A PROBLEM ON COMPLETENESS OF EXPONENTIALS

A. POLTORATSKI

ABSTRACT. Let μ be a finite positive measure on the real line. For $a > 0$ denote by $\mathcal{E}_a$ the family of exponential functions

$$\mathcal{E}_a = \{e^{ist}|\ s \in [0, a]\}.$$

The exponential type of μ is the infimum of all numbers a such that the finite linear combinations of the exponentials from $\mathcal{E}_a$ are dense in $L^2(\mu)$. If the set of such a is empty, the exponential type of μ is defined as infinity. The well-known type problem asks to find the exponential type of μ in terms of μ.

In this note we present a solution to the type problem and discuss its relations with known results.

1. Introduction

1.1. **Completeness of exponentials.** Let μ be a finite positive Borel measure on $\mathbb{R}$. Let us consider the family $\mathcal{E}_\Lambda$ of exponential functions $\exp(i\lambda t)$ on $\mathbb{R}$ whose frequencies λ belong to a certain set $\Lambda \subset \mathbb{C}$:

$$\mathcal{E}_\Lambda = \{\exp(i\lambda t)|\ \lambda \in \Lambda\}.$$

One of the classical problems of Harmonic analysis is to find conditions on μ and Λ that ensure completeness, i.e. density of finite linear combinations, of functions from $\mathcal{E}_\Lambda$ in $L^2(\mu)$.

Versions of this problem were considered by many prominent analysts. The case when Λ is a sequence and μ is Lebesgue measure on an interval was solved by Beurling and Malliavin in the early sixties [4, 5]. The so-called Beurling-Malliavin theory, created to treat that problem, is considered to be one of the deepest parts of the 20th century Harmonic Analysis.

Other cases of the problem and its multiple reformulations were studied by Wiener, Levinson, Kolmogorov, Krein and many others. Such an extensive interest is largely due to the fact that it is naturally related to other fields of classical analysis, such as stationary Gaussian processes and prediction theory, spectral problems for differential operators, approximation theory, signal processing, etc. Despite considerable efforts by the analytic community many important cases of the problem remain open.

1.2. **The type problem.** One of such open cases is the so-called type problem. Consider a family $\mathcal{E}_a = \mathcal{E}_{[0,a]}$ of exponential functions whose frequencies belong to the interval from 0 to a. If μ is a finite positive measure on $\mathbb{R}$ we denote by $\mathbf{G}^2_\mu$ its exponential type that is defined as

$$\mathbf{G}^2_\mu = \inf\{\ a > 0 \mid \mathcal{E}_a \text{ is complete in } L^2(\mu)\ \} \tag{1.1}$$

if the set of such a is non-empty and as infinity otherwise. The type problem asks to calculate $\mathbf{G}^2_\mu$ in terms of μ.

This question first appears in the work of Wiener, Kolmogorov and Krein in the context of stationary Gaussian processes (see [17, 18] or the book by Dym and McKean [11]). If μ is a spectral measure of a stationary Gaussian process, the property that $\mathcal{E}_a$ is complete in $L^2(\mu)$ is equivalent to the property that the process at any time can be predicted from the data for the time period from 0 to a. Since any even measure is a spectral measure of a stationary Gaussian process, and vice versa, this reformulation is nearly equivalent.

The type problem can also be restated in terms of the Bernstein weighted approximation, see for instance Koosis [15]. Important connections with spectral theory of second order differential operators were studied by Gelfand and Levitan [12] and Krein [18, 19].

For more on the history and connections of the type problem see, for instance, a note by Dym [10] or a recent paper by Borichev and Sodin [6].

1.3. **General case** $p \neq 2$. The family $\mathcal{E}_a$ is incomplete in $L^2(\mu)$ if and only if there exists a function $f \in L^2(\mu)$ orthogonal to all elements of $\mathcal{E}_a$. Expanding to other $1 \leqslant p \leqslant \infty$ we define

$$\mathbf{G}^p_\mu = \sup\{\ a \mid \exists\ f \in L^p(\mu), \int f(x)e^{i\lambda x}d\mu(x) = 0, \forall\ \lambda \in [0,a]\ \}. \tag{1.2}$$

We put $\mathbf{G}^p_\mu = 0$ if the set in (1.2) is empty. By duality, for $1 < p \leqslant \infty$, $\mathbf{G}^p_\mu$ can still be defined as the infimum of a such that $\mathcal{E}_a$ is complete in $L^q(\mu)$, $\frac{1}{p} + \frac{1}{q} = 1$. Cases $p \neq 2$ were considered in several papers, see for instance articles by Koosis [16] or Levin [21] for the case $p = \infty$ or [29] for $p = 1$.

Since μ is a finite measure we have

$$\mathbf{G}^p_\mu \leqslant \mathbf{G}^q_\mu \text{ for } p \geqslant q. \tag{1.3}$$

Apart from this obvious observation, the problems of finding $\mathbf{G}^p_\mu$ for different p's were generally considered non-equivalent. One of the consequences of theorem 2, section 3.1, is that, in some sense, there are only two significantly different cases, $p = 1$ (the gap problem) and $1 < p \leqslant \infty$ (the general type problem).

In this paper we restrict our attention to the class of finite measures. The formal reason for that is the fact that μ has to be finite for exponentials to belong to $L^2(\mu)$. This obstacle can be easily overcome if instead of $\mathcal{E}_a$ one considers E_a, the set of Fourier transforms of smooth functions supported on $[0, a]$. All elements of E_a decay fast at infinity and one one can ask about the density of E_a in $L^p(\mu)$ for wider sets of μ, see for instance [6]. One of such traditional sets is the class of Poisson-finite measures satisfying

$$\int \frac{d|\mu|(x)}{1 + x^2} < \infty.$$

However, due to the reasons similar to lemma 1 below, considering such a wider set of measures will not change the problem and most of the statements will remain the same or analogous. Note that if μ is Poisson-finite then $\mu/(1 + x^2)$ is finite and vice versa.

1.4. **The gap problem.** One of the important cases is the so-called gap problem, $p = 1$. Here one can reformulate the question as follows.

Let X be a closed subset of the real line. Denote

$$\mathbf{G}_X = \sup\{\ a \mid \exists \mu \neq 0,\ \operatorname{supp}\mu \subset X, \hat{\mu} = 0 \text{ on } [0, a]\ \}.$$

Here and in the rest of the paper $\hat{\mu}$ denotes the (inverse) Fourier transform of a finite measure μ on $\mathbb{R}$:

$$\hat{\mu}(z) = \int_{\mathbb{R}} e^{izt} d\mu(t).$$

As was shown in [29], for any finite measure μ on $\mathbb{R}$, $\mathbf{G}_\mu^1$, as defined in the previous section, depends only on its support:

$$\mathbf{G}_\mu^1 = \mathbf{G}_{\operatorname{supp}\mu}.$$

This property separates the gap problem from all the cases $p > 1$.

For a long time both the gap problem and the type problem were considered by experts to be "transcendental," i.e. not having a closed form solution. Following an approach developed in [23] and [24], a solution to the gap problem was recently suggested in [29], see section 2.3. Some of definitions and results from [29] are used in the present paper.

1.5. **Known examples.** We say that a function f on $\mathbb{R}$ is Poisson-summable if it is summable with respect to the Poisson measure Π,

$$d\Pi = dx/(1 + x^2).$$

We say that a sequence of real numbers $A = \{a_n\}$ is discrete if it does not have finite accumulation points. We always assume that a discrete sequence is enumerated in the natural increasing order: $a_n \geqslant a_{n-1}$. Since the sequences considered here have $\pm\infty$ as their density points, the indices run over $\mathbb{Z}$. In most of our statements and definitions, the sequences do not

have multiple points. We call a discrete sequence $\{a_n\} \subset \mathbb{R}$ separated if $|a_n - a_k| > c$ for some $c > 0$ and any $n \neq k$.

A classical result by Krein [17] says that if $d\mu = w(x)dx$ and $\log w$ is Poisson-summable then $\mathbf{G}_\mu^p = \infty$ for all p. A partial inverse, proved by Levinson and McKean, holds for even monotone w, see section 4.5.

A theorem by Duffin and Schaeffer [9] implies that if μ is a measure such that for any $x \in \mathbb{R}$

$$\mu([x - L, x + L]) > d$$

for some $L, d > 0$ then $\mathbf{G}_\mu^2 \geqslant 2\pi/L$, see section 4.7.

For discrete measures, in the case $\operatorname{supp} \mu = \mathbb{Z}$, a deep result by Koosis shows an analogue of Krein's result: if $\mu = \sum w(n)\delta_n$ where $\sum \frac{\log w(n)}{1+n^2} > -\infty$ then $\mathbf{G}_\mu^p = 2\pi$ for all p [16]. Not much was known about supports other than $\mathbb{Z}$ besides a recent result from [25], which implies that if

$$\mu = \sum \frac{\delta_{a_n}}{1 + a_n^2}$$

for a separated sequence $A = \{a_n\} \subset \mathbb{R}$ then $\mathbf{G}_\mu^p = 2\pi D_*(A)$, where D_* is the interior Beurling-Malliavin density of A, see section 2.3 for the definition. We generalize these results in sections 3.2 and 3.4.

In addition to these few examples, classical theorems by Levinson-McKean, Beurling and de Branges show that if a measure has long gaps in its support or decays too fast, then $\mathbf{G}_\mu^p = 0$, see section 4. Examples of measures of positive type can be constructed using the results by Benedicks [2], see section 4.8. The most significant recent development, that allows one to modify existing examples, is the result by Borichev and Sodin [6], which says that "exponentially small" changes in weight or support do not change the type of a measure, see section 4.6.

1.6. **Approach and goals of the paper.** The problems discussed above belong to the area often called the Uncertainty Principle in Harmonic Analysis [13]. A new approach developed by N. Makarov and the author in [23, 24] allows one to study this area with modern tools of analytic function theory and singular integrals. Together with traditional methods, such as de Branges' theory of Hilbert spaces of entire functions or the Beurling-Malliavin results, these techniques have produced some new ideas and developments. Among them is an extension of the Beurling-Malliavin theory [24], a solution to the Pólya-Levinson problem on sampling sets for entire functions of zero type [25] and a solution to the gap problem [29]. In the present paper we continue to apply the same approach.

We focus on the type problem, the problem of finding $\mathbf{G}_\mu^2$ in terms of μ. Our main results are theorem 2 and its corollaries contained in section 3. In most of our statements, treating $p > 1$, $p \neq 2$ did not require any additional efforts, and hence they were formulated for general $p > 1$. The case $p = 1$,

studied in [29], provided us with some useful definitions and statements, see section 2.3.

1.7. **Contents.** The paper is organized as follows:

- Section 2 contains preliminary discussions including the basics of the so-called Clark theory, definitions of Beurling-Malliavin densities and a further discussion of the gap problem.
- In section 3 we state the main results of the paper.
- Section 4 discusses connections of our results with classical theorems by Duffin and Schaeffer, Beurling, de Branges, Krein, Levinson and McKean as well as more recent results by Benedicks, Borichev and Sodin.
- Section 5 contains several lemmas needed for the main proofs.
- In section 6 we give the proofs of the main results.

2. Preliminaries

2.1. **Clark theory.** By H^2 we denote the Hardy space in the upper half-plane $\mathbb{C}_+$. We say that an inner function $\theta(z)$ in $\mathbb{C}_+$ is meromorphic if it allows a meromorphic extension to the whole complex plane. The meromorphic extension to the lower half-plane $\mathbb{C}_-$ is given by

$$\theta(z) = \frac{1}{\theta^{\#}(z)}$$

where $\theta^{\#}(z) = \bar{\theta}(\bar{z})$.

