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Completeness of Sets of Complex Exponentials

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TO NORMAN LEVINSON, WITH APPRECIATION AND AFFECTION

Since its appearance in 1940, Levinson's "Gap and Density Theorems" has been an inspiring source of methods and results in classical analysis. Aside from a few digressions, this expository account is restricted to just two of the topics in that book: conditions for L^p completeness on a closed finite segment of the real axis, and conditions for completeness on every segment shorter than the given one. The first topic is exemplified by Theorems 2 and 8, both of which are taken from (Levinson, 1940). The second topic originates in Theorem 30, also from (Levinson, 1940), and culminates in the Beurling–Malliavin formula for the completeness radius.

In accordance with this plan, the following will be excluded: completeness properties on an infinite interval, closure or completeness on regions other than a segment, closure or completeness relative to a weight other than 1, and study of the nature of the space spanned when the set is not complete. Also excluded are nearly all theorems in which a separation condition plays an essential role; insofar as completeness alone is the issue, such conditions are extremely restrictive, and do not reflect the structure of the main problems.

It is a major objective to present the subject simply, and minimal knowledge is presupposed. We take for granted such matters as Jensen's formula, Hadamard's factorization theorem, the dual relation of closure and completeness, and the fact that the Hilbert transform is an L^2 isometry. A summary of some of the additional results needed is given in Section 2.

1. NOTATION

Throughout this paper D, ϵ, and p denote constants, with $D \geqslant 0$, $\epsilon > 0$, $p \geqslant 1$. As usual, $1/p + 1/q = 1$. Although the quantity E introduced below depends on p, this dependence is not built into the notation. In general, "interval" means "finite interval."

We use $\{\lambda_n\}$ and $\{\mu_n\}$ for sequences of real or complex numbers. It is said that $\{\lambda_n\}$ has *completeness interval* I or $I(\lambda)$ if $\{\exp i\lambda_n x\}$ is complete L^p on every

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interval of length less than I and on no larger interval. By convention $I = 0$ if the set is not complete on any interval and $I = \infty$ if it is complete on every finite interval. When a λ is repeated, we require a zero of corresponding multiplicity in the entire function $F(z)$ which vanishes at the λ_n ; see (1). This means that, if closure rather than completeness is in question, the functions

$$e^{i\lambda x}, xe^{i\lambda x}, \ldots, x^{m-1}e^{i\lambda x}$$

are available for the approximation. Infinite multiplicity is excluded.

The set $\{\lambda_n\}$ has *excess* E or $E(\lambda)$ on a given closed interval if $\{\exp i\lambda_n x\}$ remains complete when E terms $\exp(i\lambda x)$ are removed but not when $E + 1$ terms are removed. The deficiency is defined similarly, or as a negative excess. By convention $E = \infty$ if arbitrarily many terms can be removed without losing completeness, and $E = -\infty$ if arbitrarily many terms can be adjoined without getting completeness. Like many of the fundamental concepts in the theory of completeness, the notion of excess is due to Paley and Wiener.

Often λ is used for a typical λ_n or, as in $E(\lambda)$, to suggest the whole sequence $\{\lambda_n\}$. This use will be clear from context. Likewise no confusion should result if we say "$\{\lambda\}$ is complete" instead of "$\{\exp(i\lambda_n x)\}$ is complete." Occasionally $I(\pm\lambda)$ is used to denote the completeness interval associated with $\{\pm\lambda_n\}$.

If $\{\lambda\}$ is complex then $n \geqslant 1$ and Λ is the *unsigned* counting function; that is, $\Lambda(t)$ is the number of λ satisfying $|\lambda| \leqslant t$. But if $\{\lambda\}$ is real then $-\infty < n < \infty$ and Λ is the *signed* counting function; that is, $\Lambda(u)$ is the number of λ on $(0, u]$, counted negatively for negative u. The restriction $\lambda \neq 0$ implied by this definition will do no harm.

The above conventions are sometimes altered when two real sequences λ and μ are being considered, or when the counting function for a single real sequence is being compared with Du. In such cases we may denote the signed counting function of λ and μ by Λ_λ and Λ_μ , respectively, and then

$$\Lambda(u) = \Lambda_\lambda(u) - \Lambda_\mu(u) \qquad \text{or} \qquad \Lambda(u) = \Lambda_\lambda(u) - Du,$$

as the case may be. The notation agrees with that above if $\{\mu\}$ is empty or if $D = 0$.

We denote by B the class of locally integrable complex-valued functions ϕ such that

$$\int_{-\infty}^{\infty} (|\phi(x)|/(1 + x^2))\, dx < \infty,$$

and by B^+ the subclass admitting a majorant $\phi(|x|)$ where $\phi(r)$ is increasing for $r \geqslant 0$ and $\phi(|x|) \in B$.

Many of the results were presented in lectures, reports, or seminars long before publication in any recognized journal. In known cases of this kind we attach the earlier date to the theorem and give the later reference elsewhere in the text.

2. Theorems for Future Reference

Here we state without proof a few results which have both a general importance in complex analysis, and a specific importance in the study of completeness. The first of these is

Theorem A (Paley and Wiener, 1934). *If $F(z)$ is an entire function of exponential type a satisfying $F(x) \in L^2$ on the real axis, then F can be represented in the form*

$$F(z) = \int_{-a}^{a} e^{izt} f(t)\, dt, \qquad f \in L^2[-a, a]. \tag{0}$$

A relatively simple proof and an extension to several complex variables can be found in (Plancherel and Pólya, 1937). Analogs for $p \neq 2$ are in (Boas, 1954) but are not used here.

The converse is easy; the above formula obviously defines a function of type a, and $F(x) \in L^2$ by the Plancherel theorem. In fact,

$$(1/2\pi) \int_{-\infty}^{\infty} |F(x + iy)|^2\, dx = \int_{-a}^{a} (e^{-yt} |f(t)|)^2\, dt \leqslant e^{2|y|a} \int_{-a}^{a} |f(t)|^2\, dt.$$

For the next result, let $P(z)$ be an entire function of finite type and assume without loss of generality that $P(0) \neq 0$. The Hadamard factorization theorem gives

$$P(z) = e^{\alpha z} \prod (1 - (z/\lambda_n))\, e^{z/\lambda_n},$$

where λ are the zeros of P. Following Levinson, we introduce

$$\tilde{\alpha} = \operatorname{Re} \alpha, \qquad 1/\tilde{\lambda} = \operatorname{Re}(1/\lambda), \qquad \tilde{P}(z) = e^{\tilde{\alpha} z} \prod (1 - (z/\tilde{\lambda}_n))\, e^{z/\tilde{\lambda}_n}.$$

The type of $\tilde{P}$ is denoted by $\tilde{T}$.

