Theorem 1.2 in Ch. 3) that if $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces then the operator acts from H^1 to L^1, $\|Tf\|_{L^1(\mathbb{R}^N, \mathcal{X})} \leq C \|f\|_{H^1(\mathbb{R}^N, \mathcal{Y})}$.

In fact, in [9] a much more general situation was considered. The kernel was only assumed to satisfy a weaker version of condition (2), and there was no condition (1). Moreover, it was assumed that T was bounded in some L^p, $1 < p < \infty$, not necessarily $p = 2$.

Note, that condition (1) in some form is required for the proof of $T1$ and Tb theorems, but if one assumes that T is bounded in some L^p, $1 < p < \infty$ the condition (2) alone is sufficient for the action from H^1 to L^1.

Also, in [9] the theorem was proved for the case when $\mathcal{X}$ and $\mathcal{Y}$ are arbitrary Banach spaces, if one defines H^1 via atomic decomposition.

It is well known that for the case of Hilbert-space-valued functions all the definitions of H^1 (via atomic decompositions, via maximal function, via different square functions, via Riesz transforms) are equivalent[1], so one can use the result from [9] without worrying about what definition of H^1 is used.

In this paper we are considering the case when $\mathcal{X} = \mathbb{C}$ and $\mathcal{Y} = \mathcal{H} = \ell^2 \otimes L^2(\Omega, \mathbb{P})$, so we can say that K takes values in the Hilbert space $\mathcal{H}$.

Operators with such kernels also act naturally from $L^2(\mathbb{R}^N; \mathcal{K}) \to L^2(\mathbb{R}^N; \mathcal{H} \otimes \mathcal{K})$, where $\mathcal{K}$ is a Hilbert space, and we will need this interpretation later in Section 2. Indeed, with each vector $h \in \mathcal{H}$ we can associate an operator $\mathcal{K} \ni f \mapsto f \otimes h \in \mathcal{K} \otimes \mathcal{H}$ (and the norm of this operator is $\|h\|$). So, if K is ah $\mathcal{H}$-valued Calderón–Zygmund kernel, then a Calderón–Zygmund operator on $L^2(\mathbb{R}^N; \mathcal{K})$ is a bounded operator $T : L^2(\mathbb{R}^N; \mathcal{K}) \to L^2(\mathbb{R}^N; \mathcal{H} \otimes \mathcal{K})$ such that

$$\langle Tf, g \rangle_{L^2(\mathbb{R}^N; \mathcal{H} \otimes \mathcal{K})} = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \langle K(x,y) \otimes f(y), g(y) \rangle_{\mathcal{H} \otimes \mathcal{K}} \, dy dx.$$

for all compactly supported f and g with separated supports.

1.2.2. *Why $\mathbf{S}$ is a Calderón–Zygmund operator?* To find the kernel K of $\mathbf{S}$ we need to compute $\mathbf{S}\delta_y$, where δ_y is the unit mass at $y \in \mathbb{R}^N$:

$$K(x,y) = \left\{ \Delta_k^{\mathcal{D}(\omega)} \delta_y(x) \right\}_{k \in \mathbb{Z}, \ \omega \in \Omega} \in \ell^2 \otimes L^2(\Omega, \mathbb{P})$$

To get that expression rigorously, one needs to approximate δ_y by appropriate bump functions; notice that $\Delta_k^{\mathcal{D}_\omega} \delta_y$ is well defined, i.e. does not depend on the choice of approximating sequence.

Notice that for a dyadic cube Q, $\Delta_Q \delta_y(x) \neq 0$ only if $x \in Q, y \in Q$. Therefore $\Delta_Q \delta_y(x) = 0$ if $\ell(Q) < |x - y|_\infty$, so $\Delta_k^{\mathcal{D}(\omega)} \delta_y(x) = 0$ if $2^k < |x - y|_\infty$.

Noticing that $\|\Delta_k^{\mathcal{D}_\omega} \delta_y\|_\infty \leq C 2^{-kN}$ and summing the geometric series we get the property (1) of Calderón–Zygmund kernels.

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[1]Unfortunately, the author cannot point to a paper where all such equivalences are proved; but following the proofs for the scalar-valued case, one can see that everything works for the case of Hilbert-space valued case as well.

To show property (2) notice that $\Delta_k^{\mathcal{D}(\omega)}\delta_y(x) = \Delta_k^{\mathcal{D}(\omega)}\delta_y(x_0)$ if all 3 points x, x_0, y are in the same cube $Q \in \mathcal{D}(\omega)$, $\ell(Q) = 2^k$ and the points x, x_0 are in the same "child" of Q.