Each inner function $\theta(z)$ determines a model subspace

$$K_\theta = H^2 \ominus \theta H^2$$

of the Hardy space $H^2(\mathbb{C}_+)$. These subspaces play an important role in complex and harmonic analysis, as well as in operator theory, see [27].

For each inner function $\theta(z)$ one can consider a positive harmonic function

$$\Re\frac{1 + \theta(z)}{1 - \theta(z)}$$

and, by the Herglotz representation, a positive measure μ such that

$$\Re\frac{1 + \theta(z)}{1 - \theta(z)} = py + \frac{1}{\pi} \int \frac{y d\mu(t)}{(x - t)^2 + y^2}, \qquad z = x + iy, \qquad (2.1)$$

for some $p \geqslant 0$. The number p can be viewed as a point mass at infinity. The measure μ is Poisson-finite, singular and supported on the set where non-tangential limits of θ are equal to 1. The measure $\mu + p\delta_\infty$ on $\hat{\mathbb{R}}$ is called the Clark measure for $\theta(z)$.

Following standard notations, we will sometimes denote the Clark measure defined in (2.1) by μ_1. More generally, if $\alpha \in \mathbb{C}, |\alpha| = 1$ then μ_α is the measure defined by (2.1) with θ replaced by $\bar{\alpha}\theta$.

Conversely, for every positive singular Poisson-finite measure μ and a number $p \geqslant 0$, there exists an inner function $\theta(z)$ satisfying (2.1).

Every function $f \in K_\theta$ can be represented by the formula

$$f(z) = \frac{p}{2\pi i}(1 - \theta(z)) \int f(t)\overline{(1 - \theta(t))}dt + \frac{1 - \theta(z)}{2\pi i} \int \frac{f(t)}{t - z}d\mu(t). \quad (2.2)$$

If the Clark measure does not have a point mass at infinity, the formula is simplified to

$$f(z) = \frac{1}{2\pi i}(1 - \theta(z))Kf\mu$$

where $Kf\mu$ stands for the Cauchy integral

$$Kf\mu(z) = \int \frac{f(t)}{t - z}d\mu(t).$$

This gives an isometry of $L^2(\mu)$ onto K_θ. Similar formulas can be written for any μ_α corresponding to θ.

In the case of meromorphic $\theta(z)$, every function $f \in K_\theta$ also has a meromorphic extension in $\mathbb{C}$, and it is given by the formula (2.2). The corresponding Clark measure is discrete with atoms at the points of $\{\theta = 1\}$ given by

$$\mu(\{x\}) = \frac{2\pi}{|\theta'(x)|}.$$

If $\Lambda \subset \mathbb{R}$ is a given discrete sequence, one can easily construct a meromorphic inner function θ satisfying $\{\theta = 1\} = \Lambda$ by considering a positive Poisson-finite measure concentrated on Λ and then choosing θ to satisfy (2.1). One can prescribe the derivatives of θ at Λ with a proper choice of pointmasses.

For more details on Clark measures and further references the reader may consult [30].

2.2. Interior and exterior densities.
A sequence of disjoint intervals $\{I_n\}$ on the real line is called *long* (in the sense of Beurling and Malliavin) if

$$\sum_n \frac{|I_n|^2}{1 + \text{dist}^2(0, I_n)} = \infty, \quad (2.3)$$

where $|I_n|$ stands for the length of I_n. If the sum is finite, we call $\{I_n\}$ *short*.

One of the obvious properties of short sequences is that $|I_n| = o(\text{dist}(0, I_n))$ as $n \to \infty$. In particular, $\text{dist}(0, I_n)$ can be replaced with any $x_n \in I_n$ in the above definition.

Following [5] we say that a discrete sequence $\Lambda \subset \mathbb{R}$ is *a-regular* if for every $\epsilon > 0$ any sequence of disjoint intervals $\{I_n\}$ that satisfies

$$\left| \frac{\#(\Lambda \cap I_n)}{|I_n|} - a \right| \geqslant \epsilon$$

for all n, is short.

A slightly different *a*-regularity can be defined in the following way, that is more convenient in some settings. For a discrete sequence $\Lambda \subset \mathbb{R}$ we denote by $n_\Lambda(x)$ its counting function, i.e. the step function on $\mathbb{R}$, that is constant between any two points of Λ, jumps up by 1 at each point of Λ and is equal to 0 at 0. We say that Λ is *strongly a-regular* if

$$\int \frac{|n_\Lambda(x) - ax|}{1 + x^2} < \infty.$$

Conditions like this can be found in many related results, see for instance [7] or [15]. Even though *a*-regularity is not equivalent to strong *a*-regularity, in the following definitions of densities changing "*a*-regular" to "strongly *a*-regular" will lead to equivalent definitions.

The interior BM (Beurling-Malliavin) density is defined as

$$D_*(\Lambda) := \sup\{ \ a \mid \exists \ a\text{-regular subsequence} \ \ \Lambda' \subset \Lambda \ \}. \qquad (2.4)$$

If the set is empty we put $D_*(\Lambda) = 0$. Similarly, the exterior BM density is defined as

$$D^*(\Lambda) := \inf\{ \ a \mid \exists \ a\text{-regular supsequence} \ \ \Lambda' \supset \Lambda \ \}. \qquad (2.5)$$

If no such sequence exists, $D^*(\Lambda) = \infty$.

It is interesting to observe that after the two densities were simultaneously introduced over fifty years ago, the exterior density immediately became one of the staples of harmonic analysis and spectral theory, mostly due to its appearance in the celebrated Beurling-Malliavin theorem, see [5], [13] or [15]. Meanwhile, the interior density remained largely forgotten until its recent comeback in [25] and [29]. It will continue to play an important role in our discussions below.

2.3. The gap problem and d-uniform sequences.

Let $\Lambda = \{\lambda_1, ..., \lambda_n\}$ be a finite set of points on $\mathbb{R}$. Define

$$E(\Lambda) = \sum_{\lambda_k, \lambda_j \in \Lambda} \log |\lambda_k - \lambda_l|. \qquad (2.6)$$

According to the 2D Coulomb law, the quantity $E(\Lambda)$ can be interpreted as potential energy of the system of "flat electrons" placed at Λ, see [29]. That observation motivates the term we use for the condition (2.9) below.

The following example is included to illustrate our next definition.

Key example:

Let $I \subset \mathbb{R}$ be an interval and let $\Lambda = d^{-1}\mathbb{Z} \cap I$ for some $d > 0$. Then

$$\Delta = \#\Lambda = d|I| + O(1)$$

and

$$E = E(\Lambda) = \sum_{1 \leqslant m \leqslant \Delta} \log\left[d^{-\Delta+1}(m-1)!(\Delta-m)!\right] = \Delta^2 \log|I| + O(|I|^2)$$

(2.7)

as follows from Stirling's formula. Here the notation $O(\cdot)$ corresponds to the direction $|I| \to \infty$.

Remark 1. *The uniform distribution of points on the interval does not maximize the energy $E(\Lambda)$ but comes within $O(|I|^2)$ from the maximum, which is negligible for our purposes, see the main definition and its discussion below. It is interesting to observe that the maximal energy for k points is achieved when the points are placed at the endpoints of I and the zeros of the Jacobi $(1,1)$-polynomial of degree $k-2$, see for example [14].*

Let

$$... < a_{-2} < a_{-1} < a_0 = 0 < a_1 < a_2 < ...$$

be a discrete sequence of real points. We say that the intervals $I_n = (a_n, a_{n+1}]$ form a short partition of $\mathbb{R}$ if $|I_n| \to \infty$ as $n \to \pm\infty$ and the sequence $\{I_n\}$ is short.

Main Definition:

Let $\Lambda = \{\lambda_n\}$ be a discrete sequence of real points. We say that Λ is d-uniform if there exists a short partition $\{I_n\}$ such that

$$\Delta_n = d|I_n| + o(|I_n|) \quad \text{for all} \quad n \quad \text{(density condition)}$$

(2.8)

as $n \to \pm\infty$ and

$$\sum_n \frac{\Delta_n^2 \log|I_n| - E_n}{1 + \text{dist}^2(0, I_n)} < \infty \quad \text{(energy condition)}$$

(2.9)

where Δ_n and E_n are defined as

$$\Delta_n = \#(\Lambda \cap I_n) \quad \text{and} \quad E_n = E(\Lambda \cap I_n) = \sum_{\lambda_k, \lambda_l \in I_n, \ \lambda_k \neq \lambda_l} \log|\lambda_k - \lambda_l|.$$

Remark 2. *Note that the series in the energy condition is positive: every term in the sum defining E_n is at most $\log|I_n|$ and there are less than Δ_n^2 terms.*

As follows from the example above, the first term in the numerator of (2.9) is approximately equal to the energy of Δ_n electrons spread uniformly over I_n. The second term is the energy of electrons placed at $\Lambda \cap I_n$. Thus the energy condition is a requirement that the placement of the points of Λ is

close to uniform, in the sense that the work needed to spread the points of Λ uniformly on each interval is summable with respect to the Poisson weight. For a more detailed discussion of this definition see [29]

In [29], d-uniform sequences were used to solve the gap problem mentioned in the introduction. Recall that with any closed $X \subset \mathbb{R}$ one can associate its (spectral) gap characteristic $\mathbf{G}_X$ defined as in section 1.4. The main result of [29] is the following statement:

Theorem 1. [29] *Let X be a closed set on $\mathbb{R}$. Then*
$$\mathbf{G}_X = \sup\{\ d \mid X \ \text{contains a } d - \text{uniform sequence }\}.$$

Recall that, as was proved in [29], $\mathbf{G}_X = \mathbf{G}_\mu^1$ for any μ such that $\operatorname{supp}\mu = X$. The following simple observations will also be useful to us in the future:

Remark 3.

- *If Λ is a d-uniform sequence then $D_*(\Lambda) = d$, as follows easily from the density condition* (2.8).
- *Among other things, the energy condition ensures that the points of Λ are not too close to each other. In particular, if Λ is d-uniform for some $d > 0$ and $\Lambda^* = \{\lambda_{n_k}\}$ is a subsequence such that for all k,*
$$\lambda_{n_k+1} - \lambda_{n_k} \leqslant e^{-c|\lambda_{n_k}|}$$
for some $c > 0$, then $D_*(\Lambda^*) = 0$.
- *An exponentially small perturbation of a d-uniform sequence contains a d-uniform subsequence. More precisely, if $c > 0$ and Λ is a d-uniform sequence then any sequence $A = \{\alpha_n\}$ such that $|\lambda_n - \alpha_n| \leqslant e^{-c|\lambda_n|}$ contains a d-uniform subsequence A^* consisting of all α_{n_k} such that*
$$\lambda_{n_k+1} - \lambda_{n_k} \geqslant e^{-(c-\varepsilon)|\lambda_{n_k}|}.$$
- *As discussed in* [29], *the energy condition always holds for separated sequences. If Λ is separated then it is d-uniform if and only if $D_*(\Lambda) = d$.*

2.4. **Polynomial decay.** In this section we prove a version of the well-known property that adding or removing polynomial decay cannot change the type of a measure.

Lemma 1. *Let μ be a finite positive measure on $\mathbb{R}$ and let $\alpha > 0$. Consider the measure ν satisfying*
$$d\nu(x) = \frac{d\mu(x)}{1 + |x|^\alpha}.$$

Then for any $1 \leqslant p \leqslant \infty$
$$\mathbf{G}_\mu^p = \mathbf{G}_\nu^p.$$

Proof. Since $d\nu/d\mu \leqslant 1$, one only needs to show that $\mathbf{G}_\mu^p \leqslant \mathbf{G}_\nu^p$. Suppose that $f \in L^p(\mu)$ is such that $\bar{f}\mu$ annihilates all $e^{iaz}, a \in (0, d)$. This is equivalent to the property that the Cauchy integral $Kf\mu$ is divisible by e^{idz} in $\mathbb{C}_+$, i.e. it decays like e^{idz} along the positive imaginary axis $i\mathbb{R}_+$, see for instance lemma 2 in [25].