Theorem B (Levinson, 1935). *Suppose the above function P is of type T and satisfies* $\log |P(x)| \leqslant o(|x|)$. *Suppose also*

$$\log^+ |P(x)| \in B \quad \textit{or} \quad \lim_{r \to \infty} \int_1^r \frac{\log |P(-x)\, P(x)|}{x^2}\, dx \quad \textit{exists.}$$

Then the following hold:

(i) $\log |P(x)| \in B, \sum |\operatorname{Im}(1/\lambda_n)| < \infty$.

(ii) $|\tilde{P}(x)| \leqslant |P(x)|, \log |\tilde{P}(x)| \in B, \tilde{T} \leqslant T$.

(iii) *If Λ^- is the unsigned counting function for λ in the left half plane and Λ^+ for λ in the right half plane, then there exist*

$$\lim_{r\to\infty} (\Lambda^-(r)/r) = \lim_{r\to\infty} (\Lambda^+(r)/r) = D \leqslant T/\pi.$$

Since (ii) shows that $\tilde{P}$ satisfies the same hypothesis as P, the corresponding conclusion (iii) holds also for $\tilde{P}$. However, this need not be separately stated.

Part (i) follows from Carleman's formula, $|\tilde{P}(x)| \leqslant |P(x)|$ is obvious, and most of the difficulty is in the proof of (iii). Besides (Levinson, 1940) see (Titchmarsh, 1927; Paley and Wiener, 1934), and especially (Boas, 1954), which contains a clear, scholarly, and complete discussion of this whole subject.

When (iii) holds the Phragmén–Lindelöf function

$$h(\theta) = \limsup_{r\to\infty} \frac{\log |P(re^{i\theta})|}{r}$$

has a regular behavior, and this gives a clue to the following:

THEOREM C (Ahlfors and Heins, 1949). *Let P_1 and P_2 be functions of exponential types T_1 and T_2 such that* $\log |P_1(x)| \in B$. *Let the Phragmén–Lindelöf functions satisfy*

$$h_1(0) = 0, \qquad h_1(\pi) = 0, \qquad h_1(\pi/2) = T_1, \qquad h_2(\phi) = T_2.$$

Then the type T_{12} of P_1P_2 satisfies

$$T_{12} \geqslant (T_1^2 + 2T_1T_2 \sin\phi + T_2^2)^{1/2}.$$

For proof see (Boas, 1954). A result of Kahane and Rubel which is stated below (Theorem 57) shows that the condition $\log |P(x)| \in B$ is sharp.

The following deep result will be needed in only a few of our theorems, but is indispensable for final determination of the completeness interval:

THEOREM D (Beurling and Malliavin, 1961). *Let $P(z)$ be an entire function of exponential type such that* $\log |P(x)| \in B$. *Then there exists an entire function $M(z) \not\equiv 0$ of type $<\epsilon$ such that $M(x)$ and $M(x)\,P(x)$ are bounded on the real axis.*

In (Beurling and Malliavin, 1967) it is stated without proof that $M(x)$ can be chosen so as to have only real zeros λ_n satisfying $|\lambda_n - \lambda_m| > 1/\epsilon$ for $m \neq n$. For proof of the theorem, see (Beurling and Malliavin, 1962) or also (Kahane, 1962; de Branges, 1968). Because the proof is both difficult and nonconstructive, we make a distinction between results which use Theorem D and those which do not.

Once $M(x)\,P(x)$ is bounded, further reduction can be achieved with ease. For example, removing one zero from M makes $MP \in L^2$, and hence Theorem A gives a representation (1). Removing another zero, we can make f absolutely continuous, as seen in the proof of Theorem 1 below. These remarks lead to a striking reformulation of Theorem D, which is stated here because of its collateral interest:

THEOREM E (Beurling and Malliavin, 1961). *Let* $1 \leqslant p \leqslant \infty$, *let* $a > 0$, *and let* $B(a, p)$ *denote the class of functions of form* (1) *below with* $f \in L^p$. *Then the following classes of entire functions are identical:*

(i) *entire functions of exponential type satisfying* $\log |F(x)| \in B$,

(ii) *entire functions of form* F_1/F_2 *where* $F_2 \in B(a, p)$ *and* $F_1 \in B(b, p)$ *for some* b.

3. Closure and Completeness

The set of functions $\{e^{i\lambda_n x}\}$ is incomplete $L^p[-a, a]$ if there is a nontrivial function $f \in L^p$ orthogonal to all of them. This means that there is a function of form

$$F(z) = \int_{-a}^{a} e^{izt} f(t)\, dt, \qquad f \in L^p, \qquad \|f\|_p > 0, \tag{1}$$

such that $F(\lambda_n) = 0$. If there is no such function, then the set is complete. It follows that the study of completeness is virtually identical with the study of the zeros of certain entire functions. As stated in (Levinson, 1940) this observation goes back to (Szász, 1916); see Theorem 7.

The above set is closed $L^p[-a, a]$ if every $f \in L^p$ on this interval can be approximated in L^p norm by linear combinations of the functions $e^{i\lambda_n x}$. Duality shows that closure L^p is equivalent to completeness L^q for $1 < p < \infty$. The slight lack of symmetry for $p = 1$ or $p = \infty$ is overwhelmed by the effect of adding or removing a single λ, and is not emphasized here. Our objective is to show how seemingly nontrivial results can be obtained by very elementary arguments, if one goes back and forth at pleasure between closure and completeness.

THEOREM 1 (Schwartz, 1943). *Suppose* $\{e^{i\lambda_n x}\}$ *is not complete* L^p *in a given interval. Then if one term* $e^{i\lambda x}$ *is removed, the remaining set is not complete with respect to the class of functions which have derivatives in* L^p *and vanish at the ends of the interval.*

Roughly speaking, Theorem 1 indicates that dropping a single λ corresponds to a variation of the class from L^1 (the worst) to the class of absolutely continuous functions (the best). All theorems which pertain to the excess, E, do in fact exhibit this general character.