The probability that it fails for a given k can be estimated above by $C|x - x_0|2^{-k}$. Using the estimate $\|\Delta_k^{\mathcal{D}_\omega}\delta_y\|_\infty \le C2^{-kN}$ we conclude that

$$\int_\Omega |\Delta_k^{\mathcal{D}(\omega)}\delta_y(x) - \Delta_k^{\mathcal{D}(\omega)}\delta_y(x_0)|^2 d\mathbb{P}(\omega) \le C2^{-2kN}|x - x_0|2^{-k} = C|x - x_0|2^{-2kN-k}.$$

Let k_0 be the maximal $k \in \mathbb{Z}$ such that $2^k < \min\{|x - y|_\infty, |x_0 - y|_\infty\}$. Then $\Delta_k^{\mathcal{D}_\omega}\delta_y(x) = \Delta_k^{\mathcal{D}_\omega}\delta_y(x_0) = 0$ for $k < k_0$, so

$$\big|K(x, y) - K(x_0, y)\big|^2 \le C \sum_{k \ge k_0} 2^{-2kN-k}|x - x_0| \le C|x - x_0|2^{-2k_0N-k_0}$$

$$\le C|x - x_0||x_0 - y|^{-2N-1}.$$

Interchanging x and y and repeating the above reasoning we also get that

$$\big|K(y, x) - K(y, x_0)\big|^2 \le C|x - x_0||x_0 - y|^{-2N-1}.$$

This means condition (2) holds with $\delta = 1/2$.

The proof for $\widetilde{\mathbf{S}}$ is absolutely the same.

1.3. A remark about conditions $\mathbf{S1} = 0$, $\mathbf{S}^*\mathbf{1} = 0$.

Material in this section is not needed for the proof of the main results. However, it might be of interest for specialists; one can use it to present a different proof of the main results, without employing the cited above in Section 1.2.1 result from [9] result about Calderón–Zygmund operators with operator-valued kernels.

Note that the operator $\mathbf{S}$, introduced above satisfies the conditions $\mathbf{S1} = 0$ and $\mathbf{S}^*\mathbf{1} = 0$ (more precisely, the second condition should read as $\mathbf{S}^*\mathbf{1}e = 0$ for all $e \in \mathcal{H}$), which are well known to everybody familiar with $T(1)$-theorem.

If one formally plugs 1 into $\mathbf{S}$ or $\mathbf{S}^*$, the result will be 0. Of course, it is only a formal reasoning, for 1 is not in the domain of $\mathbf{S}$, but any reasonable interpretation of $\mathbf{S1}$ gives the same result. For example it is not hard to show that

(1.2) $\mathbf{S1}_Q \to 0, \quad \mathbf{S}^*\mathbf{1}_Q e \to 0 \qquad$ as $\ell(Q) \to \infty$

uniformly on compact subsets, where cubes Q are centered at 0.

It is also easy to see that $\widetilde{\mathbf{S}}\mathbf{1} = 0$, but unfortunately $\widetilde{\mathbf{S}}^*\mathbf{1} \ne 0$. However, it is easy to modify $\widetilde{\mathbf{S}}$ to make $\widetilde{\mathbf{S}}^*\mathbf{1} = 0$.

Namely, let φ be a function on the cube $[0, 1)^N$ taking values ± 1 and such that $\int_Q \varphi \, dx = 0$ and let $\varphi_Q(x) = \varphi((x - x_Q)/\ell(Q))$, where x_Q is the base of Q, i.e. the point in Q with smallest coordinates.

Define the square function $\widetilde{\mathbf{S}}$ by

$$\widetilde{\mathbf{S}}f(k, \omega, x) := \varphi_Q(x)U_Q\Delta_Q f \in \mathbb{C}^{2^N},$$

where $Q \in \mathcal{D}(\omega)$ is the cube of size 2^k containing x.

The function φ_Q in the definition of $\widetilde{\mathbf{S}}$ is introduced to insure that $\widetilde{\mathbf{S}}^*\mathbf{1} = 0$. Now it is easy to show that $\widetilde{\mathbf{S}}\mathbf{1} = 0$, $\widetilde{\mathbf{S}}^*\mathbf{1} = 0$ (in the sense of (1.2)).

Calderón–Zygmund operators satisfying $T\mathbf{1} = 0$ and $T^*\mathbf{1} = 0$ map $H^1 \to H^1$. To show that one, for example can consider matrix of such an operator in the wavelet basis. It was shown in [6] that under rather mild assumption about wavelet basis, the coefficient space of

H^1 in this basis is the Triebel–Lizorkin space $\dot{\mathbf{f}}_1^{0,2}$; see Appendix (Section 3) for the definition. In [6] the scalar-valued case was treated, but one can easily see that everything works for the Hilbert-space valued case, and one just get the vector-valued space $\dot{\mathbf{f}}_1^{0,2}$. Moreover, in [5] the H^1 spaces with values in UMD Banach spaces were characterized in terms of coefficient in the wavelet basis; in the Hilbert-space-valued case the result gives exactly $\dot{\mathbf{f}}_1^{0,2}$ with values in the Hibert space.

Using the standard estimates with Calderón–Zygmund kernels one can see that if a Calderón–Zygmund operator T (even with the operator-valued kernel) satisfies $T\mathbf{1} = 0$, $T^*\mathbf{1} = 0$, then its matrix in the wavelet basis is what is called in [3] *almost diagonal* for $\dot{\mathbf{f}}_p^{0,q}$, $1 \le p, q < \infty$; cf. Section 3 below for the definition.