Let $N \geqslant \alpha$ be an integer. It is enough to prove the statement for $N = 1$: the general case will follow by induction.

First let us assume that $Kf\mu$ has at least one zero a in $\mathbb{C} \setminus \mathbb{R}$. It is well-known, and not difficult to verify, that then the measure $\frac{f}{x-a}\mu$ satisfies

$$K\left(\frac{f}{x-a}\mu\right) = \frac{Kf\mu}{z-a}.$$

Hence the Cauchy integral in the left-hand side still decays like e^{idz} along $i\mathbb{R}_+$ and therefore the measure still annihilates $e^{iaz}, a \in (0, d)$. It is left to notice that

$$f(x)\frac{1+|x|^\alpha}{x-a} \in L^p(\nu).$$

If $Kf\mu$ does not have any zeros outside of $\mathbb{R}$, note that the measure $\eta = e^{-i\varepsilon x}f\mu$ satisfies

$$K\eta = e^{-i\varepsilon z}Kf\mu$$

(see for instance theorems 3.3 and 3.4 in [28]) and therefore

$$K(f\mu - c\eta) = K(1 - ce^{-i\varepsilon x})f\mu$$

has infinitely many zeros in $\mathbb{C} \setminus \mathbb{R}$ for any $c, |c| \neq 1$, while still decaying like $e^{i(d-\varepsilon)z}$ along $i\mathbb{R}_+$. $\square$

Let μ be a finite measure on a separated sequence X, with point masses decaying polynomially. Lemma 1 together with elementary estimates imply that then $\mathbf{G}_\mu^2 = \mathbf{G}_\mu^1$. Hence in this case theorem 2 below becomes the statement from [25] mentioned above:

$$\mathbf{G}_\mu^2 = 2\pi D_*(X).$$

3. Main Results

3.1. **Main Theorem.** Let τ be a finite positive measure on the real line. We say that a function $W \geqslant 1$ on $\mathbb{R}$ is a τ-*weight* if W is lower semi-continuous, tends to ∞ at $\pm\infty$ and $W \in L^1(\tau)$.

Theorem 2. *Let μ be a finite positive measure on the line. Let $1 < p \leqslant \infty$ and $a > 0$ be constants.*

Then $\mathbf{G}_\mu^p \geqslant a$ *if and only if for any μ-weight W and any $0 < d < a$ there exists a d-uniform sequence $\Lambda = \{\lambda_n\} \subset \operatorname{supp} \mu$ such that*

$$\sum \frac{\log W(\lambda_n)}{1 + \lambda_n^2} < \infty. \tag{3.1}$$

We postpone the proof until section 6.

One of the immediate corollaries of the above statement is that the p-type of a measure, $\mathbf{G}_\mu^p$ for $1 < p \leqslant \infty$, does not depend on p, which may come as a surprise to some of the experts. Further corollaries of theorem 2 and its connections with classical results are discussed in the following sections.

3.2. Discrete Measures with log-summable masses.

If μ is a measure satisfying $\mathbf{G}_\mu^p \geqslant 2\pi d$ for some $1 \leqslant p \leqslant \infty$, then its support contains a $(d - \varepsilon)$-uniform sequence for any ε, $d > \varepsilon > 0$. Conversely, for $p = 1$, if $\operatorname{supp} \mu$ contains a $(d - \varepsilon)$-uniform sequence for every ε, $d > \varepsilon > 0$ then $\mathbf{G}_\mu^1 \geqslant 2\pi d$, by theorem 1. For $p > 1$ the latter statement is false, as follows, for instance, from Levinson's theorem 7 below. The correct "inverse" in the case of discrete measures is the following theorem that will provide a crucial step in the proof of theorem 2.

Theorem 3. *Let $B = \{b_n\}$ be a sequence that contains a d-uniform subsequence. Let $w(n)$ be a positive function of n such that*

$$\sum w(n) < \infty$$

and

$$\sum \frac{\log w(n)}{1 + b_n^2} > -\infty.$$

Then the measure

$$\mu = \sum w(n) \delta_{b_n}$$

supported on B satisfies

$$\mathbf{G}_\mu^\infty \geqslant 2\pi d. \tag{3.2}$$

The proof is given in section 6.

3.3. A general sufficient condition.

As a corollary of theorem 3 we obtain the following sufficient condition for general measures. The condition seems to be reasonably sharp, as it is satisfied by all examples of measures with positive type existing in the literature.

Theorem 4. *Let μ be a finite positive measure on $\mathbb{R}$. Let $A = \{a_n\}$ be a d-uniform sequence of real numbers such that*

$$\sum \frac{\log \mu((a_n - \varepsilon_n, a_n + \varepsilon_n))}{1 + n^2} > -\infty, \tag{3.3}$$

where

$$\varepsilon_n = \frac{1}{3} \min \left((a_{n+1} - a_n), (a_n - a_{n-1}) \right).$$

Then $\mathbf{G}_\mu^\infty \geqslant 2\pi d.$

Proof. For each $\tau \in [0,1]$ let us define a discrete measure ν_τ as follows. The measure ν_τ has exactly one pointmass of the size

$$\mu((a_n - \varepsilon_n, a_n + \varepsilon_n))$$

in each interval

$$(a_n - \varepsilon_n, a_n + \varepsilon_n)$$

at the point x_n^τ chosen as

$$x_n^\tau = \inf\{ \ a \mid \mu((a_n - \varepsilon_n, a)) \geqslant \tau\mu((a_n - \varepsilon_n, a_n + \varepsilon_n)) \ \}.$$

Notice that $\{x_n^\tau\}$ is a d-uniform sequence. In view of (3.3) and theorem 3, ν_τ satisfies

$$\mathbf{G}_{\nu_\tau}^\infty \geqslant 2\pi d.$$

Then

$$\nu = \int_0^1 \nu_\tau d\tau$$

satisfies $d\nu/d\mu \leqslant 1$ and therefore

$$\mathbf{G}_\mu^\infty \geqslant \mathbf{G}_\nu^\infty \geqslant 2\pi d.$$

$\square$

3.4. A formula for the type of measures on separated sequences.
For some natural classes of measures the conditions of theorem 2 can be significantly simplified. One of such classes is the class of discrete measures on separated sequences.

In [16] Koosis showed that if $\mu = \sum w(n)\delta_n$ is a measure supported on $\mathbb{Z}$, with pointmasses satisfying

$$\sum \frac{\log w(n)}{1 + n^2} > -\infty,$$

then $\mathbf{G}_\mu^p = 2\pi$ for all $1 \leqslant p \leqslant \infty$. A result from [25] implies that if

$$\mu = \sum \frac{\delta_{a_n}}{1 + a_n^2}$$

for a separated sequence $A = \{a_n\} \subset \mathbb{R}$ then $\mathbf{G}_\mu^p = 2\pi D_*(A)$.

Theorems 1 and 2 allow to extend these results and give an exact formula for the type of a general measure concentrated on a separated sequence.

Note that for $p = 1$, $\mathbf{G}_\mu^1 = D_*(\Lambda)$ for any separated sequence Λ and any measure $\mu, \operatorname{supp} \mu = \Lambda$ by theorem 1. For $p > 1$ we have

Theorem 5. *Let $\Lambda = \{\lambda_n\}$ be a separated sequence and let*

$$\mu = \sum w(n)\delta_{\lambda_n}$$

be a finite positive measure supported on Λ. Define

$$D = \sup D_*(\Lambda'),$$

where the supremum is taken over all subsequences $\Lambda' \subset \Lambda$ satisfying

$$\sum_{\lambda_n \in \Lambda'} \frac{\log w(n)}{1 + n^2} > -\infty. \tag{3.4}$$

Then

$$G_\mu^p = 2\pi D$$

for all $1 < p \leqslant \infty$.

Proof. Suppose that $G_\mu^p > 2\pi D$ for some $D > 0, p > 1$. Define the μ-weight W as $W(\lambda_n) = (\mu(\{\lambda_n\})(1 + \lambda_n^2))^{-1}$. Then by theorem 2 there exists a subsequence $\Lambda' \subset \Lambda$ such that $D_*(\Lambda') > D$ and (3.4) is satisfied.

In the opposite direction the statement follows directly from theorem 3 and remark 3. □

4. Connections with classical results and further corollaries

One of the goals of this section is to give examples of applications of theorem 2 by discussing its corollaries that include classical results on the type problem. Due to this reason, we prefer to deduce each of the statements directly from the results of the last section, rather than obtaining them from each other, even when the latter approach may shorten the proof.

In our estimates we write $a(n) \lesssim b(n)$ if $a(n) < Cb(n)$ for some positive constant C, not depending on n, and large enough $|n|$. Similarly, we write $a(n) \asymp b(n)$ if $ca(n) < b(n) < Ca(n)$ for some $C \geqslant c > 0$. Some formulas will have other parameters in place of n or no parameters at all.

4.1. **Beurling's Gap Theorem.**

Theorem 6 (Beurling [3]). *If μ is a finite measure supported on a set with long gaps and the Fourier transform of μ vanishes on an interval, then $\mu \equiv 0$.*

Proof. If $\operatorname{supp}\mu$ has long gaps than for every short partition of $\mathbb{R}$ infinitely many intervals of the partition must be contained in the gaps of $\operatorname{supp}\mu$. Therefore $\operatorname{supp}\mu$ does not contain a sequence satisfying the density condition (2.8) for any $d > 0$. □

4.2. **Levinson's Gap Theorem.**

Theorem 7 (Levinson [22]). *Let μ be a finite measure on $\mathbb{R}$ whose Fourier transform vanishes on an interval. Denote*

$$M(x) = |\mu|((x, \infty)).$$

If $\log M$ is not Poisson-summable on $\mathbb{R}_+$ then $\mu \equiv 0$.

Proof. Suppose that $\log M$ is not Poisson-summable on $\mathbb{R}_+$. Without loss of generality, $M(0) = 1$. Let $0 = a_0 \leqslant a_1 \leqslant a_2 \leqslant ...$ be the sequence of points such that

$$a_n = \inf\{ \ a \mid M(a) \leqslant 3^{-n} \ \}.$$

Define a $|\mu|$-weight W as 2^n on each $(a_{n-1}, a_n]$, $a_{n-1} < a_n$.

Since $\hat{\mu}$ vanishes on an interval, there exists a sequence $\Lambda \subset \operatorname{supp} \mu$ satisfying the density condition (2.8) with some $a > 0$ on a short partition $I_n = (b_n, b_{n+1}]$, such that (3.1) holds. WLOG $b_0 = 0$. Notice that $\log W$ is an increasing step function on $\mathbb{R}_+$ satisfying $\log W \gtrsim -\log M$. Also, since $\{I_n\}$ is short, $cb_{n+1} \leqslant b_n$ for some $0 < c < 1$ and all $n > 0$. Hence,

$$\sum \frac{\log W(\lambda_n)}{1 + \lambda_n^2} \gtrsim \sum_{n=1}^{\infty} \frac{\log W(b_n)|I_n|}{1 + b_n^2}$$

$$\gtrsim \sum_{n=1}^{\infty} \frac{\log W(cb_{n+1})|I_n|}{1 + b_n^2} \gtrsim \int_0^{\infty} \frac{-\log M(cx)dx}{1 + x^2} = \infty.$$

$\square$

Levinson's result above was later improved by Beurling [3] who showed that instead of vanishing on an interval $\hat{\mu}$ may vanish on a set of positive Lebesgue measure with the same conclusion. Note that an analogous improvement cannot be made in Beurling's own gap theorem above, as illustrated by Kargaev's counterexample, see [15, vol. 1, p. 305].