Theorem 1 for closure is due to Schwartz, but the dual given here is implicit in (Levinson, 1940). Let $F(\lambda) = 0$ in (1). Writing $e^{izt} = e^{i(z-\lambda)t}e^{i\lambda t}$ and integrating by parts, we get

$$F(z)/(z-\lambda) = \int_{-a}^{a} e^{izt}\left(-ie^{-i\lambda t}\int_{-a}^{t} e^{i\lambda s}f(s)\,ds\right)dt. \tag{2}$$

This gives Theorem 1.

THEOREM 2 (Levinson, 1940). *The completeness of the set $\{e^{i\lambda_n x}\}$ is not altered if a λ is changed to some other number, μ.*

For proof consider $F(z)/(z-\lambda) = G(z)/(z-\mu)$ where G is related to g in the same way as F is related to f. Transforming both integrals as in (2) gives Levinson's formula

$$g(t) = f(t) + i(\mu - \lambda)\, e^{-i\lambda t}\int_{-a}^{t} e^{i\lambda\tau}f(\tau)\,d\tau. \tag{3}$$

Since g is orthogonal to the new set, and $g \in L^p$, Theorem 2 follows. We shall find that the usefulness of (3) goes beyond the simple application indicated here.

THEOREM 3 (after Schwartz, 1943). *The set $\{e^{i\lambda_n x}\}$ is closed L^p on a given interval if, and only if, it is possible to approximate some function $e^{i\lambda x}$ other than those already present.*

In (Redheffer, 1961) this was deduced from results of Schwartz, but a simpler direct proof was found by Straus. The new proof changes the logic of the subject, since the results of Schwartz then follow from Theorem 3 as seen below.

Here is Straus' proof: If $e^{i\lambda x}$ can be approximated, dividing by $e^{i\lambda x}$ gives an approximation for 1 with nonzero exponents. Integration gives approximations for x, x^2, and so on, and we conclude that $e^{i\lambda x}p(x)$ can be approximated for any polynomial p. Since $e^{-i\lambda x}f(x)$ can be approximated by a polynomial, Theorem 3 follows.

A set of vectors in a normed linear space is *free* if no one of them is in the closure of the space spanned by finite linear combinations of the others. Theorem 3 gives

THEOREM 4 (Schwartz, 1943). *If a set $\{e^{i\lambda_n x}\}$ with distinct λ_n is not closed on a given interval, then it is free on that interval.*

A set of vectors is *linked* if every one of them is in the closure of the space spanned by the others.

THEOREM 5 (Schwartz, 1943). *On a given interval, every set* $\{e^{i\lambda_n x}\}$ *with distinct* λ_n *is either free or it is linked.*

If the set is not closed, the conclusion follows from Theorem 4. If it is closed, let one term $e^{i\lambda_n x}$ be in the closure of the space spanned by the others. Then the set remains closed when this term is dropped. By the dual of Theorem 2 it remains closed when any term is dropped. This shows that any term is in the closure of the space spanned by the other terms, and hence the set is linked.

We now give an elementary result of a somewhat different type. It will be recalled that $E = -\infty$ on a given interval if arbitrarily many terms can be adjoined without getting completeness. One could equally well define $E = -\infty$ to mean that infinitely many terms can be adjoined without getting completeness. In a like manner, $E = \infty$ could be defined to mean that infinitely many terms can be dropped without losing completeness.

THEOREM 6 (Redheffer, 1961; Peterson, 1973). *The two definitions of* $E = -\infty$ *are equivalent, and the two definitions of* $E = \infty$ *are equivalent.*

The result for $E = -\infty$ is deduced from Hadamard's factorization theorem in (Redheffer, 1968). Peterson's proof of the result for $E = \infty$ is as follows. Drop all λ_j which are 0, and then approximate 1 within 1 by a finite sum of terms $c_j \exp(i\lambda_j x)$. Next, drop some later λ_j, and approximate 1 within $\frac{1}{2}$ by another finite sum. Then drop some still later λ_j, and approximate 1 within $\frac{1}{3}$, and so on. The final set approximates 1 within $1/n$ for every n, hence is complete by Theorem 3.

As stated above, the general line of thought connecting closure with zeros of entire functions originates in (Szász, 1916). We conclude this introductory account by presenting the original Szász theorem, together with two others of similar character:

THEOREM 7 (Szász, 1916; Levin, 1956; Redheffer, 1961). *Let* $\lambda_n \neq 0$ *and*

$$\sum \frac{1}{|\lambda_n|^{1+\epsilon}} = \infty \qquad \text{or} \qquad \sum \left| \operatorname{Im} \frac{1}{\lambda_n} \right| = \infty \qquad \text{or} \qquad \sum \frac{1}{|\lambda_n|^{\theta(n)}} = \infty$$

where, in the last case, $\theta(n) \geqslant 0$, $\sum n^{-\theta(n)} < \infty$, *and* $|\lambda_n|$ *increases with n. Then* $I(\lambda) = \infty$.

Indeed, if $I(\lambda) < \infty$, then λ would be among the zeros μ of a function (1). The first result follows because the convergence exponent of the μ's is 1, and the second follows from Theorem B(i). The third holds because Jensen's theorem gives $|\mu_n| \geqslant \epsilon n$ as in (5) of the following section.