And it was shown in [3] that *almost diagonal* operators are bounded on all $\dot{\mathbf{f}}_p^{0,q}$, $1 \le p, q < \infty$. Since the almost diagonality is a condition on the magnitude of the entries, the result holds for vector-valued Triebel–Lizorkin spaces. Of course, instead of considering a wavelet basis, one can consider a frame decomposition, given by what is called in [3] *φ-transform*; all the estimates will be the same.

2. Proof for the multi-parameter case

Proof of the main result for multi-parameter case (Theorem 0.2) follows the lines of the proof for the one parameter.

Without loss of generality we can assume that the probability space Ω is represented as a product, $(\Omega, \mathbb{P}) = (\Omega_1 \times \Omega_2 \times \ldots \times \Omega_n, \mathbb{P}_1 \times \mathbb{P}_2 \times \ldots \times \mathbb{P}_n)$ and that the random dyadic grid $\mathcal{D}_k(\omega)$ on X_k depends only on ω_k.

For a dyadic lattice $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2 \times \ldots \times \mathcal{D}_n$ on $X_1 \times X_2 \times \ldots \times X_n$ define the vector-valued square function $\mathbf{S}_\mathcal{D}$ taking values in $\ell^2(\mathbb{Z}^n)$

$$\mathbf{S}_\mathcal{D} f(\mathbf{k}, x) := \Delta_{\mathbf{k}}^\mathcal{D} f(x), \qquad \mathbf{k} \in \mathbb{Z}^n, \ x \in \mathbb{R}^N$$

Consider the space $\mathcal{L} = L^1(\ell^2(\mathbb{Z}^n) \otimes L^2(\Omega, \mathbb{P}))$,

$$\|f\|_\mathcal{L} := \int_{\mathbb{R}^N} \left(\int_\Omega \sum_{\mathbf{k} \in \mathbb{Z}^n} |f(\mathbf{k}, \omega, x)|^2 d\mathbb{P}(\omega) \right)^{1/2} dx.$$

It is an L^1 space with values in the Hilbert space $\ell^2(\mathbb{Z}^n) \otimes L^2(\Omega, \mathbb{P})$. Note, that this Hilbert space can be decomposed as $\ell^2(\mathbb{Z}^n) \otimes L^2(\Omega, \mathbb{P}) = (\ell^2 \otimes L^2(\Omega_1, \mathbb{P}_1)) \otimes (\ell^2 \otimes L^2(\Omega_2, \mathbb{P}_2)) \otimes \ldots \otimes (\ell^2 \otimes L^2(\Omega_n, \mathbb{P}_n))$; here $\ell^2 = \ell^2(\mathbb{Z})$.

Define the vector-valued square function, taking values in the space $\ell^2(\mathbb{Z}^n) \otimes L^2(\Omega, \mathbb{P})$ by

$$\mathbf{S}f(\mathbf{k}, \omega, x) = \mathbf{S}_{\mathcal{D}(\omega)} f(\mathbf{k}, x), \qquad x \in \mathbb{R}^N, \ \mathbf{k} \in \mathbb{Z}^n, \omega \in \Omega.$$

Note, that $\mathbf{S}$ can be decomposed as a tensor product $\mathbf{S} = \mathbf{S}_1 \otimes \mathbf{S}_2 \otimes \ldots \otimes \mathbf{S}_n$, where $\mathbf{S}_k$ is the one parameter square function defined by (1.1) in variables $x_k \in X_k = \mathbb{R}^{N_k}$, $\omega_k \in \Omega_k$.

Clearly, as in the one parameter case, we have for the multi-parameter square functions $\|\mathbf{S}f\|_1 = \int_X \left[\mathbb{E}_\omega(|S_{\mathcal{D}(\omega)} f(x)|^2) \right]^{1/2} dx$.

Similarly, for the square function $\widetilde{S}$ one can construct its vector version $\widetilde{\mathbf{S}}$ with values in $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n$, $\mathcal{H}_k = \ell^2 \otimes L^2(\Omega_k, \mathbb{P}_k) \otimes \mathbb{C}^{2^{N_k}}$. Again, it is easy to see that $\|\widetilde{\mathbf{S}}f\|_{L^1(X;\mathcal{H})} = \int_X \left[\mathbb{E}_\omega(|\widetilde{S}_{\mathcal{D}(\omega)} f(x)|^2) \right]^{1/2} dx$.

As it was already discussed in Section 1.2, operators $\mathbf{S}_k$ are (Hilbert-space-valued) one-parameter Calderón–Zygmund operators, so $\mathbf{S}$ is the tensor product of such operators. And

it is probably immediately clear to experts, that such operators map $H^1(X_1 \otimes X_2 \otimes \ldots \otimes X_n)$ to L^1.