4.3. **A hybrid theorem.** Beurling's and Levinson's Gap Theorems compliment each other by treating measures with sparse supports and fast decay correspondingly. In this section we suggest a hybrid theorem that combines the features of both statements. In comparison with Beurling's result it shows that the measure does not have to be zero on a long sequence of intervals, it just has to be small on it. In regard to Levinson's theorem, our statement says that the measure does not have to decay fast along the whole axis, just along a large enough set. One can show that the statement is sharp in both scales.

Theorem 8. *Let μ be a finite measure on $\mathbb{R}$ whose Fourier transform vanishes on an interval. Suppose that there exists a sequence of disjoint intervals $\{I_n\}$ such that*

$$\sum \frac{|I_n| \min\left(|I_n|, \log \frac{1}{|\mu|(I_n)}\right)}{1 + \operatorname{dist}^2(I_n, 0)} = \infty. \tag{4.1}$$

Then $\mu \equiv 0$.

Proof. We can assume that $|I_n| \to \infty$ because any subsequence of intervals with uniformly bounded lengths can be deleted from $\{I_n\}$ without changing (4.1). Define the $|\mu|$-weight W as

$$W = \left(|\mu|(I_n)(1 + \operatorname{dist}^2(I_n, 0))\right)^{-1}$$

on each I_n. Suppose that there exists a sequence $\Lambda \subset \operatorname{supp}\mu$, $\mathbf{G}_\Lambda = 2a > 0$ satisfying (3.1). Let

$$N = \{\, n \mid \#(\Lambda \cap I_n) > a|I_n| \}.$$

Note that the sequence $\{I_n\}_{n \notin N}$ cannot be long because otherwise Λ will not contain any subsequences satisfying the density condition with $d > a$ on any short partition. Therefore the part of the sum in (4.1) corresponding to $n \notin N$ is finite and

$$\sum \frac{\log W(\lambda_n)}{1 + \lambda_n^2} \geqslant \sum_{n \in N} \frac{\log W(\lambda_n)}{1 + \lambda_n^2} \gtrsim \sum_{n \in N} \frac{|I_n| \log \frac{1}{|\mu|(I_n)}}{1 + \lambda_n^2} = \infty.$$

$\square$

4.4. De Branges' Gap Theorem.

Theorem 9 (de Branges, theorem 63 [7])**.** *Let $K(x)$ be a continuous function on $\mathbb{R}$ such that $K(x) \geqslant 1$, $\log K$ is uniformly continuous and Poisson-unsummable. Then there is no nonzero finite measure μ on $\mathbb{R}$ such that*

$$\int_{-\infty}^{\infty} K d|\mu| < \infty \tag{4.2}$$

and $\hat\mu$ vanishes on an interval.

Proof. Suppose that μ satisfies (4.2) and its Fourier spectrum has a gap. Since K is a μ-weight there must exist a $d > 0$ and a d-uniform sequence $\Lambda \subset \operatorname{supp}\mu$ satisfying (3.1) with K in place of W. Since Λ has positive interior density and $\log K$ is uniformly continuous, (3.1) implies that $\log K$ is Poisson-summable. $\square$

4.5. **A theorem by Krein, Levinson and McKean.** Our next statement combines results by Krein (part I, $p = 2$) and by Levinson and McKean (part II, $p = 2$).

Theorem 10 (Krein [17], Levinson-McKean [11])**.** *Let μ be a finite measure on $\mathbb{R}$, $\mu = w(x)dx$ where $w(x) \geqslant 0$. Then*

I) If $\log w$ is Poisson-summable then for any $1 \leqslant p \leqslant \infty$, $\mathbf{G}_\mu^p = \infty$.

II) If $\log w$ is monotone and Poisson-unsummable on a half-axis $(-\infty, x)$ or (x, ∞) for some $x \in \mathbb{R}$ then for any $1 < p \leqslant \infty$, $\mathbf{G}_\mu^p = 0$.

Proof. If $\log w$ is Poisson-summable, denote by $H(z)$ the outer function in $\mathbb{C}_+$ satisfying $|H| = w$ on $\mathbb{R}$. Then for any $a > 0$ the measure $\eta = e^{-iax}\bar{H}(x)dx$ annihilates all exponentials with frequencies from $[0, a)$. (Here we use the fact that the integral over $\mathbb{R}$ for any function from $H^1(\mathbb{C}_+)$ is 0.) Since $|\eta| = \mu$, it follows that $\mathbf{G}_\mu^p = \infty$ for any $1 \leqslant p \leqslant \infty$.

In the opposite direction, suppose that $\log w$ is Poisson-unsummable and monotone on $\mathbb{R}_+$. Since for any $f \in L^p(\mu)$, $p > 1$, the function $\log(fw)$ is Poisson-unsummable as well, we will simply assume that the Fourier transform of μ itself vanishes on an interval and arrive at a contradiction.

Consider a μ-weight $W(x) = (w(x)(1+x^2))^{-1}$. If $\mathbf{G}_\mu^p > 2\pi d > 0$, there exists a d-uniform sequence Λ satisfying (3.1). Suppose that Λ satisfies (2.8) on a short partition $I_n = (b_n, b_{n+1}]$, $b_0 = 0$. Then, similarly to the proof of corollary 7, for some $0 < c < 1$, $cb_{n+1} < b_n$. Together with monotonicity of $\log w = -\log W - \log(1 + x^2)$, we obtain

$$\sum \frac{\log W(\lambda_n)}{1 + \lambda_n^2} \gtrsim \sum_{n=0}^{\infty} \frac{\log W(b_n)|I_n|}{1 + b_n^2} + \text{const}$$

$$\gtrsim \sum_{n=0}^{\infty} \frac{\log W(cb_{n+1})|I_n|}{1 + b_n^2} + \text{const} \gtrsim \int_0^{\infty} \frac{-\log w(cx)dx}{1 + x^2} + \text{const} = \infty.$$

$\square$

4.6. **A result by Borichev and Sodin on stability of type.** If $I \subset \mathbb{R}$ is an interval and $D > 0$ is a constant we denote by DI the interval concentric with I of length $D|I|$. Following [6], for $\delta > 0$ and $x \in \mathbb{R}$, we denote

$$I_{x,\delta} = [x - e^{-\delta|x|}, x + e^{-\delta|x|}].$$

If μ and ν are two finite positive measures on $\mathbb{R}$ we write $\mu \preccurlyeq \nu$ if there exist constants $\delta > 0$, $C > 0$, and $l > 0$, such that, for all $x \in R$,

$$\mu(I_{x,\delta}) \leqslant C(1 + |x|)^l \left(\nu(2I_{x,\delta}) + e^{-2\delta|x|} \right).$$

Instead of finite measures [6] deals with a wider class of polynomially growing measures and uses the corresponding definition of type. However, in view

of the statements like lemma 1 above, such differences are not essential for the type problem and the corresponding results are equivalent.

Theorem 11. [6] *If $\mu \preccurlyeq \nu$ then $\mathbf{G}_\mu^2 \leqslant \mathbf{G}_\nu^2$.*

Proof. Let $\{a_n\}_{n\in\mathbb{Z}}$ be a strictly increasing discrete sequence of real points satisfying $a_{-n} = -a_n$ and

$$a_{n+1} - a_n = 2e^{-\delta b_n}, \; b_n = \frac{a_{n+1} + a_n}{2} \text{ for all } n \in \mathbb{N},$$

where $\delta > 0$ is the constant from the definition of the relation $\mu \preccurlyeq \nu$. Denote $I_n = (a_n, a_{n+1}]$. Let W be a ν-weight. Then the step-function W^* defined as

$$W^*(x) = 1 + (1 + |b_n|)^{-l} \left[\frac{1}{\nu(2I_n) + e^{-2\delta b_n}} \int_{2I_n} W d\nu \right] \text{ on each } I_n$$

is a μ-weight, as follows from the condition $\mu \preccurlyeq \nu$. Assume that $\mathbf{G}_\mu^2 = 2\pi d > 0$. Then there exists an $(d - \varepsilon)$-uniform sequence $\Lambda \in \operatorname{supp}\mu$, that satisfies (3.1) with W^*. Our goal is to modify Λ into an $(d - \varepsilon)$-uniform sequence in $\operatorname{supp}\nu$ satisfying (3.1) with W.

Notice that WLOG we can assume that each interval $2I_n$ contains at most one point of Λ, see remark 3.

Now for each $\lambda_n \in 2I_{k_n}$ choose a point $\alpha_n \in 2I_{k_n} \cap \operatorname{supp}\nu$ such that

$$W(\alpha_n) \leqslant \frac{1}{\nu(2I_n)} \int_{2I_n} W d\nu.$$

WLOG

$$\int_{2I_n} W d\nu \geqslant e^{-\frac{3}{2}\delta b_n}$$

for all n: otherwise we can increase the weight W to satisfy this condition and it will still remain ν-summable. If W is such a weight, then the interior density of the subsequence of Λ that falls in the intervals I_n satisfying $\nu(2I_n) \leqslant e^{-2\delta b_n}$ must be zero: otherwise the sum (3.1) for Λ and W^* would diverge. We can assume that Λ does not have such points. Then

$$\log W(\alpha_n) \leqslant \log W^*(\lambda_n) + 2l \log(1 + |\lambda_n|)$$

and therefore $A = \{\alpha_n\}$ satisfies (3.1) with W. By remark 3, A has a $(d - \varepsilon)$-uniform subsequence. Hence $\mathbf{G}_\nu^2 \geqslant \mathbf{G}_\mu^2 - 2\pi\varepsilon$. $\qquad\square$

Notice that our proof is p-independent, i.e. $\mathbf{G}^2$ can be replaced with $\mathbf{G}^p$ for any $1 < p \leqslant \infty$ in the Borichev-Sodin result.

4.7. **A sufficient condition by Duffin and Schaeffer.** Our next statement is formulated in [9, 6] for Poisson-finite measures. Here we present an equivalent finite version.

Theorem 12. *Let μ be a finite positive measure on $\mathbb{R}$ such that for any $x \in \mathbb{R}$*
$$\mu([x - L, x + L]) > d(1 + x^2)^{-1}$$
for some $L, d > 0$. Then $\mathbf{G}_\mu^2 \geqslant \pi/L$.

Proof. If $\varepsilon > 0$ consider $a_n = n(2L + \varepsilon)$. Then in every interval $(a_n - L, a_n + L)$ there exists a subinterval I_n of the length ε satisfying
$$\mu(I_n) \geqslant \frac{d\varepsilon}{L(1 + a_n^2)}.$$
It is left to apply theorem 4 to the sequence of centers of I_n. $\square$

4.8. **Benedicks' result on unions of intervals.** The following reslut contained in [2] provides non-trivial examples of measures with positive type. Until now, only a few examples of this kind existed in the literature.

Theorem 13. [2] *Let $... < a_{-1} < a_0 = 0 < a_1 < a_2 < ...$ be a discrete sequence of points and let $I_n = (a_n, a_{n+1}]$ be the corresponding partition of $\mathbb{R}$. Suppose that there exist positive constants C_1, C_2, C_3 such that*

1) if
$$C_1^{-1} a_{2n+1} < a_{2k+1} < C_1 a_{2n+1},$$
for some n, k, then
$$C_2^{-1} |I_{2n+1}| < |I_{2k+1}| < C_2 |I_{2n+1}|;$$

2) for all n
$$C_1^{-1} |a_{2n+1}| < |a_{2n-1}| < C_1 |a_{2n+1}|;$$

3) for all n
$$|I_{2n+1}| > C_3 \max(|I_{2n}|, 1);$$

4)

$$\sum \frac{|I_{2n+1}|^2}{1 + a_{2n+1}^2} \left[\log_+ \frac{|I_{2n+1}|}{|I_{2n}|} + 1\right] < \infty. \tag{4.3}$$

Then for any real number $A > 0$ and $1 \leqslant p < \infty$ there exists a nonzero function
$$f \in L^1(\mathbb{R}) \cap L^p(\mathbb{R}) \cap C^\infty(\mathbb{R}), \quad \text{supp } f \subset \cup I_{2n},$$
such that $\hat{f} = 0$ on $[0, A]$.