4. A Completeness Theorem of Levinson

Although the following theorem is somewhat more general than that in the reference cited, the essential ideas are unchanged. We take λ complex, with unsigned counting-function Λ:

Theorem 8 (after Levinson, 1936). *The sequence* $\{e^{i\lambda_n x}\}$ *is complete* L^p *on an interval of length* $2\pi D$ *if*

$$\limsup_{r\to\infty}\left(\int_1^r \frac{\Lambda(t)-2Dt}{t}\,dt+\frac{\log r}{q}\right)>-\infty.$$

Let $a=\pi D$ in (1) and, denying the conclusion, assume $F(\lambda_n)=0$. Clearly

$$|F(z)|\leqslant\int_{-(a-\delta)}^{a-\delta}e^{-yt}\,|f(t)|\,dt+\int_{\text{out}}e^{-yt}\,|f(t)|\,dt,$$

where δ is a small positive number, $z=x+iy$, and "out" means "the part of $[-a,a]$ outside the interval of integration used in the previous integral." By the Hölder inequality, if $\|f\|$ is small,

$$|F(z)|\leqslant e^{a|y|}\,|\,y\,|^{-1/q}(e^{-\delta|y|}+\eta), \tag{4}$$

where $\eta\to 0$ as $\delta\to 0$. Consequently,

$$\begin{aligned}\int_{-\pi}^{\pi}\log|F(re^{i\theta})|\,d\theta\leqslant&\int_{-\pi}^{\pi}ar\,|\sin\theta|\,d\theta-(1/q)\int_{-\pi}^{\pi}\log r\,d\theta\\&-(1/q)\int_{-\pi}^{\pi}\log|\sin\theta|\,d\theta+\int_{-\pi/3}^{\pi/3}\log(e^{-\delta r/2}+\eta)\,d\theta\\&+\int_{\text{out}}\log(1+\eta\theta)\,d\theta.\end{aligned}$$

The first two integrals on the right are easy, the third is convergent, and the last tends to 0 with η. But the fourth integral can be made less than any preassigned negative number by first choosing δ so η is small and then taking r to be large. Accordingly,

$$(1/2\pi)\int_{-\pi}^{\pi}\log|F(re^{i\theta})|\,d\theta\leqslant 2Dr-((\log r)/q)-\phi(r), \tag{5}$$

where $\phi(r)\to\infty$ as $r\to\infty$. Theorem 2 allows us to assume $|\lambda_n|\geqslant 1$. Then Jensen's formula gives

$$\int_1^r(\Lambda(t)/t)\,dt\leqslant(1/2\pi)\int_{-\pi}^{\pi}\log|F(re^{i\theta})|\,d\theta \tag{6}$$

when we note that $\Lambda(t)$ is the counting function for some, though perhaps not all, of the zeros of F. Relations (5) and (6) together contradict the hypothesis.

THEOREM 9 (Levinson, 1936). *The set* $\{e^{i\lambda_n t}\}$ *is complete* $L^p[-\pi, \pi]$ *if* $|\lambda_n| \leqslant |n| + 1/(2q)$, *and the constant* $1/(2q)$ *in this assertion is sharp.*

The completeness follows from Theorem 8. A counterexample is obtained by noting that the set $\{1, e^{ix(n+c)}\}$ is orthogonal to the function

$$(\cos \tfrac{1}{2}x)^{2c-1} \sin \tfrac{1}{2}x$$

and that this function belongs to $L^p[-\pi, \pi]$ if $c > 1/(2q)$.

The above simplification of Levinson's proof of Theorem 8 consists in neglecting certain estimates on the imaginary axis which are not really needed. The possibility of this simplification was noticed independently in (Levin, 1956) and (Redheffer, 1961). However, Levin also observed that calculations similar to those of Levinson lead to another theorem:

THEOREM 10 (after Levin, 1956). *Let* $\{\lambda_n\}$ *be the zeros of an entire function* $P(z)$ *of exponential type which satisfies*

$$\liminf_{|y|\to\infty} e^{-\pi|y|} \, |y|^{1/q} |P(iy)| > 0.$$

Then the set $\{e^{i\lambda_n x}\}$ *is complete* $L^p[-\pi, \pi]$.

Levin omits the factor $|y|^{1/q}$ and asserts completeness for all $p \geqslant 1$.

For proof, suppose the set is not complete. Then the Hadamard factorization theorem gives $F(z) = P(z)\, M(z)$ for some entire function M; and M has finite type by the criterion of Lindelöf. The estimate (4) shows that $|M(iy)| = o(1)$. By Theorem C (or also by Theorem B) the type of M is 0, and using $M(iy) = o(1)$ again, we conclude that $M = 0$; cf. Theorem 12 below. This completes the proof of Theorem 10.

5. THEOREMS OF BERNSTEIN TYPE

Besides generalizing familiar results on Fourier series, Levinson's Theorem 9 gives examples of sets which are complete L^p on a given interval but are not complete L^r for any $r < p$. Another application of Levinson's theorem was noticed in (Redheffer, 1953):

THEOREM 11 (Boas, 1936; Duffin and Schaeffer, 1937, 1938). *Let* $g(z)$ *be an entire function of exponential type* 1, *real on the real axis, and satisfying*

$$|g(x)|^2 \leqslant 1, \qquad -\infty < x < \infty.$$

Then this inequality remains valid if $|g'(x)|^2$ *is added to the left side.*

Failure of the conclusion means $|g(x_0)|^2 + |g'(x_0)|^2 > 1$ at some x_0. By reflection and translation we can assume that the graph of $g(x)$ intersects the graph of $\cos x$ as shown in Fig. 1. For $z = x$ the function

$$F(z) = (g(z) - \cos z)/(z - x_0) \tag{7}$$

belongs to L^2, hence by the Paley–Wiener theorem it can be represented in the form (0) with $a = 1$. On the other hand Fig. 1 shows that $F(x)$ in (7) has a zero λ_0 and further zeros λ_n, where the numeration can be so arranged that $|\lambda_n| \leqslant \pi n$ for $n = \pm 1, \pm 2, \ldots$. Theorem 9 indicates that $f = 0$, which contradicts the initial assumption.

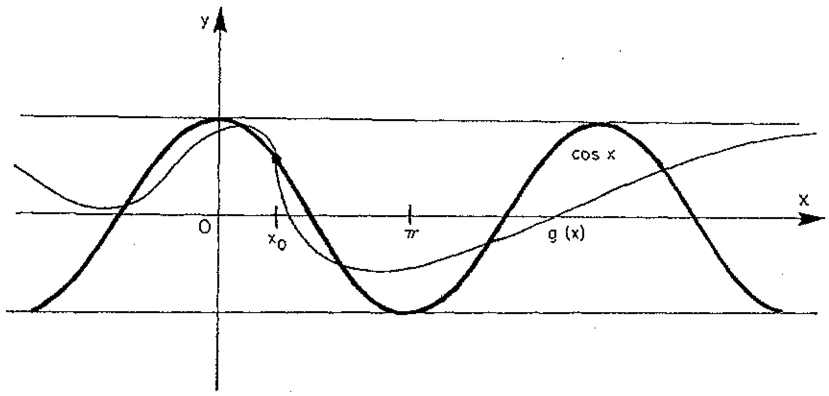

FIGURE 1

THEOREM 12 (Bernstein, 1923). *Let $f(z)$ be an entire function of type* 1 *which satisfies $|f(x)| \leqslant 1$ for x real. Then $|f'(x)| \leqslant 1$ for x real.*

For proof let $f(x) = u(x) + iv(x)$ and let α and β be real constants with $\alpha^2 + \beta^2 = 1$. By Theorem 11 we have

$$(\alpha u + \beta v)^2 + (\alpha u' + \beta v')^2 \leqslant 1, \qquad -\infty < x < \infty. \tag{8}$$

Thus $|\alpha u' + \beta v'| \leqslant 1$ and Theorem 12 follows from the converse to the Schwarz inequality.