One way to see that is to notice that $\mathbf{S}$ is a trivial case of multi-parameter Calderón–Zygmund operators, and according to Theorem 2.2 in [7] such operators map $H^1(X_1 \otimes X_2 \otimes \ldots \otimes X_n)$ to L^1. Of course, one needs to use a Hilbert space valued version of the theorem, but it is clear to the specialists, that the proof from [7] works in this case. It is also clear that while Theorem 2.2 in [7] is stated for $\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$, the proof works for $X_1 \otimes X_2 \otimes \ldots \otimes X_n$, $X_k = \mathbb{R}^{N_k}$.

For the reader who is not well familiar with multi-parameter H^1 spaces we present below an alternative proof, which exploits the tensor product structure of $\mathbf{S} = \mathbf{S}_1 \otimes \mathbf{S}_2 \otimes \ldots \otimes \mathbf{S}_n$; it uses only theory of one parameter H^1-spaces. Of course, we will need the theory of H^1-spaces with values in a Hilbert space, but we need the vector valued theory in the above multi-parameter reasoning as well.

There is one more reason for the presenting the one-parameter proof below: the reasoning above gives the estimate $\int_X \left[\mathbb{E}_\omega (|S_{\mathcal{D}(\omega)} f(x)|^2) \right]^{1/2} dx \leq C \|f\|_{H^1(X)}$. The opposite estimate follows from the inequality $\|f\|_{H^1} \leq \int_X |S_{\mathcal{D}} f(x)| dx$, which is trivial in one-parameter case. In multi-parameter case, the same estimate, while true and known to specialists, requires some some work to prove it. The one-parameter approach presented below gives a reasonably simple proof of this estimate.

2.1. Multi-parameter H^1-spaces.

Recall, cf [10, III.4.4] that for for one-parameter Hardy space $H^1(\mathbb{R}^N)$ the norm $\|S^{\mathrm{L}} f\|_1$, where S^{L} is the Lusin square function (aka Lusin Area Integral)

$$S^{\mathrm{L}} f(x) = \int_{\Gamma_x} |\nabla f(y,t)|^2 t^{1-n} dy dt$$

(here $\Gamma_x := \{(y,t) : y \in \mathbb{R}^N, t \geq 0, |y - x| < t\}$, and $f(y,t)$ is the harmonic extension of f from $\mathbb{R}^N$ to $\mathbb{R}^{N+1}_+ := \mathbb{R}^N \times \mathbb{R}_+ = \{(x,t) : x \in \mathbb{R}^N, t \in \mathbb{R}_+\}$) gives an equivalent norm.

One can consider the *vectorization* $\mathbf{S}^{\mathrm{L}}$ of S^{L} as follows. Let $\Gamma = \Gamma_0$ and define

$$\mathbf{S}^{\mathrm{L}} f(x,y,t) = t^{(1-n)/2} \nabla f(x+y,t), \qquad x \in \mathbb{R}^N, (y,t) \in \Gamma.$$

By construction $\mathbf{S}^{\mathrm{L}} f(x, \cdot, \cdot) \in L^2(\Gamma) \otimes \mathbb{C}^2$ and $\|\mathbf{S}^{\mathrm{L}} f(x, \cdot, \cdot)\|_{L^2(\Gamma) \otimes \mathbb{C}^2} = S^{\mathrm{L}} f(x)$, therefore $\|\mathbf{S}^{\mathrm{L}} f\|_{L^1(L^2(\Gamma))} = \|S^{\mathrm{L}} f\|_1$.

One can define multi-parameter square functions $\vec{S}^{\mathrm{L}}$ and $\vec{\mathbf{S}}^{\mathrm{L}}$ by

$$\vec{S}^{\mathrm{L}} f(x_1, x_2, \ldots, x_n) :=$$

$$\left[\int_{\Gamma_{x_1} \times \Gamma_{x_2} \times \ldots \times \Gamma_{x_n}} |\nabla_1 \nabla_2 \ldots \nabla_n f(y_1, t_1, y_2, t_2, \ldots, y_n, t_n)|^2 \times \right.$$

$$\left. \times t_1^{1-N_1} t_2^{(1-N_2)} \ldots t_n^{1-N_n} dy_1 dt_1 dy_2 dt_2 \ldots dy_n dt_n \right]^{1/2};$$

here $f(y_1, t_1, y_2, t_2, \ldots, y_n, t_n)$ is the harmonic in each variable (y_k, t_k), $y_k \in \mathbb{R}^{N_k}$, $t_k \in \mathbb{R}_+$ extension of f from $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \ldots \times \mathbb{R}^{N_n}$ to $\mathbb{R}^{N_1+1}_+ \times \mathbb{R}^{N_2+1}_+ \times \ldots \times \mathbb{R}^{N_n+1}_+$ and ∇_k is the gradient in the variable (y_k, t_k).

Following [1] we say that $f \in H^1(X) = H^1(X_1 \otimes X_2 \otimes \ldots \otimes X_n)$ if $\vec{S}^{\mathrm{L}} f \in L^1(X)$ and $\|\vec{S}^{\mathrm{L}} f\|_1$ defines one of the possible equivalent norms in $H^1(X)$.