Here we will not concern ourselves with the condition $f \in C^\infty$. The rest of the statement, i.e. the existence of $f \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ follows from theorem 2. Moreover, conditions 1 and 2 prove to be redundant.

Proof. Let $\{b_n\}_{n\in\mathbb{Z}}$ be a sequence of positive integers, monotonically increasing to ∞ as $n \to \infty$ and as $n \to -\infty$, such that if one replaces $|I_{2n+1}|$ in (4.3) with $b_n|I_{2n+1}|$ the series still converges. Consider the sup-partition of $\{I_n\}$ defined in the following way. Let

$$n_0 = 0, n_{k+1} - n_k = b_{n_k}$$

for $k > 0$ and

$$n_{k+1} - n_k = b_{n_{k+1}}$$

for $n < 0$. Define $J_k = (a_{2n_k}, a_{2n_{k+1}}]$. By 3, the new partition satisfies the property $|J_n| \to \infty$ and, because of monotonicity of b_n,

$$\sum \frac{|J_n|^2}{1 + \text{dist}^2(0, J_n)} \left[\log_+ \frac{|J_n|}{|J_n \cap (\cup I_{2k})|} + 1 \right] < \infty. \qquad (4.4)$$

In particular $\{J_n\}$ is short.

Let C be a large positive number. By $[\cdot]$ we will denote the integer part of a real number. Define a sequence Λ as follows. On each $J_k = (a_{2n_k}, a_{2n_{k+1}}]$ place $N = [C|J_k|]$ points of Λ inside $J_k \cap (\cup I_{2n})$ so that

$$\lambda_{m_k} < \lambda_{m_{k+1}} < ... < \lambda_{m_k+N}$$

and

$$|(\cup I_{2n}) \cap (a_{2n_k}, \lambda_{m_k}]| = |(\cup I_{2n}) \cap (\lambda_{m_k+N}, a_{2n_{k+1}}]| = |(\cup I_{2n}) \cap (\lambda_l, \lambda_{l+1}]|,$$

for all $l, m_k \leqslant l < m_k + N - 1$.

Then conditions 3 and 4 of the theorem imply that Λ satisfies the energy condition (2.9) on J_n and that $D_*(\Lambda) = C$. Also the measure

$$\nu = \chi_{\cup I_{2n}} \Pi$$

and Λ satisfy conditions of theorem 4. Therefore $\mathbf{G}_\nu^p \geqslant 2\pi C$ for any $1 \leqslant p \leqslant \infty$ which implies the existence of the desired function f satisfying $\hat{f} = 0$ on $(0, 2\pi C)$.

$\square$

Notice that our proof actually produces $f \in L^\infty$. If, in addition to the conditions of the theorem, $|I_{2n}| > \text{const} > 0$, then the remaining property $f \in C^\infty$ can be added with little effort. One would need to construct f supported on $\cup \frac{1}{3} I_{2n}$ and then consider a convolution $f * \phi$ with a C^∞-function ϕ with small support. In the general case f can be "smoothed out" using functions with exponentially decreasing size of support and involving arguments like theorem 11.

5. Proofs: Auxiliary Statements

This section contains the results that will be needed to prove our main theorems 2 and 3.

5.1. **A measure with positive type.** The following lemma is essentially proved, but not explicitly stated in [29].

Lemma 2. *Let* $A = \{a_n\}$ *be a discrete increasing sequence of distinct real numbers that does not have infinitely large gaps, i.e.* $a_{n+1} - a_n < C$ *for some* $0 < C < \infty$. *Denote by* b_n *the middle of the interval* (a_n, a_{n+1}), $b_n = (a_n + a_{n+1})/2$. *Suppose that the sequence* A *is* d-*uniform for some* $d > 0$. *Then there exists a finite positive measure supported on* $B = \{b_n\}$,

$$\mu = \sum \beta_n \delta_{b_n},$$

satisfying

$$0 < \beta_n \leqslant \frac{\sqrt{a_{n+1} - a_n}}{1 + a_n^2}, \tag{5.1}$$

such that $\mathbf{G}_\mu^\infty \geqslant 2\pi d$.

Proof. Let θ be the meromorphic inner function constructed for the sequence A as in lemma 5 from [29]. By construction, the Clark measure $\nu = \mu_{-1}$ corresponding to θ is supported on B and satisfies

$$\nu(\{b_n\}) \lesssim a_{n+1} - a_n, \tag{5.2}$$

see the estimate (7.3) in [29].

Let $c = d - \varepsilon$. As was proved in [29], if θ satisfies the conditions of lemma 5, [29], and A is d-uniform, then there exists $f \in K_\theta$ that is divisible by e^{icz} in $\mathbb{C}_+$. (This is one of the main steps in the proof of theorem 2, [29]. See the part from the fourth line before claim 1 to the end of part I of the proof.)

Then, by the Clark representation, $2\pi i f = (1 + \theta) K f \nu$. Since $1 + \theta$ is outer, $K f \nu$ is divisible by e^{icz} in C_+. Because ε is arbitrary, by lemma 1, the measure $\mu = |f| \nu / (1 + x^2)$ satisfies $\mathbf{G}_\mu^\infty \geqslant 2\pi d$. Since $f \in L^2(\nu)$ and ν satisfies (5.2), considering a constant multiple of μ if necessary, we obtain (5.1). $\qquad\qquad\square$

5.2. **Construction of an auxiliary sequence.** To apply our previous lemma in the main proofs we will need the following

Lemma 3. *Let* $B = \{b_n\}$ *be a* d-*uniform sequence satisfying* (2.8) *and* (2.9) *on a short partition* $\{I_n\}$. *Let* $w(n)$ *be a positive bounded function on* $\mathbb{Z}$ *such that*

$$\sum \frac{\log w(n)}{1 + n^2} > -\infty. \tag{5.3}$$

Then for any $\varepsilon > 0$ *there exists a discrete sequence* $A = \{a_n\}$ *satisfying:*

1) $a_{n+1} - a_n < 1/\varepsilon$.

2) Define the sequence $C = \{c_k\}$ *as* $c_k = \frac{a_{k+1} + a_k}{2}$. *Then the sequence* $B^* = B \cap C$ *satisfies*

$$\#(B^* \cap I_n) \geqslant (d - e)|I_n|$$

for large enough $|n|$.

3) If $b_n = c_k$, i.e. b_n is the middle of (a_k, a_{k+1}), then $a_{k+1} - a_k \leqslant w(n)$.

4) A is $2d$-uniform

5) $D^*(C \setminus B) \leqslant d + \varepsilon$.

Proof. Denote

$$l_n = \min(b_{n+1} - b_n, b_n - b_{n-1}, w(n)).$$

Consider the sequence $P = \{p_n\}$ defined as

$$p_{2n} = b_n - \frac{1}{3}l_n, \quad p_{2n+1} = b_n + \frac{1}{3}l_n.$$

Choose a large $L >> 1/\varepsilon$. Define the sequence Q as follows: if $p_{2n+2} - p_{2n+1} > L$, insert $M = [(p_{2n+2} - p_{2n+1})/L]$ points of Q into the interval (p_{2n+1}, p_{2n+2}) uniformly, i.e. at the points

$$p_{2n+1} + k\frac{p_{2n+1} - p_{2n+2}}{M+1}, \quad k = 1, 2, ...M.$$

Now put $A = P \cup Q$.

By our construction the sequence A satisfies

$$2\#(B \cap I_n) - 2 \leqslant \#(A \cap I_n) \leqslant 2\#(B \cap I_n) + \varepsilon|I_n|.$$

To make A satisfy the more precise density condition (2.8) with $2d$ we may need to delete some points of B on each interval I_n and consider a smaller sequence B^* in place of B in the above construction. Note that we would have to delete at most $\varepsilon|I_n|$ points from B on each I_n and that B^* will satisfy the energy condition (2.9) as a subsequence of B. After such an adjustment, A will satisfy 1), 2), 3) and the density condition (2.8) with $2d$.

Note that A satisfies the energy condition on $\{I_n\}$. Indeed, let us denote $\Delta_n = \#(P \cap I_n)$ and $\Gamma_n = \#(Q \cap I_n)$. Then

$$\#(A \cap I_n)^2 \log|I_n| - \sum_{a_n, a_k \in A \cap I_n} \log|a_n - a_k| =$$

$$\left(\Delta_n^2 \log|I_n| - \sum_{a_n, a_k \in P \cap I_n} \log|a_n - a_k|\right) +$$

$$\left(\Gamma_n^2 \log|I_n| - \sum_{a_n, a_k \in Q \cap I_n} \log|a_n - a_k|\right) +$$

$$2\left(\Delta_n \Gamma_n \log|I_n| - \sum_{a_n \in P \cap I_n, a_k \in Q \cap I_n} \log|a_n - a_k|\right) =$$

$$I + II + III.$$

To estimate I notice that for any $p_{2k} \in P \cap I_n$,

$$-\log(p_{2k+1} - p_{2k}) \leqslant -\log w(k),$$

by our choice of points p_{2k}, p_{2k+1}. The rest of the terms in I can be estimated by the similar terms for B^*, i.e.

$$I \lesssim \left(\#(B^* \cap I_n)^2 \log |I_n| - \sum_{b_n, b_k \in B^* \cap I_n} \log |b_n - b_k| \right)$$

$$- \sum_{p_{2k} \in P \cap I_n} \log w(k) + O(|I_n|^2).$$

Since B^* satisfies the energy condition and because of (5.3) and shortness of the partition, I will give finite contribution to the energy sum in (2.9).

To estimate II notice that points in Q are at a distance at least $L/2$ from each other. Therefore

$$II \lesssim \left(\Gamma_n^2 \log |I_n| - \sum_{0 \leqslant n,k \leqslant \Gamma_n} \log |n - k| \right) + O(\Gamma_n^2) =$$

$$\Gamma_n^2 \log \frac{|I_n|}{\Gamma_n} + O(\Gamma_n^2)$$

after estimating the sum via Stirling's formula. Notice that since $\Gamma_n < |I_n|$ and

$$\log \frac{|I_n|}{\Gamma_n} < \frac{|I_n|}{\Gamma_n},$$

the last quantity will also give finite contribution to (2.9).

Finally, III can be estimated similarly to II. Just notice that any point a_j in P is at a distance at least $L/2$ from Q and therefore

$$\Gamma_n \log |I_n| - \sum_{a_k \in Q \cap I_n} \log |a_j - a_k| \lesssim \Gamma_n \log \frac{|I_n|}{\Gamma_n} + O(|I_n|^2).$$

Summing over all $a_j \in P \cap I_n$ and recalling that $\#(P \cap I_n) = \Delta_n \lesssim |I_n|$ we again get a finite quantity in (2.9).

To prove 5), let us split C into two subsequences:

$$C_1 = \{(a_n + a_{n+1})/2 \mid a_n, a_{n+1} \in P\} \text{ and } C_2 = C \setminus C_1.$$

Notice that $C_1 \setminus B^*$ has at most one point between each two points of B^*. Therefore,

$$D^*(C_1 \setminus B) \leqslant D^*(B) \leqslant d + \varepsilon.$$

Also, if $2/L << \varepsilon$ then $D^*(C_2) < \varepsilon$, because any two points of C_2 are at a distance at least $L/2$ from each other. $\qquad \square$

5.3. **Existence of extremal measure with a spectral gap.** The lemma in this section can be viewed as a version of de Branges' theorem 66 from [7]. The last section of [29] contains a discussion of that theorem and its equivalent reformulations.