THEOREM 13 (Achieser, 1953). *Let $f(x)$ be an entire function of type* 1 *which satisfies $|f(x)| \leqslant 1$ for x real. Let α and β be any real constants with $\alpha^2 + \beta^2 = 1$. Then $|\alpha f(x) + \beta f'(x)| \leqslant 1$ for x real.*

With $f = u + iv$ as above, the desired conclusion is $A\alpha^2 + 2B\alpha\beta + C\beta^2 \geqslant 0$ where A, B, C are readily written in terms of u, v, u', v'. That $A \geqslant 0$ and $C \geqslant 0$ follows from Theorem 12. The remaining condition $B^2 \leqslant AC$ turns out to be algebraically equivalent to the corresponding condition for (8). The latter holds, hence Theorem 13 holds.

The proof in (Achieser, 1953) is entirely different from that here, and gets Theorem 11 at the end rather than at the beginning.

Naturally, these theorems apply to functions of arbitrary type by change of scale. In particular, if the type is 0, Theorem 12 gives the conclusion $f'(x) = 0$, hence $f(x) = \text{const}$. This was used in the proof of Theorem 10.

6. Comparison of Excesses

In the following results it is very important that *no regularity* is assumed for the individual sequences $\{\lambda_n\}$ and $\{\mu_n\}$.

Theorem 14 (after Alexander and Redheffer, 1967). *We have* $I(\lambda) = I(\mu)$ *and* $E(\lambda) = E(\mu)$ *if*

$$\sum_{n=1}^{\infty} \frac{|\lambda_n - \mu_n|}{1 + |\operatorname{Im} \lambda_n| + |\operatorname{Im} \mu_n|} < \infty.$$

The result in the reference cited is slightly weaker, because it is based on a weaker inequality (9). The same applies to Theorem 19.

For proof, let us assume that $\{\lambda_n\}$ is incomplete L^p for given p and show $\{\mu_n\}$ is also incomplete. Starting with F in (1) we set $F_0 = F$, $f_0 = f$, and

$$f_n(x) = f_{n-1}(x) + i(\mu_n - \lambda_n)\, e^{i\lambda_n x} \int_{-a}^{x} e^{i\lambda_n t} f_{n-1}(t)\, dt.$$

Since this construction agrees with (3), the conclusion will follow if f_n has a limit f, $f \not\equiv 0$, $f \in L^p$.

By an inequality which is discussed in the next section,

$$\| f_n - f_{n-1} \| \leqslant \| f_{n-1} \| \, | \lambda_n - \mu_n | \min(I, |\sigma_n|^{-1}) \tag{9}$$

where $\sigma_n = \operatorname{Im} \lambda_n$. The desired convergence follows from this together with the hypothesis. Hence, $I(\mu) \leqslant I(\lambda)$, $E(\mu) \leqslant E(\lambda)$ and equality holds by symmetry. The result allows multiple roots.

Although Theorem 14 is much weaker than Theorem 9 when $\mu_n = n$, it was discovered by Peterson that the convergence criterion is sharp in the following sense: The class of weights $w(n)$ such that, with suitable indexing,

$$\sum | \lambda_n - \mu_n | \, w(n) < \infty \Rightarrow E(\lambda) = E(\mu),$$

are precisely those with $\inf w(n) > 0$. According to Peterson, a sequence $\{\lambda_n\}$

is canonically indexed if $0 \leqslant n < m$ implies $|\lambda_n| \leqslant |\lambda_m|$ and $|\lambda_{-n}| \leqslant |\lambda_{-m}|$. The sequence is separated if $|\lambda_n - \lambda_m| \geqslant \epsilon$ for $m \neq n$.

THEOREM 15 (Peterson, 1974). *If* $\inf w(n) = 0$ *there exist real, separated sequences* $\{\lambda_n\}$ *and* $\{\mu_n\}$ *such that, when* $\{\lambda_n\}$ *is canonically indexed,*

$$\sum |\lambda_n - \mu_n| w(n) < \infty \qquad \text{but} \qquad -\infty < E(\lambda) < E(\mu) < \infty.$$

The proof involves an explicit construction, making use of Theorem 63 below. As a contrast to Theorem 15, we have:

THEOREM 16 (Peterson, 1974). *There exists a weight with* $\inf w(n) = 0$ *such that*

$$\sum |\lambda_n - \mu_n| w(n) < \infty \Rightarrow |E(\lambda) - E(\mu)| < \infty.$$

Theorem 16 follows from (12) and Theorem 20. We can take $w(n) = 1$ for $n \neq j!$ and $w(n) = 1/\log(j + 2)$ for $n = j!$

The following result gives $E(\lambda) = E(\mu)$ without requiring $\lim |\lambda_n - \mu_n| = 0$:

THEOREM 17 (Elsner, 1969; Peterson, 1975). *Let* $p = 2$ *and let*

$$\operatorname{Re} \lambda_n = \operatorname{Re} \mu_n , \qquad |\operatorname{Im} \lambda_n - \operatorname{Im} \mu_n| \leqslant \text{const.}$$

Then $I(\lambda) = I(\mu)$ *and* $E(\lambda) = E(\mu)$.

Combining Theorem 14 with Theorem 17 gives a refinement of both. Further refinement is obtained by use of the following:

THEOREM 18 (Peterson, 1975). *Let* λ_n *be indexed so that* $|\lambda_n|$ *increases with* $|n|$, *let* $|\lambda_n - \mu_n| = O(1/|n|)$, *and let* $p = 2$. *Then* $I(\lambda) = I(\mu)$ *and* $E(\lambda) = E(\mu)$.