We also define the vector-valued linear square function $\vec{\mathbf{S}}^{\mathrm{L}}$ as $\vec{\mathbf{S}} f(x) = \mathbf{S}_1^{\mathrm{L}} \otimes \mathbf{S}_2^{\mathrm{L}} \otimes \ldots \otimes$
$\mathbf{S}_n^{\mathrm{L}} f(x) \in \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n$, where $\mathbf{S}_k^{\mathrm{L}}$ it one-parameter square function defined above
taken in the variable x_k, $\mathcal{H}_k = L^2(\Gamma_k) \otimes \mathbb{C}^2$ and Γ_k is the cone in $\mathbb{R}_+^{N_k+1}$ with the vertex at 0.

Again, by the construction $\|\vec{\mathbf{S}}^{\mathrm{L}} f(x)\|_{\mathcal{H}} = |\vec{S}^{\mathrm{L}} f(x)|$, so $\|\vec{\mathbf{S}}^{\mathrm{L}} f\|_{L^1(X;\mathcal{H})}$ gives the norm in
$H^1(X)$.

We will use $\vec{\mathbf{S}}^{\mathrm{L}}$ to define multi-parameter H^1 with values in a Hilbert space $\mathcal{K}$; in this case
$\vec{\mathbf{S}}^{\mathrm{L}} f(x) \in \mathcal{H} \times \mathcal{K}$. We introduced such spaces only for notational purposes, so while most of
the theory of multi-parameter H^1-spaces can be transfered to the Hilbert-space-valued case,
we do not need this.

2.2. Proof of estimate $\int_X \left[\mathbb{E}_\omega(|S_{\mathcal{D}(\omega)} f(x)|^2) \right]^{1/2} dx \le C \|f\|_{H^1(X)}$.
Consider a multi-param-
eter square function $\widetilde{\mathbf{S}}_2 \otimes \ldots \otimes \widetilde{\mathbf{S}}_n$, where each $\widetilde{\mathbf{S}}_k$, $2 \le k \le n$ is either one-parameter $\mathbf{S}^{\mathrm{L}}$ or
one-parameter "random" square function $\mathbf{S}$, defined in (1.1), taken in the variable x_k.

Assume that the choice of $\widetilde{\mathbf{S}}_k$ is fixed. For a scalar-valued f the function $\widetilde{\mathbf{S}}_2 \otimes \ldots \otimes \widetilde{\mathbf{S}}_n f$
takes values in $\widetilde{\mathcal{H}}^1 := \widetilde{\mathcal{H}}_2 \otimes \ldots \otimes \widetilde{\mathcal{H}}_n$, where each $\widetilde{\mathcal{H}}_k$ is either $L^2(\Gamma_k) \otimes \mathbb{C}^2$ or $\ell^2 \otimes L^2(\Omega_k, \mathbb{P}_k)$,
depending on what square function $\widetilde{\mathbf{S}}_k$ is.

Let $\mathbf{S}_k^{\mathrm{L}}$, $\mathbf{S}_k$ be the Lusin and "random" square functions, taken in the variable x_k, and let
$\mathcal{H}_k := L^2(\Gamma_k) \otimes \mathbb{C}^2$ and $\mathcal{H}_k' := \ell^2 \otimes L^2(\Omega_k, \mathbb{P}_k)$ be the corresponding target spaces.

Lemma 2.1.

$$\int_X \|\mathbf{S}_1 \otimes \widetilde{\mathbf{S}}_2 \otimes \ldots \otimes \widetilde{\mathbf{S}}_n f(x)\|_{\mathcal{H}_1' \otimes \widetilde{\mathcal{H}}^1} dx \le C \int_X \|\mathbf{S}_1^{\mathrm{L}} \otimes \widetilde{\mathbf{S}}_2 \otimes \ldots \otimes \widetilde{\mathbf{S}}_n f(x)\|_{\mathcal{H}_1 \otimes \widetilde{\mathcal{H}}^1} dx$$

Since the tensor products of square functions we consider does not depend on the order
(the square functions, taken in different variables obviously commute), the above lemma tells
us that one can replace a factor $\mathbf{S}_k^{\mathrm{L}}$ by $\mathbf{S}_k$ in $\widetilde{\mathbf{S}}_1 \otimes \widetilde{\mathbf{S}}_2 \otimes \ldots \otimes \widetilde{\mathbf{S}}_n f$ and increase the norm by
at most the factor C.

Starting with $\mathbf{S}_1^{\mathrm{L}} \otimes \mathbf{S}_2^{\mathrm{L}} \otimes \ldots \otimes \mathbf{S}_n^{\mathrm{L}} f$ and applying Lemma 2.1 successively to each factor, we
get

$$\int_X \|\mathbf{S}_1 \otimes \mathbf{S}_2 \otimes \ldots \otimes \mathbf{S}_n f(x)\|_{\mathcal{H}'} dx \le C \int_X \|\mathbf{S}_1^{\mathrm{L}} \otimes \mathbf{S}_2^{\mathrm{L}} \otimes \ldots \otimes \mathbf{S}_n^{\mathrm{L}} f(x)\|_{\mathcal{H}} dx,$$

which is exactly the desired estimate (here $\mathcal{H}' = \mathcal{H}_1' \otimes \mathcal{H}_2' \otimes \ldots \otimes \mathcal{H}_n'$ and $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n$).