Here and throughout the rest of the paper we will use the standard notation $S(z) = e^{iz}$ for the exponential inner function in the upper half-plane. In general, $S^a(z) = e^{iaz}$ is inner in $\mathbb{C}_+$ if $a > 0$ and inner in $\mathbb{C}_-$ if $a < 0$.

Lemma 4. *Let μ be a finite complex measure such that $\hat{\mu} \equiv 0$ on $[0,a]$. Let W be a $|\mu|$-weight. Then there exists a finite measure $\nu = \sum \alpha_n \delta_{\lambda_n}$ concentrated on a discrete sequence $\Lambda = \{\lambda_n\}$ such that*

1) $\Lambda \subset \operatorname{supp} \mu$;

2) W is a $|\nu|$-weight;

3) $\hat{\nu} \equiv 0$ on $[0,a]$;

4) $K\nu$ has no zeros in $\mathbb{C}$, $K\nu/S^a$ is outer in $\mathbb{C}_+$ and $K\nu$ is outer in $\mathbb{C}_-$.

Proof. It will be more convenient for us to assume that $\hat{\mu} \equiv 0$ on a symmetric interval $[-a,a]$. Then $\bar{\mu}$ has the same property. Hence we can assume that the measure is real (otherwise consider $\mu \pm \hat{\mu}$).

Consider the following set of measures on $\operatorname{supp} \mu$:

$$M_W = \{\, \nu \mid \int W d|\nu| \leqslant 1, \ \hat{\nu} = 0 \text{ on } [-a,a], \ \operatorname{supp} \nu \subset \operatorname{supp} \mu, \nu = \bar{\nu}\}.$$

Notice that the set is non-empty, because $\mu \in M_W$, and convex. It is also $*$-weakly closed in the space of all finite measures on $\operatorname{supp} \mu$. Therefore by the Krein-Milman theorem it has an extreme point. Let ν be such a point. We claim that it is the desired measure.

First, let us note that $\hat{\nu} \equiv 0$ on $[-a,a]$. It is well known that this property is equivalent to the property that ν annihilates the Payley-Wiener class PW_a, i.e. that for any bounded $f \in PW_a$,

$$\int f d\nu = 0,$$

see for instance the last section of [29].

Next, let us show that the set of real $L^\infty(|\nu|)$-functions h, such that $\widehat{h\nu} \equiv 0$ on $[-a,a]$, is one-dimensional and therefore $h = c \in \mathbb{R}$. (This is equivalent to the statement that the closure of PW_a in $L^1(|\nu|)$ has deficiency 1, i.e. the space of its annihilators is one dimensional.)

Let there be a bounded real h such that $\widehat{h\nu} \equiv 0$ on $[-a,a]$. WLOG $h \geqslant 0$, since one can add constants, and $\int W|h|d|\nu| = 1$. Choose $0 < \alpha < 1$ so that $0 \leqslant \alpha h < 1$. Consider the measures $\nu_1 = h\nu$ and $\nu_2 = (1-\alpha)^{-1}(\nu - \alpha\nu_1)$. Then both of them belong to M_W and $\nu = \alpha\nu_1 + (1-\alpha)\nu_2$ which contradicts the extremality of ν.

Now let us show that ν is discrete. Let g be a continuous compactly supported real function on $\mathbb{R}$ such that $\int g d\nu = 0$. By the previous part, there exists a sequence $f_n \in PW_a$, $f_n \to g$ in $L^1(|\nu|)$. Indeed, otherwise there would exist a function $h \in L^\infty(|\nu|)$ annihilating all $f \in PW_a \cap L^1(|\nu|)$ and such that $\int hgd|\nu| = 1$. Since $\int g d\nu = 0$, $h \neq const$ and we would obtain a contradiction with the property that the space of annihilators is one-dimensional.

Since ν annihilates PW_a and $(f_n(z) - f_n(w))/(z - w) \in PW_a$ for every fixed $w \in \mathbb{C} \setminus \mathbb{R}$,

$$0 = \int \frac{f_n(z) - f_n(w)}{z - w} d\nu(z) = K f_n \bar{\nu}(w) - f_n(w) K\nu(w)$$

and therefore

$$f_n(w) = \frac{K f_n \nu}{K\nu}(w).$$

Taking the limit,

$$f = \lim f_n = \lim \frac{K f_n \nu}{K\nu} = \frac{K g \nu}{K\nu}.$$

Since all of f_n are entire, one can show that the limit function f is also entire. Indeed, first notice that there exists a positive function $V \in L^1(|\nu|)$ such that $f_{n_k}/V \to g/V$ in $L^\infty(|\nu|)$, for some subsequence $\{f_{n_k}\}$. To find such a V first choose f_{n_k} so that $||f_{n_k} - g||_{L^1(|\nu|)} < 3^{-k}$ and then put

$$V = 1 + \sum 2^k |f_{n_k} - g|.$$

Denote $F_k = f_{n_k}/V$ and $\eta = V|\nu|$. Then F_k converge in $L^2(\eta)$ and by the Clark theorem $(1 - I)KF_k\eta$ converge in $H^2(\mathbb{C}_+)$, where I is the inner function whose Clark measure is η. Notice that

$$f_{n_k} = \frac{K f_{n_k} \nu}{K\nu} = \frac{K F_k \eta}{K\nu} = \frac{(1 - I)KF_k\eta}{(1 - I)K\nu}.$$

Now let T be a large circle in $\mathbb{C}$ such that $|(1 - I)K\nu| > const > 0$ on T. Denote $T_\pm = T \cap \mathbb{C}_\pm$ and let m_T be the Lebesgue measure on T. Since $(1 - I)KF_k\eta$ converge in $H^2(\mathbb{C}_+)$, f_{n_k} converge in $L^1(T_+, m_T)$. Similarly, f_{n_k} converge in $L^1(T_-, m_T)$. By the Cauchy formula it follows that f_{n_k} converge normally inside T and therefore f is analytic inside T. Since such a circle T can be chosen to surround any bounded subset of $\mathbb{C}$, f is entire.

Since the numerator in the representation

$$f = \frac{K g \bar{\nu}}{K\bar{\nu}}$$

is analytic outside the compact support of g, the measure in the denominator must be singular outside of that support: Cauchy integrals of non-singular measures have jumps at the real line on the support of the a.c. part, which would contradict the property that f is entire. Choosing two different functions g with disjoint supports we conclude that ν is singular.

Moreover, since f is entire, the zero set of f has to be discrete. Since ν is singular, $K\nu$ tends to ∞ nontangentially in $\mathbb{C}_+$ at ν-a.e. point and $f = 0$ at ν-a.e. point outside of the support of g. Again, by choosing two different g with disjoint supports, we can see that ν is concentrated on a discrete set.

It remains to verify 4. Since we chose to deal with the symmetric interval $[-a, a]$, we need to show that $K\nu/S^{\pm a}$ are outer in $\mathbb{C}_\pm$ correspondingly.

Let J be the inner function corresponding to $|\nu|$ ($|\nu|$ is the Clark measure for J). Denote

$$G = \frac{1}{2\pi i}(1 - J)K\nu \in K_J.$$

As was mentioned in section 2.1, G has non-tangential boundary values $|\nu|$-a.e. and

$$\nu = G|\nu|.$$

Since $K\nu$ is divisible by S^a in $\mathbb{C}_+$, G is divisible by S^a in $\mathbb{C}_+$. Suppose that $G = S^a U H$ for some inner U. Since the measure ν is real, $\bar{G} = G$, $|\nu|$-a.e.

Let $F \in K_J$ be the function such that $\bar{J}G = \bar{F}$. Since $J = 1$, $|\nu|$-a.e., $F = \bar{G} = G$, $|\nu|$-a.e. Since functions in K_J are uniquely determined by their traces on the support of the Clark measure $|\nu|$, $F = G = S^a U H$. Notice that the function $h = S^a(1 + U)^2 H$ also belongs to K_J:

$$\bar{J}h = \bar{J}S^a(1 + U)^2 H = (\bar{J}G)\bar{U}(1 + U)^2 = \bar{G}\bar{U}(1 + U)^2$$

$$= \overline{S^a(1 + U)^2 H} = \bar{h} \in \overline{H^2}(\mathbb{C}_+),$$

because $\bar{U}(1 + U)^2$ is real a.e. on $\mathbb{R}$. Denote by γ the measure from the Clark representation of h, i.e.

$$\gamma = h|\nu|, \quad h = \frac{1}{2\pi i}(1 - J)K\gamma.$$

Then

$$\gamma = h|\nu| = \bar{U}(1 + U)^2 G|\nu| = \bar{U}(1 + U)^2 \nu.$$

The Cauchy integral of γ is divisible by S^a in $\mathbb{C}_+$ because h is divisible by S^a in $\mathbb{C}_+$. Since $\bar{U}(1 + U)^2$ is real, a constant multiple of γ belongs to M_W. Since U is non-constant and $|\nu|$ is the Clark measure for J, γ is not a constant multiple of ν. Again we obtain a contradiction with the property that the space of annihilators is one-dimensional.

Thus $G/S^a \in K_J$ is outer in C_+. Since $J\bar{G} = \bar{G}$, the pseudocontinuation of G does not have an inner factor except S^{-a} in $\mathbb{C}_-$ as well. Hence $K\nu/S^{\pm a}$ is outer in $\mathbb{C}_\pm$.

If G has a zero at $x = a \in \mathbb{R}$ outside of spec_J then

$$\frac{G}{x - a} \in K_J$$

and the measure

$$\gamma = \frac{G}{x - a}|\nu|$$

leads to a similar contradiction with the property that the space of annihilators is one-dimensional, since $(x - a)^{-1}$ is bounded and real on the support of ν. Since $G = \frac{1}{2\pi i}(1 - J)K\nu$, $K\nu$ does not have any zeros on $\mathbb{R}$.   □

Remark 4. *A statement similar to lemma 9 from [29], where S^a was replaced with an arbitrary inner function, can also be formulated in the case of lemma 4.*

5.4. **Estimates of $\log|\theta|$ for a meromorphic inner function.**

Lemma 5. *Consider a short partition of $\mathbb{R}$, $\{I_n\}$. Consider the set of circles $T_n = \{z \mid |z - \xi_n| = 2|I_n|\}$ where $\xi_n \in I_n$, and disks $D_n = \{z \mid |z - \xi_n| \leqslant 4|I_n|\}$. Let θ be a meromorphic inner function such that*

$$\sum_n \frac{\#\left(\{\theta = 0\} \cap D_n\right)|I_n|}{1 + \mathrm{dist}^2(0, I_n)} < \infty. \tag{5.4}$$

Then the integrals

$$p_n = \int_{T_n} |\log|\theta(z)||\, d|z|$$

satisfy

$$\sum_n \frac{p_n}{1 + \mathrm{dist}^2(0, I_n)} < \infty. \tag{5.5}$$

Proof. Suppose that $\theta = S^a B$ for some $a > 0$ and some Blaschke product $B, \{B = 0\} = \{a_n\} \subset \mathbb{C}_+$. Then

$$\log|\theta| = \log|S^a| + \log|B|.$$

The integrals of $|\log|S^a||$ are summable because

$$|\log|S^a|| \lesssim |I_n|$$

on T_n and the sequence $\{I_n\}$ is short. To estimate the integral of $|\log|B||$ notice that

$$|\log|B|(z)| = \sum_{a_n \in D_n} \left|\log\frac{|z - a_n|}{|z - \bar{a}_n|}\right| + \sum_{a_n \notin D_n} \left|\log\frac{|z - a_n|}{|z - \bar{a}_n|}\right|.$$

Elementary estimates show that for any $a_n \in D_n$

$$\int_{T_n} \left|\log\frac{|z - a_n|}{|z - \bar{a}_n|}\right| d|z| \lesssim |I_n|$$

and, because of (5.4), such integrals will give a finite contribution to the sum in (5.5). For each term in the second sum

$$\int_{T_n} \left|\log\frac{|z - a_n|}{|z - \bar{a}_n|}\right| d|z| \lesssim \int_{I_n} \frac{\Im a_n dx}{(\Re a_n - x)^2 + (\Im a_n)^2} \asymp \int_{I_n} \frac{\Im a_n dx}{(\Re a_n - x)^2 + 1} dx.$$

Since, as follows from the Blaschke condition,

$$\sum_n \frac{\Im a_n}{(\Re a_n - x)^2 + 1} \in L^1_\Pi$$

we again get a finite sum in (5.5). □

5.5. A version of the first BM theorem.
The following lemma is essentially a version of the so-called first Beurling-Malliavin theorem, see also [24].