The proofs of Theorems 17 and 18 are discussed next.

As in the proof of Theorem 14 we assume that $\{\lambda_n\}$ is not complete and try to show the same for $\{\mu_n\}$. Without loss of generality $\{\lambda_n\}$ can be taken to be the zero set for $F(z)$ in (0). Theorem B then shows $\lim n/\lambda_n = \alpha$ as $|n| \to \infty$, where α is real. Since virtually all comparison theorems entail $|\lambda_n - \mu_n| = o(n)$ it follows that $\lim n/\mu_n = \alpha$. This gives the needed estimate in the complex plane for the canonical product associated with μ, and all that remains is to estimate the product on the real axis. A similar remark applies to many theorems given later, and may serve to show why estimates in the complex plane are not emphasized here.

To prove Theorem 17, let $\{\lambda_n\}$ be not complete, so that λ_n are the zeros of $F(z)$ in (0). We set $\lambda = \rho + i\sigma$, $\mu = \lambda + i\tau$ and assume $|\tau| \leqslant M$ where M is constant. Since it is possible to change first the terms with $\sigma \leqslant 0$, and then those with $\sigma > 0$, there is no loss of generality in assuming $\sigma \leqslant 0$ in the following estimates.

Clearly,

$$\left| 1 - \frac{x}{\mu} \right|^2 = \frac{(\rho - x)^2 + (\sigma + \tau)^2}{(\rho - x)^2 + (\sigma - M)^2} \left| \frac{\lambda}{\mu} \right|^2 \left| 1 - \frac{x + iM}{\lambda} \right|^2.$$

Since $\sigma \leqslant 0$ implies $|\sigma + \tau| \leqslant |\sigma - M|$, the factor in the center involving x does not exceed 1 in magnitude. For large $|n|$ Theorem B gives $|\rho_n| \sim \alpha |n|$, and it is easily checked that the product of the above expressions converges. This shows that the new function F^* associated with μ satisfies

$$|F^*(x)| \leqslant (\text{const}) \, |F(x + iM)|.$$

Hence $F^* \in L^2$ and $E(\mu) \leqslant E(\lambda)$. The conclusion follows by symmetry.

This surprisingly simple argument is due in the main to Peterson. The method of Elsner also involves comparison with $F(x + iM)$, but is more difficult, and requires the additional assumption that $|\sigma| \leqslant M$ for the terms changed.

In Theorem 18, which is discussed next, let us first change the imaginary part of λ_n to agree with that of μ_n. This can be done (under much lighter hypothesis) by Theorem 17. Theorem 17 also enables us to assume $|\sigma| \geqslant 1$, where $\lambda = \rho + i\sigma$ as above.

Accordingly, let $\mu = \lambda + \tau$ where τ is real. Then

$$\left| 1 - \frac{x}{\mu} \right|^2 = \frac{(\rho - x + \tau)^2 + \sigma^2}{(\rho - x)^2 + \sigma^2} \left| \frac{\lambda}{\mu} \right|^2 \left| 1 - \frac{x}{\lambda} \right|^2.$$

The hypothesis $|\lambda_n - \mu_n| = O(1/|n|)$ shows that the product of $|\lambda/\mu|$ converges, as before. Furthermore the factor in the center involving x can be written $1 + \eta$ where

$$|\eta| = \frac{|2\tau(\rho - x) + \tau^2|}{(\rho - x)^2 + \sigma^2} \leqslant (\text{const}) \frac{|\tau|}{1 + |\rho - x|}.$$

Here we have used $|\tau| \leqslant$ const and $|\sigma| \geqslant 1$. The desired condition

$$|F^*(x)| \leqslant (\text{const}) \, |F(x)|$$

will hold if

$$\sup_{-\infty<x<\infty} \sum_{-\infty}^{\infty} \frac{|\tau_n|}{1 + |\rho_n - x|} < \infty.$$

To check this, we have $|\tau_n| = O(1/|n|)$ by hypothesis, and Theorem B enables us to assume $|\rho_n| \sim (\text{const}) \, |n|$. Thus, $|\rho_n|$ has the order of magnitude of $|n|$. We consider terms of three types, characterized informally as follows:

$$|\rho| < |x|/2, \qquad |x|/2 \leqslant |\rho| < 2|x|, \qquad 2|x| \leqslant |\rho|.$$

The number of terms of the first type is $O(|x|)$, and the denominator has the order of $|x|$. The number of terms of the second type is $O(|x|)$, and the numerator has the order of $1/|x|$. For terms of the third type $|\rho - x| \geqslant |\rho|/2$, and a bound independent of x is obtained again. This completes the proof.

As pointed out by Peterson, other comparison theorems can be obtained by the same technique and in particular, the method gives a new proof of Theorem 14 for the case $p = 2$.

7. Comparison of Excesses, Continued

Proof of Theorem 14 hinges on an inequality of form

$$\left| e^{-i\lambda x} \int_{-a}^{x} e^{i\lambda t} f(t)\, dt \right|_p \leqslant \|f\|_p\, \kappa(p, I, \lambda),$$

where $I = 2a$ is the length of the interval $[-a, a]$ on which both norms are computed, and where it is assumed that the integral on the left vanishes at $x = a$. We set $\lambda = \rho + i\sigma$.

The results of this section are extremely sensitive to minute changes in the value of κ, and the crude estimate $\kappa = \min(I, |\sigma|^{-1})$ used for (9) is not appropriate. It can be shown that $\kappa(p, I, \lambda) = \kappa(p, I, |\sigma|)$ and

$$\kappa(2, I, \lambda) = \frac{I}{[\pi^2 + (I\sigma)^2]^{1/2}}, \qquad \kappa(\infty, I, \lambda) = \kappa(1, I, \lambda) = \frac{\tanh(I\sigma/2)}{\sigma}.$$

The above values for $p = \infty$ and $p = 1$ were found by Straus. By interpolation these values apply for any p and hence $\kappa(p, I, \lambda) \leqslant \min(I/2, 1/|\sigma|)$. Furthermore, if p is an even integer and f is real,

$$\kappa(p, I, 0) = \frac{p}{(p-1)^{1/p}} \frac{I}{2\pi} \sin\frac{\pi}{p}.$$

Since the known value is least for $p = 2$, we take $p = 2$ here. The methods apply to any p.