Proof of Lemma 2.1. Let us introduce notation $x = (x_1, x^1) \in X = X_1 \times X^1$, where $x^1 = (x_2, x_3, \ldots, x_n) \in X^1 := X_2 \times \ldots \times X_n$.

Consider the vector-valued function

$$\mathbf{S}_1^{\mathrm{L}} \otimes \widetilde{\mathbf{S}}_2 \otimes \ldots \otimes \widetilde{\mathbf{S}}_n f = \mathbf{S}_1^{\mathrm{L}} \otimes (\widetilde{\mathbf{S}}_2 \otimes \ldots \otimes \widetilde{\mathbf{S}}_n) f.$$

If

$$\int_X \|\mathbf{S}_1^{\mathrm{L}} \otimes (\widetilde{\mathbf{S}}_2 \otimes \ldots \otimes \widetilde{\mathbf{S}}_n) f(x)\|_{\mathcal{H}_1' \otimes \widetilde{\mathcal{H}}^1} dx < \infty,$$

we conclude that for almost all x^1

$$(2.1) \qquad \widetilde{\mathbf{S}}_2 \otimes \ldots \otimes \widetilde{\mathbf{S}}_n f(\cdot, x^1) \in H^1(X_1; \widetilde{\mathcal{H}}^1),$$

and

$$(2.2) \qquad \int_X \|\mathbf{S}_1^{\mathrm{L}} \otimes (\widetilde{\mathbf{S}}_2 \otimes \ldots \otimes \widetilde{\mathbf{S}}_n) f(x)\|_{\mathcal{H}_1 \otimes \widetilde{\mathcal{H}}^1} dx$$
$$= \int_{X^1} \|\widetilde{\mathbf{S}}_2 \otimes \ldots \otimes \widetilde{\mathbf{S}}_n f(\,\cdot\,, x^1)\|_{H^1(X_1; \widetilde{\mathcal{H}}^1)} dx^1.$$

Note, that we have in (2.1) the usual one-parameter H^1-space (although vector-valued).

As we discussed above in Section 1.2, $\mathbf{S}_1$ is a vector-valued Calderón–Zygmund operator, so it maps one-parameter H^1 to L^1 (even in the Hilbert-space-valued case), so for almost all x^1 we have

$$(2.3) \qquad \int_{X_1} \|\mathbf{S}_1 \otimes (\widetilde{\mathbf{S}}_2 \otimes \ldots \otimes \widetilde{\mathbf{S}}_n) f(x_1, x^1)\|_{\mathcal{H}_1' \otimes \widetilde{\mathcal{H}}^1} dx_1$$
$$\le C \|\widetilde{\mathbf{S}}_2 \otimes \ldots \otimes \widetilde{\mathbf{S}}_n f(\,\cdot\,, x^1)\|_{H^1(X_1; \widetilde{\mathcal{H}}^1)}$$
$$= C \int_{X_1} \|\mathbf{S}_1^{\mathrm{L}} \otimes (\widetilde{\mathbf{S}}_2 \otimes \ldots \otimes \widetilde{\mathbf{S}}_n) f(x_1, x^1)\|_{\mathcal{H}_1 \otimes \widetilde{\mathcal{H}}^1} dx_1$$

Integrating over X^1 and taking into account (2.2), we get the conclusion of the lemma. $\square$

2.3. **Estimate** $\|f\|_{H^1(X)} \le C \int_X |S_{\mathcal{D}} f(x)| dx$**.** Proof of this estimate follows the lines of Section 2.2 almost word by word. It is based on the following analogue of Lemma 2.1, which allows us replace one-parameter Lusin square functions by the dyadic ones.

Consider again a multi-parameter square function $\widetilde{\mathbf{S}}_2 \otimes \ldots \otimes \widetilde{\mathbf{S}}_n$, where now each $\widetilde{\mathbf{S}}_k$, $2 \le k \le n$ is either one-parameter $\mathbf{S}^{\mathrm{L}}$ or one-parameter dyadic square function $\mathbf{S}_{\mathcal{D}_k}$, defined in (1.1), taken in the variable x_k. We assume here that in each X_k dyadic lattices $\mathcal{D}_k$ are fixed.