Lemma 6. *Let $\{I_n\}$ be a long sequence of intervals and let c be a positive constant. Denote by I_n' and I_n'' the intervals of the length $c|I_n|$ adjacent to I_n from the left and from the right correspondingly. Let u be a real function on $\mathbb{R}$ such that*

$$\Delta_n = \sup_{I_n''} u - \inf_{I_n'} u \geqslant d|I_n|$$

for all n and for some $d > 0$. Then u is not a harmonic conjugate of a Poisson-summable function.

Proof. Note that if $\tilde{u} \in L_\Pi^1$ then $f = e^{-iu+\tilde{u}}$ is an outer function in the Smirnov class in $\mathbb{C}_+$. Moreover, f belongs to the kernel $N^+[e^{iu}]$ of the Toeplitz operator with the symbol e^{iu} in the Smirnov class. This contradicts a Toeplitz version of the first BM theorem, see section 4.4 of [24]. □

5.6. An estimate for an extremal discrete measure of positive type.
In this section we show that that a discrete measure of positive type, like in the statement of lemma 4, must have log-summable pointmasses. We start with the following elementary statement that can be easily verified.

Lemma 7. *Let $\{I_n\}$ be a short sequence of intervals and let $C > 1$. Denote*

$$l_n = \sum_{I_m \cap CI_k \neq \emptyset} |I_m|.$$

Then

$$\frac{l_n|I_n|}{1 + \operatorname{dist}^2(0, I_n)} < \infty.$$

Our main statement in this section is

Lemma 8. *Let ν be a finite measure*

$$\nu = \sum \alpha_n \delta_{\lambda_n}$$

on a discrete sequence $\Lambda = \{\lambda_n\}$, such that $\alpha_n \neq 0$, $\hat{\nu} \equiv 0$ on $[0, 2\pi d]$, $K\nu$ does not have any zeros in $\mathbb{C}$, $K\nu/S^d$ is outer in $\mathbb{C}_+$ and $K\nu$ is outer in $\mathbb{C}_-$. Then for any $\varepsilon > 0$, Λ contains a $(d - \varepsilon)$-uniform subsequence and

$$\sum \frac{\log|\alpha_n|}{1 + n^2} > -\infty. \tag{5.6}$$

Proof. The statement that Λ contains a $(d-\varepsilon)$-uniform subsequence follows from the property that $\mathbf{G}_\Lambda \geqslant d$ and theorem 1.

To establish (5.6), let us first show that there exists a short partition $\{I_k\}$ of $\mathbb{R}$ such that Λ satisfies (2.8) with d on that partition.

Let J be the inner function whose Clark measure is $|\nu|$. Then by the Clark theorem the function

$$Q(z) = (1 - J)K\nu$$

belongs to K_J. It follows from the properties of $K\nu$ that $Q = S^d O$ in $\mathbb{C}_+$ for some outer O and $\bar{J}Q = \bar{O}$. Therefore the argument of O satisfies $u = 2\arg O = \arg J - dx$. Notice that $\arg J$ is a growing function that is equal, up to a bounded term, to the counting function of Λ. Also, since $O \in H^2$, $\tilde{u} \in L^1_\Pi$. If the desired short partition $\{I_k\}$, where Λ satisfies (2.8), does not exist then there exists a long sequence of intervals $\{J_k\}$ such that

$$|\#(\Lambda \cup J_k) - d|J_k|| \geqslant c_1|J_k| \tag{5.7}$$

for each k and for some $c_1 > 0$. First, let us assume that the difference in the left-hand side is positive for a long subsequence of $\{J_k\}$. Let J'_k, J''_k denote the intervals of the length $c_2|J_k|, 0 < c_2 << c_1$, adjacent to J_k from the left and from the right correspondingly. Since u' is bounded from below we get that

$$\Delta_k = \inf_{J''_k} u - \sup_{J'_k} u > c_3|J_k|$$

for some $c_3 > 0$ on a long subsequence of $\{J_k\}$, if c_2 is small enough. By lemma 6, this contradicts the property that $\tilde{u} \in L^1_\Pi$. If the difference in (5.7) is negative for a long subsequence of $\{J_k\}$ then lemma 6 can be applied to $-u$ and the intervals J'_k, J''_k chosen so that $J'_k, J''_k \subset J_k$, $|J'_k| = |J''_k| = c_2|J_k|$, J'_k shares its left endpoint with J_k and J''_k shares its right endpoint with J_k, to arrive at the same contradiction. Hence a short partition where Λ satisfies (2.8) with d does exist.

Let $\{I_k\}$ be such a partition. Let $\lambda_{n_k} \in I_k$ be such that

$$\log_- \alpha_{n_k} = \max_{\lambda_n \in I_k} \log_- \alpha_n.$$

Suppose that (5.6) is not satisfied. Then

$$\sum_k \#(\Lambda \cap I_k)\frac{\log_- \alpha_{n_k}}{1 + n_k^2} \asymp \sum_k |I_k|\frac{\log_- \alpha_{n_k}}{1 + n_k^2} = \infty. \tag{5.8}$$

Consider $\mu = \sum_n \delta_{\lambda_n}$ the counting measure of Λ. Since Λ satisfies (2.8), μ is Poisson-finite. Let θ be the inner function such that μ is its Clark measure. Since ν is finite, it can be represented as $\nu = f\mu$ with $f \in L^2(\nu)$. Hence, by Clark theory,

$$F = \frac{1}{2\pi i}(1 - \theta)K\nu \in K_\theta$$

with $F(\lambda_n) = f(\lambda_n) = \alpha_n$.

For each k consider the disk

$$D_k = \{z \mid |z - \lambda_{n_k}| < 2|I_k|\}$$

and its boundary circle $T_k = \partial D_k$. Notice that for each k, F does not have zeros in D_k. It does have poles at the points $\bar{a}_n \in \mathbb{C}_-$, where $A = \{a_n\}$ are the zeros of θ in $\mathbb{C}_+$. Hence in D_k the function F admits factorization $F = H_k/B_k$, where B_k is the finite Blaschke product in D_k ($|B_k| = 1$ on T_k) with zeros at $\bar{A} \cap D_k$, and H_k is analytic without zeros in D_k.

Denote by $2D_k$ the disk concentric with D_k of twice the radius of D_k. We claim that the numbers $\#(A \cap 2D_k)$ satisfy

$$\sum \frac{\#(A \cap 2D_k)|I_k|}{1 + \text{dist}^2(0, I_k)} < \infty \tag{5.9}$$

Indeed, if $a_n \in 2D_k$ then the argument of the Blaschke factor

$$\frac{z - a_n}{z - \bar{a}_n}$$

increases by at least π on the diameter $d_k = 2D_k \cap \mathbb{R}$ of $2D_k$. Since overall that increase is at most

$$2\pi(\#(\Lambda \cap d_k) + 2) \lesssim \sum_{I_m \cap d_k \neq \emptyset} |I_m|$$

we obtain (5.9) from lemma 7.

Notice that

$$-\int_{T_k} \log_- |F(z)| d|z| \leqslant \int_{T_k} \log |F(z)| d|z| = \int_{T_k} \log |H_k(z)| d|z| \lesssim |I_k| \log \alpha_{n_k} \tag{5.10}$$

by Jensen's inequality, because F has only poles and no zeros in D_k. At the same time, since $F \in K_\theta$, it belongs to $H^2(\mathbb{C}_+)$ and is equal to $\theta\bar{G}, G \in H^2(\mathbb{C}_+)$ in $\mathbb{C}_-$. Denote by $T_k^{\pm}$ the upper and lower halves of T_k. Since the absolute value of an H^2 function is bounded by

$$\text{const} + \text{const} \, |y|^{-1/2}$$

inside the half-plane, we have

$$\int_{T_k} \log_+ |F(z)| d|z| \leqslant \int_{T_k^+} \log_+ |F(z)| d|z| + \int_{T_k^-} \log_+ |G(z)| d|z|$$

$$+ \int_{T_k^-} \log_+ |\theta(z)| d|z| \lesssim |I_k| + v_k, \tag{5.11}$$

where $\sum v_k/(1 + a_{n_k}^2) < \infty$ by lemma 5.

Since $H_k \neq 0$ in D_k, $\log |H|$ is harmonic in D_k. Hence its values on I_k can be recovered from the values of $\log |H_k| = \log |F|$ on T_k via the Poisson formula.

By (5.11), the Poisson integral of $\log_+ |F|$ will deliver a small contribution, i.e. on each I_k it will be equal to a function h_k^+ such that

$$\sum \int_{I_k} h_k^+(x) d\Pi(x) < \infty.$$

On the other hand, the Poisson integral of $\log_- |F|$ in D_k, restricted to I_k, will be equal to h_k^-, where $h_k^-(x) \asymp \log \alpha_{n_k}$ for all $x \in I_k$ by (5.10). Hence by (5.8)

$$\sum \frac{\int_{I_k} \log |H_k| dx}{1 + \operatorname{dist}^2(0, I_k)} = -\infty.$$

Furthermore, (5.9) implies that

$$\sum_k \frac{|I_k| \deg B_k}{1 + \operatorname{dist}^2(0, I_k)} < \infty.$$

Thus

$$\int_{\mathbb{R}} \log |F(x)| d\Pi \asymp \sum_k \frac{\int_{I_k} (\log |B_k(x)| + \log |H_k(x)|)\, dx}{1 + \operatorname{dist}^2(0, I_k)} \lesssim$$

$$\sum_k \frac{|I_k| \deg B_k + \int_{I_k} \log |H_k(x)| dx}{1 + \operatorname{dist}^2(0, I_k)} = -\infty$$

and we obtain a contradiction. $\qquad\square$

Remark 5. *Using results of [29] one can prove a slightly stronger statement that Λ itself is d-uniform.*

5.7. **Equivalence of completeness in L^p and C_W.** The theorem we discuss in this section relates the type problem to Bernstein's study of weighted uniform approximation, see [15] or [6].

Consider a weight W, i.e. a lower semicontinuous function $W : \mathbb{R} \to [1, \infty)$ that tends to ∞ as $x \to \pm\infty$. We define C_W to be the space of all continuous functions on $\mathbb{R}$ satisfying

$$\lim_{x \to \pm\infty} \frac{f(x)}{W(x)} = 0.$$

We define the norm in C_W as

$$\|f\| = \|fW^{-1}\|_\infty.$$

The following is a well known result by A. Bakan. For reader's convenience we supply a short proof.

Theorem 14 ([1]). *Let μ be a finite positive measure on $\mathbb{R}$. Then the system of exponentials $\mathcal{E}_d$ is complete in $L^p(\mu)$ for some $1 \leqslant p \leqslant \infty$ if and only if there exists a μ-weight $W \in L^p(\mu)$ such that $\mathcal{E}_d$ is complete in C_W.*

Proof. If $\mathcal{E}_d$ is complete in C_W for some μ-weight $W \in L^p(\mu)$ then for any bounded continuous function f there exists a sequence $\{S_n\}$ of finite linear combinations of exponentials from $\mathcal{E}_d$ such that S_n/W converges to f/W uniformly. Then S_n converges to f in $L^p(\mu)$. Hence $\mathcal{E}_d$ is complete in $L^p(\mu)$.