In the following theorems it is assumed that $\{\lambda_n\}$ is numbered so that $|\lambda_n|$ is an increasing function of n, $1 \leqslant n < \infty$.

Theorem 19 (after Alexander and Redheffer, 1967). *Let $\{\lambda_n\}$ have excess $E(\lambda)$ on an interval of length I and define*

$$\epsilon_n = \frac{|\mu_n - \lambda_n|}{[(\pi/I)^2 + |\operatorname{Im} \lambda_n|^2]^{1/2}}.$$

Suppose further that $p = 2$ *and*

$$\sum_{n=1}^{\infty} \frac{(1 + \epsilon_1)(1 + \epsilon_2) \cdots (1 + \epsilon_n)}{n^2} < \infty. \tag{10}$$

Then $I(\mu) \leqslant I(\lambda)$ *and* $E(\mu) \leqslant E(\lambda) + 1$.

For proof, we can remove enough λ's and μ's to make $E(\lambda) \leqslant 1$, so that the function $f = f_0$ in the argument leading to Theorem 14 satisfies the conditions of Theorem 1. It turns out that a recursion formula for f_n' holds, as well as for f_n, and the two together lead to Theorem 19.

Formulation of a symmetric criterion is easier when the term $| \operatorname{Im} \lambda_n |$ is not exploited, as in Theorem 20 below. We assume that m is a positive integer and state the following:

THEOREM 20. *If criterion* (10) *holds with* $p = 2$ *and*

$$\epsilon_n = (I/m\pi) \mid \lambda_n - \mu_n \mid, \tag{11}$$

then $I(\lambda) = I(\mu)$ *and* $| E(\lambda) - E(\mu)| \leqslant m$.

For proof, join λ_n to μ_n by a straight line which is divided into m equal parts. In m steps we can get from λ to μ, and Theorem 20 follows.

The hypothesis holds if ϵ_n in (11) satisfies

$$\limsup_{n \to \infty} \frac{\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n}{\log n} < 1. \tag{12}$$

However, this is a much more restrictive assumption than (10). If $\delta(n)$ is any function with $\delta(n) = o(n)$, then we can find a sequence ϵ_n satisfying (10) and satisfying $\epsilon_n > \delta(n)$ for infinitely many n. By contrast, (12) requires $\epsilon_n < \log n$.

An unsolved problem in the theory of completeness is to characterize all sequences $\{\epsilon_n(p, m)\}$ such that, with suitable numeration,

$$| \lambda_n - \mu_n | \leqslant \epsilon_n(p, m) \Rightarrow | E(\lambda) - E(\mu)| \leqslant m.$$

In this connection, we remark that the question whether $| \lambda_n - \mu_n | = o(1)$ suffices for $E(\lambda) = E(\mu)$ is left open in the theorems above. However, if

$$\lambda_n = n + (1/2q) + (\alpha/\log n),$$

it turns out that $E(\pm\lambda)$ drops by 1 when the constant α increases from values below $1/(2q)$ to values above. Accordingly, the condition

$$| \lambda_n - \mu_n | \leqslant \epsilon/\log | n |$$

is not enough to assure $E(\lambda) = E(\mu)$ even if the sequences λ and μ are very regular.

8. Levinson's Theorem on Nonharmonic Fourier Series

Theorem 9 shows that the set $\{\lambda_n\}$ is complete on $[-\pi, \pi]$ if $|\lambda_n - n| \leqslant 1/2q - \delta$, $\delta \geqslant 0$. If $\delta > 0$ and λ_n is real it was discovered by Levinson that a much stronger conclusion is true. To explain this result, we recall that the sequences $\{f_n\}$, $\{g_n\}$ of elements of a Hilbert space $\mathscr{H}$ form a biorthogonal system if

$$(f_n, g_m) = \delta_{mn}.$$

The system is closed if each system $\{f_n\}$, $\{g_n\}$ is closed; this means that linear combinations of the f_n, and also of the g_n, are both dense in $\mathscr{H}$. In that case there are two biorthogonal developments

$$f = \sum (f, g_n) f_n, \qquad f = \sum (f, f_n) g_n \tag{13}$$

valid whenever the second series converges. A theorem of Paley and Wiener asserts that if $\{f_n\}$ differs only slightly from a complete orthonormal sequence $\{\phi_n\}$, in the sense that

$$\left\| \sum a_n(\phi_n - f_n) \right\|^2 \leqslant \theta^2 \sum |a_n|^2 \tag{14}$$

holds for every finite sequence $\{a_n\}$ and for some constant $\theta < 1$, then there exists a sequence $\{g_n\}$ which forms with $\{f_n\}$ a closed biorthogonal system, and furthermore, the developments (13) converge. A proof based on simple calculations with inner products can be found in (Riesz-Nagy, 1955).

Paley and Wiener showed that this applies to $f_n = \exp(i\lambda_n x)$ with real λ_n if $|\lambda_n - n| \leqslant c$ with $c < 1/\pi^2$. Furthermore, the convergence and summability properties of the biorthogonal series are the same as those of the ordinary Fourier series, at least on every closed subinterval of $(-\pi, \pi)$.

For brevity, we summarize this entire complex of assertions by saying that $\{e^{i\lambda_n x}\}$ forms a basis for nonharmonic Fourier series.

Theorem 21 (Levinson, 1936). *Let $1 < p \leqslant 2$ and let $\{\lambda_n\}$ be a real sequence such that $|\lambda_n - n| \leqslant c$ where c is constant. If $c < 1/(2q)$ then $\{\exp(i\lambda_n x)\}$ forms a basis for nonharmonic Fourier series, but if $c = 1/(2q)$ then this conclusion no longer follows.*

The proof depends on a very careful estimation of a certain canonical product, and is much more difficult than any of the proofs discussed up to now. The reader interested in the proof (or in a more precise statement) is referred to (Levinson, 1940).