We will use the same notation as in Section 2.2, with the only exception that now $\mathcal{H}_k' = \ell^2$

Lemma 2.2.
$$\int_X \|\mathbf{S}_1^{\mathrm{L}} \otimes \widetilde{\mathbf{S}}_2 \otimes \ldots \otimes \widetilde{\mathbf{S}}_n f(x)\|_{\mathcal{H}_1 \otimes \widetilde{\mathcal{H}}^1} dx \le C \int_X \|\mathbf{S}_{\mathcal{D}_1} \otimes \widetilde{\mathbf{S}}_2 \otimes \ldots \otimes \widetilde{\mathbf{S}}_n f(x)\|_{\mathcal{H}_1' \otimes \widetilde{\mathcal{H}}^1} dx.$$

Applying Lemma 2.1 successively to each variable, as we did in Section 2.2, we get

$$\int_X \|\mathbf{S}_1^{\mathrm{L}} \otimes \mathbf{S}_2^{\mathrm{L}} \otimes \ldots \otimes \mathbf{S}_n^{\mathrm{L}} f(x)\|_{\mathcal{H}} dx \le C \int_X \|\mathbf{S}_{\mathcal{D}_1} \otimes \mathbf{S}_{\mathcal{D}_2} \otimes \ldots \otimes \mathbf{S}_{\mathcal{D}_n} f(x)\|_{\mathcal{H}'} dx,$$

which is exactly what we need.

Proof of Lemma 2.2. Similarly to (2.2) we get

$$(2.4) \qquad \int_X \|\mathbf{S}_{\mathcal{D}_1} \otimes (\widetilde{\mathbf{S}}_2 \otimes \ldots \otimes \widetilde{\mathbf{S}}_n) f(x)\|_{\mathcal{H}_1 \otimes \widetilde{\mathcal{H}}^1} dx$$
$$= \int_{X^1} \|\widetilde{\mathbf{S}}_2 \otimes \ldots \otimes \widetilde{\mathbf{S}}_n f(\,\cdot\,, x^1)\|_{H_{\mathcal{D}_1}^1(X_1; \widetilde{\mathcal{H}}^1)} dx^1.$$

We will now use the fact that for Hilbert-space-valued functions $\|\varphi\|_{H^1} \le C \|\varphi\|_{H_{\mathcal{D}}^1}$. Again, as in the scalar-valued case, it follows from H^1-BMO duality $((H^1)^* = \mathrm{BMO}, (H_{\mathcal{D}}^1)^* = \mathrm{BMO}_{\mathcal{D}})$ and the trivial inclusion $\mathrm{BMO} \subset \mathrm{BMO}_{\mathcal{D}}$, which imply the inclusion $H_{\mathcal{D}}^1 \subset H^1$ with the desired estimates of the norms.

Using this inequality we get that for almost all x^1

$$\|\widetilde{\mathbf{S}}_2 \otimes \ldots \otimes \widetilde{\mathbf{S}}_n f(\,\cdot\,, x^1)\|_{H^1(X_1; \widetilde{\mathcal{H}}^1)} \le C \|\widetilde{\mathbf{S}}_2 \otimes \ldots \otimes \widetilde{\mathbf{S}}_n f(\,\cdot\,, x^1)\|_{H_{\mathrm{d}}^1(X_1; \widetilde{\mathcal{H}}^1)}.$$

Integrating over X^1 and using (2.2), (2.4) we get the conclusion of the lemma. $\qquad\square$

3. Appendix: some facts about H^1 and BMO spaces.

3.1. Hilbert-space-valued BMO spaces.
Let us recall that a function on $X = \mathbb{R}^N$ with values in a Hilbert space $\mathcal{H}$ belongs to the space $\text{BMO} = \text{BMO}(X, \mathcal{H})$ if

$$(3.1) \qquad \|f\|_{\text{BMO}} := \sup_Q \int_Q \|f(x) - f_Q\|_{\mathcal{H}} dx < \infty;$$

here $f_Q := |Q|^{-1} \int_Q f(x) dx$ and the supremum is taken over all cubes $Q \subset \mathbb{R}^N$.

If we fix a dyadic lattice $\mathcal{D}$ and take the supremum in(3.1) only over *dyadic* cubes $Q \in \mathcal{D}$, we get the dyadic space $\text{BMO}_{\mathcal{D}}$ associated with this lattice.

It is well known that $(H^1(\mathbb{R}^N; \mathcal{H}))^* = \text{BMO}(\mathbb{R}^N; \mathcal{H})$ and $(H^1_{\mathcal{D}}(\mathbb{R}^N; \mathcal{H}))^* = \text{BMO}_{\mathcal{D}}(\mathbb{R}^N; \mathcal{H})$; any standard proof of H^1-BMO duality would work for the Hilbert-space-valued functions.

3.2. Triebel–Lizorkin spaces $\dot{\mathbf{f}}_p^{\alpha,q}$ and equivalence of $\|Sf\|_1$ and $\|\widetilde{S}f\|_1$.
In this section we fix a dyadic lattice $\mathcal{D}$, for example take for $\mathcal{D}$ the standard dyadic lattice.