Suppose that $\mathcal{E}_d$ is complete in $L^p(\mu)$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a set of bounded continuous functions on $\mathbb{R}$, that is dense in C_W. Let $\{S_{n,k}\}$ be a family of finite linear combinations of exponentials from $\mathcal{E}_d$ such that

$$||f_n - S_{n,k}||_{L^p(\mu)} < 4^{-(n+k)}.$$

Denote

$$W = 1 + \sum_{n,k\in\mathbb{N}} 2^{n+k}|f_n - S_{n,k}|.$$

Notice that then $W \in L^p(\mu)$ and $S_{n,k}/W \to f_n/W$ uniformly as $k \to \infty$. Since $\{f_n\}$ is dense in C_W, $\mathcal{E}_d$ is complete in C_W. $\qquad\square$

6. Proofs of main results

6.1. **Proof of theorem 3.**

WLOG we can assume that B itself is a d-uniform sequence.

Fix a small $\varepsilon > 0$. Let $C = \{c_n\}$ be the sequence provided by lemma 3. Then by lemma 2 (apolied to C and w^2) there exists a finite positive measure $\nu = \sum \sigma_n \delta_{c_n}$ concentrated on C, satisfying

$$0 < \sigma_n < w(k) \text{ for } c_n = b_k \text{ and } \mathbf{G}_\nu^\infty \geqslant 2\pi(2d).$$

Let θ be the Clark inner function corresponding to ν. Then there exists a function in K_θ divisible by $S^{2\pi(2d-\varepsilon)}$ in the upper half-plane, i.e. $S^{2\pi(2d-\varepsilon)}h \in K_\theta$ for some $h \in H^2$: if $\widehat{\phi\nu} = 0$ on $[0, 2\pi(2d - \varepsilon)]$ for some $\phi \in L^\infty(\nu)$, put

$$h = \frac{1}{2\pi i}(1 - \theta)K\phi\nu.$$

By lemma 3, $D^*(C \setminus B) < d + \varepsilon$. Let J be an inner function such that $\{J = 1\} = C \setminus B$. By a version of the Beurling-Malliavin theorem, see [23] section 4.6, the kernel of the Toeplitz operator with the symbol $S^{2\pi(-d-\varepsilon)}J$ in H^∞ is non-empty, i.e. there exists a function $g \in H^\infty(\mathbb{C}_+)$ such that

$$S^{2\pi(-d-\varepsilon)}Jg \in \bar{H}^\infty.$$

Since

$$\bar{\theta}S^{2\pi(-d-\varepsilon)}JgS^{2\pi(2d-\varepsilon)}h = \bar{\theta}S^{2\pi(d-2\varepsilon)}Jgh \in \bar{H}^2,$$

we have

$$S^{2\pi(d-2\varepsilon)}Jgh \in K_\theta.$$

Since K_θ is closed under division by inner components, $S^{2\pi(d-2\varepsilon)}gh \in K_\theta$ and therefore

$$p = S^{2\pi(d-2\varepsilon)}Jgh - S^{2\pi(d-2\varepsilon)}gh = S^{2\pi(d-2\varepsilon)}(J-1)gh \in K_\theta.$$

By the Clark representation formula, $p = \frac{1}{2\pi i}(1-\theta)Kp\nu$, and since $1-\theta$ is outer, $Kp\nu$ is divisible by $S^{2\pi(d-2\varepsilon)}$ in $\mathbb{C}_+$. Notice that $p = (1-J)gh = 0$ on $C \setminus B = \{J = 1\}$ and $p \in L^\infty(\nu)$ on $B \cap C$. Therefore, if η is the restriction of ν on $B \cap C$, the existence of such p implies

$$\mathbf{G}_\eta^\infty \geqslant 2\pi(d - 2\varepsilon).$$

For any $\varepsilon > 0$, the measure η constructed as above will have a bounded density with respect to μ. Hence $\mathbf{G}_\mu^\infty \geqslant 2\pi d$. $\square$

6.2. **Proof of theorem 2.**

I) First, suppose that $\mathbf{G}_\mu^p \geqslant a$ for some $1 < p \leqslant \infty$. Then for any $d > 0$, $2\pi d < a$, there exists $f \in L^p(\mu)$ such that $\widehat{f\mu} = 0$ on $[0, 2\pi d]$. Let W be a μ-weight. Denote $V = W^{1/q}$ where $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\int V|f|d\mu < \infty.$$

Therefore by lemma 4 there exists a discrete measure $\nu = \sum \alpha_n \delta_{\lambda_n}$, $\Lambda = \{\lambda_n\} \subset \operatorname{supp}\mu$ such that $\hat{\nu} = 0$ on $[0, 2\pi d]$, V is a $|\nu|$-weight and ν satisfies the rest of the conditions of lemma 8. Then by lemma 8, Λ contains a d-uniform subsequence Λ' and α_n satisfy (5.6). Since $\sum V(\lambda_n)|\alpha_n| < C < \infty$, $V(\lambda_n) < C/|\alpha_n|$ for all n. It is left to notice that $\log W(\lambda_n) = q \log V(\lambda_n)$ and therefore

$$\sum_{\lambda_n \in \Lambda'} \frac{\log W(\lambda_n)}{1 + \lambda_n^2} < \infty.$$

II) Now suppose that $\mathbf{G}_\mu^p < d < a$ for some $1 < p \leqslant \infty$.

Since $\mathbf{G}_\mu^p < d$, by theorem 14 there exists a μ-weight W such that finite linear combinations of exponentials from $\mathcal{E}_d$ are dense in C_W. Suppose that there exists a d-uniform sequence $\Lambda = \{\lambda_n\} \subset \operatorname{supp}\mu$, satisfying (3.1). Then by theorem 3 there exists a measure $\nu = \sum \alpha_n \delta_{\lambda_n}$ such that $|\alpha_n| \leqslant W^{-1}(\lambda_n)/(1 + \lambda_n^2)$ and $\hat{\nu} = 0$ on $[0, d]$. Then the finite measure $W\nu$ annihilates all functions e^{ict}/W, $c \in [0, d]$. This contradicts completeness of $\mathcal{E}_d$ in C_W. $\square$

References

[1] BAKAN, A. *Representation of measures with polynomial denseness in $Lp(\mathbb{R}, d\mu), 0 < p < 1$, and its application to determinate moment problems,* Proc. Amer. Math. Soc., 136 (2008), no. 10, 35793589

[2] BENEDICKS, M. *The support of functions and distributions with a spectral gap,* Math. Scand., 55 (1984), 285–309

[3] BEURLING, A. *On quasianalyticity and general distributions,* Mimeographed lecture notes, Summer institute, Stanford University (1961)

[4] BEURLING, A., MALLIAVIN, P. *On Fourier transforms of measures with compact support,* Acta Math. 107 (1962), 291–302

[5] BEURLING, A., MALLIAVIN, P. *On the closure of characters and the zeros of entire functions,* Acta Math. 118 (1967), 79–93

[6] BORICHEV, A., SODIN, M.*Weighted exponential approximation and non-classical orthogonal spectral measures*, preprint, arXiv:1004.1795v1

[7] DE BRANGES, L. *Hilbert spaces of entire functions.* Prentice-Hall, Englewood Cliffs, NJ, 1968

[8] DE BRANGES, L. *Some Hilbert spaces of entire functions II,* Trans. Amer. Math. Soc. 99 (1961), 118-152

[9] DUFFIN, R., SCHAEFFER, A. *Power series with bounded coefficients,* American Journal of Mathematics, 67 (1945), 141154.

[10] DYM, H. *On the span of trigonometric sums in weighted L^2 spaces,* Linear and Complex Analysis Problem Book 3, Part II, Lecture Notes in Math., Springer, 1994, 87 – 88

[11] DYM H, MCKEAN H.P. *Gaussian processes, function theory and the inverse spectral problem* Academic Press, New York, 1976

[12] GELFAND, I, M., LEVITAN, B. M. *On the determination of a differential equation from its spectral function* (Russian), Izvestiya Akad. Nauk SSSR, Ser. Mat., 15 (1951), 309360; English translation in Amer. Math. Soc. Translation, Ser. 2, 1 (1955), 253304.

[13] HAVIN, V., JÖRICKE, B. *The uncertainty principle in harmonic analysis.* Springer-Verlag, Berlin, 1994.

[14] KEROV, S. V. *Equilibrium and orthogonal polynomials,* Algebra i Analiz, 12:6 (2000), 224237

[15] KOOSIS, P. *The logarithmic integral, Vol. I& II,* Cambridge Univ. Press, Cambridge, 1988

[16] KOOSIS, P. *A local estimate, involving the least superharmonic majorant, for entire functions of exponential type*, Algebra i Analiz 10 (1998), 4564; English translation in St. Petersburg Math. J. 10 (1999), no. 3, 441455.

[17] KREIN, M. G. *On an extrapolation problem of A. N. Kolmogorov,* Dokl. Akad. Nauk SSSR 46 (1945), 306–309 (Russian).

[18] KREIN, M. G. *On a basic approximation problem of the theory of extrapolation and filtration of stationary random processes,* Doklady Akad. Nauk SSSR (N.S.) 94, (1954), 13–16 (Russian).

[19] KREIN, M. G. *On the transfer function of a one- dimensional boundary problem of the second order* (Russian), Doklady Akad. Nauk SSSR (N.S.) 88 (1953), 405408.

[20] LEVIN, B. *Lectures on entire functions* AMS, Providence, RI, 1996

[21] LEVIN, B. *Completeness of systems of functions, quasi-analyticity and subharmonic majorants* (Russian), Issled. Linein. Oper. Teorii Funktsii, 17, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI) 170 (1989), 102156; English translation in J. Soviet Math., 63 (1993), no. 2, 171201.

[22] LEVINSON, N. *Gap and density theorems,* AMS Colloquium Publications, 26 (1940)

[23] MAKAROV, N., POLTORATSKI, A. *Meromorphic inner functions, Toeplitz kernels, and the uncertainty principle,* in *Perspectives in Analysis*, Springer Verlag, Berlin, 2005, 185–252

[24] MAKAROV, N., POLTORATSKI, A. *Beurling-Malliavin theory for Toeplitz kernels,* Invent. Math., Vol. 180, Issue 3 (2010), 443-480

[25] MITKOVSKI, M., POLTORATSKI, A. *Polya sequences, Toeplitz kernels and gap theorems,* Advances in Math., 224 (2010), pp. 1057-1070

[26] NIKOLSKII, N. K. *Bases of exponentials and the values of reproducing kernels*, Dokl. Acad. Nauk SSSR 252, (1980), 1316–1320

[27] NIKOLSKII, N. K. *Treatise on the shift operator*, Springer-Verlaag, Berlin (1986)

[28] POLTORATSKI, A. *Boundary behavior of pseudocontinuable functions*, Algebra i Analiz 5 (1993), no. 2, 189–210; translation in St. Petersburg Math. J. 5 (1994), no. 4, 389–408

[29] POLTORATSKI, A. *Spectral gaps for sets and measures*, preprint, arXiv:0908.2079.

[30] POLTORATSKI, A., SARASON, D. *Aleksandrov-Clark measures,* Recent advances in operator-related function theory, 1–14, Contemp. Math., 393, Amer. Math. Soc., Providence, RI, 2006

TEXAS A&M UNIVERSITY, DEPARTMENT OF MATHEMATICS, COLLEGE STATION, TX 77843, USA

E-mail address: alexeip@math.tamu.edu