3.2.1. Spaces $\dot{\mathbf{f}}_p^{\alpha,q}$.
Following [3] define the space $\dot{\mathbf{f}}_p^{\alpha,q}$, ($\alpha \in \mathbb{R}$, $1 \le p, p < \infty$, consisting of sequences $s = \{s_Q\}_{Q \in \mathcal{D}}$ such that

$$\|s\|_{\dot{\mathbf{f}}_p^{\alpha,q}} := \left\| \left(\sum_{Q \in \mathcal{D}} (|Q|^{-\alpha/n} |s_Q| \cdot |Q|^{-1/2} \mathbf{1}_Q)^q \right)^{1/q} \right\|_{L^p} < \infty$$

For $p = \infty$ the norm is defined using BMO-like norm

$$\|s\|_{\dot{\mathbf{f}}_\infty^{\alpha,q}} := \sup_{P \in \mathcal{D}} \left(\frac{1}{|P|} \int_P \sum_{Q \in \mathcal{D}, Q \subset P} (|Q|^{-\alpha/n} |s_Q| \cdot |Q|^{-1/2} \mathbf{1}_Q)^q \right)^{1/q}$$

We are interested in the case when the smoothness parameter $\alpha = 0$; to simplify the notation in this case we will use $\dot{\mathbf{f}}_p^q := \dot{\mathbf{f}}_p^{0,q}$.

We will need the following facts about duality for spaces $\dot{\mathbf{f}}_p^{\alpha,q}$:

$$(\dot{\mathbf{f}}_p^{\alpha,q})^* = \dot{\mathbf{f}}_{p'}^{-\alpha,q'}, \qquad 1 \le p, q < \infty;$$

here $1/p + 1/p' = 1$, $1/q + 1/q' = 1$.

3.2.2. Almost diagonal operators.
Following [3] we say that an operator A with matrix $\{a_{Q,P}\}_{Q,P \in \mathcal{D}}$ is *almost diagonal* (for spaces $\dot{\mathbf{f}}_p^q = \dot{\mathbf{f}}_p^{0,q}$) if there exists $\varepsilon > 0$ and $C < \infty$, such that

$$|a_{Q,P}| \le C \left(1 + \frac{|x_P - x_Q|}{\max\{\ell(P), \ell(Q)\}} \right)^{-N-\varepsilon} \times \min \left[\left(\frac{\ell(Q)}{\ell(P)} \right)^{(N+\varepsilon)/2}, \left(\frac{\ell(P)}{\ell(Q)} \right)^{(N+\varepsilon)/2} \right]$$

(the definition is a bit more complicated for $\dot{\mathbf{f}}_p^{s,q}$ with $s \ne 0$)

It was shown it [3] that an almost diagonal operator is bounded in all $\dot{\mathbf{f}}_p^q$ spaces, $1 \le, q, p < \infty$.

3.2.3. *Equivalence of $\|Sf\|_1$ and $\|\widetilde{S}f\|_1$.* From the above result one can easily obtain the equivalence of $\|Sf\|_1$ and $\|\widetilde{S}f\|_1$. First, since $\max_Q |\Delta_Q f|^2 \leq 2^N E_Q(|\Delta_Q f|^2)$ we have pointwise estimate $Sf(x) \leq 2^{N/2}\widetilde{S}f(x)$ and so $\|Sf\|_1 \leq 2^{N/2}\|\widetilde{S}f\|_1$.

To get the estimate $\|\widetilde{S}f\|_1 \leq C\|Sf\|_1$ let us express the conditions $Sf \in L^1$, $\widetilde{S}f \in L^1$ in terms of Triebel-Lizorkin space $\dot{\mathbf{f}}_1^2$. Namely, with each function $f \in L^1_{\mathrm{loc}}$ let us associate 2 sequences $a = \{a_Q\}_{Q \in \mathcal{D}}$ and $b = \{b_Q\}_{Q \in \mathcal{D}}$

$$a_Q = \left[\left(E_Q |\Delta_Q f|^2\right)(x)\right]^{1/2}, \qquad b_Q = (\Delta_R f)(x),$$

where R is the "parent" of Q and x is an arbitrary point in Q (the result does not depend on x). Then clearly

$$\|\widetilde{S}f\|_1 = \|a\|_{\dot{\mathbf{f}}_1^2}, \qquad \|Sf\|_1 = \|b\|_{\dot{\mathbf{f}}_1^2}.$$

Note that

$$a_R = \left(2^{-N} \sum_{Q \text{ is child of } R} |b_Q|^2\right)^{1/2} \leq 2^{-N/2} \sum_{Q \text{ is child of } R} |b_Q| =: T|b|,$$

where $|b| := \{|b_Q|\}_{Q \in \mathcal{D}}$ and for $s = \{s_Q\}_{Q \in \mathcal{D}}$

$$(Ts)_R := 2^{-N/2} \sum_{Q \text{ is child of } R} s_Q.$$

The operator T is almost diagonal (it has only finitely many "diagonals"), so

$$\|\widetilde{S}f\|_1 = \|a\|_{\dot{\mathbf{f}}_1^2} \leq \|T|b|\|_{\dot{\mathbf{f}}_1^2} \leq C\| |b| \|_{\dot{\mathbf{f}}_1^2} = C\|b\|_{\dot{\mathbf{f}}_1^2} = C\|Sf\|_1.$$

$\square$

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