Since Levinson's proof is so difficult, considerable activity has centered on the question whether the L^2 result could be obtained from (14). (Of course this does not come to grips with the L^p case, which is the main problem.) We briefly summarize results on the value of c in $|\lambda_n - n| \leqslant c$ which ensure $\theta < 1$ in (14). The value $c < 1/\pi^2$ was obtained in (Paley and Wiener, 1934), where the entire theory originates. The value $c < (\log 2)/\pi$ was given in (Duffin and Eachus, 1942) and certain aspects of this analysis are generalized in (Duffin and Schaeffer, 1952). The correct value $c < \frac{1}{4}$ was finally obtained in (Kadec, 1964), thus giving another proof of Levinson's theorem when $p = 2$. It is seen in (Young, 1974) that the condition $|\lambda_n - n| < \frac{1}{4}$ is not sufficient and this insufficiency is extended to an interpolation problem in (Young, 1975).

9. A General Completeness Criterion

The following is an immediate consequence of Theorem A:

THEOREM 22 (after Paley and Wiener, 1934). *Let $p = 2$, let $Q(z)$ be an entire function of type T with zeros λ, and let $E(\lambda) > -\infty$ on an interval of length $2T$. Then $E(\lambda) < \infty$ if, and only if, the function $P(x) = e^{cx}Q(x)$ satisfies $P(x) = O(|x|^d)$ for some real constants c, d. In that case $E(\lambda)$ is the largest integer m such that*

$$\int_{-\infty}^{\infty} \frac{|P(x)|^2}{(1+x^2)^m}\, dx = \infty.$$

For real λ and even Q this follows from results in the reference cited; see also (Levin, 1956; Redheffer, 1953, 1967). If $P(x) = O(|x|^d)$ we divide P by a polynomial R of degree $d + 1$. Theorem A gives a representation for P/R which shows $E(\lambda) < \infty$. For the other assertions, assume without loss of generality that $E = -1$. Then the set is not complete and there is a function (0) with $a = T$ such that $F(z) = Q(z)\, M(z)$. Since $E \geqslant -1$ the function M has no zeros, hence $M(z) = e^{cz}$ essentially. This shows that $P(x)$ is bounded, and also $P(x) \in L^2$. On the other hand if $xP(x) \in L^2$ Theorem A gives a representation for $xP(x)$ which shows that $E \leqslant -2$, contrary to the hypothesis $E = -1$.

We apply Theorem 22 to the proof of the following interesting theorem:

THEOREM 23 (Levin, 1956). *Let $\lambda_n - n = G(n)$, $-\infty < n < \infty$, where $G(z)\, e^{-i\pi z}$ is an entire function of type $\leqslant \pi$ and $|G(x)|$ is bounded for x real. Then the set $\{e^{i\lambda_n x}\}$ is exact L^2 on $[-\pi, \pi]$.*

The statement in the conclusion means that the set is complete and has excess $E = 0$. Theorem 23 depends on a result of independent interest which reads as follows:

THEOREM 24 (Krein and Levin, 1949). *Let $|\lambda_n - Dn| \leqslant \epsilon$ and let*

$$P(z) = \lim_{R\to\infty} \prod_{|\lambda_n| \leqslant R} \left(1 - \frac{z}{\lambda_n}\right).$$

Then $P(z)$ satisfies a condition of the form

$$0 < A(y) \leqslant |P(x + iy)| \leqslant B(y), \qquad |y| > \epsilon,$$

if and only if there exists an entire function G, of type at most π, bounded on the real axis, and satisfying

$$(-1)^n G(n) = \lambda_n - D_n, \qquad -\infty < n < \infty.$$

This is stated without proof in the reference cited and proved in (Levin, 1956). The proof is not given here.

To establish Theorem 23, note that the role of $G(z)$ in Theorem 24 is taken by $G(z)\, e^{-i\pi z}$ in Theorem 23. The corresponding function $P(z)$ in Theorem 22 agrees with that in Theorem 24 hence has the same properties. Since $|\lambda_n - n|$ is bounded, we can add finitely many zeros and conclude from Theorem 9 that $E(\lambda) > -\infty$. Accordingly, Theorem 22 is applicable.

In the discussion of Theorem A we remarked that the L^2 properties of $F(x + iy)$ are independent of y. This means that the quantity m in Theorem 22 could have been referred to $P(x + iy)$ for fixed y, just as well as to $P(x)$. With P as in Theorem 24, the largest m for $P(x + iy)$ is clearly $m = 0$, and Theorem 23 follows.

10. A Measure of the Number of a-Points

We want to assign a precise numerical value to certain infinite sets of complex numbers. If the set of all integers has the "number" ∞, the set augmented by i and $2^{1/2}$ should have the number $\infty + 2$, the set of even integers should have the number $\frac{1}{2}\infty$, and so on.

In general, it is desired that two congruent sets shall have the same number, and that if m elements are adjoined or removed, the number assigned to the set shall increase or decrease by m, respectively. Here, for simplicity, we abandon the first requirement, and declare that the number to be attached to an infinite set $\{\lambda_n\}$ is

$$N(\lambda) = I(\lambda)\, \infty + E(\lambda), \qquad p = 2. \tag{15}$$

This assigns the value $\infty 1 + 0$ to $\{2\pi n\}$ instead of $\{n\}$, but the factor 2π has no significance aside from normalization. The measure is invariant under translation and reflection, and has other desirable properties by Theorem 2. We take $p = 2$ because $E(\lambda)$ depends on p.

Measure (15) gives $N\{n\} = 2\pi\infty + 0$ whereas $\{in\}$ has $I = E = \infty$. A rotation-invariant measure which gives finite I and E for a much larger class of sets can be constructed by rotation of the plane, the new x-axis being chosen in such a direction θ that $I(\lambda e^{-i\theta})$ is finite. It turns out that the direction is unique if it exists at all, and the resulting measure assigns the same number to any two congruent sets. With a slight change in the definition of "exceptional value," the following analysis allows use of this rotation-invariant measure. The details are not given here.

As the reader will recall, an a-point of a given function F is a root of the equation $F(z) = a$. We shall use the measure introduced above to compare the number of a-points with the number of b-points. It is assumed, naturally, that $a \neq b$.

If λ_n are the roots of the equation $F(z) = 2ie^{3iz} \sin z = 0$, then $N_0 = 2\infty + 0$ is the corresponding number $N(\lambda)$. But $F(z) = a \neq 0$ leads to a certain quadratic equation, and $N_a = 4\infty + 0$ holds for the new number. The value 0 is I-exceptional. As in the Nevanlinna theory, the exceptional value is well approximated; in fact, $|F(iy) - 0| \leqslant 2e^{-2y}$ as $y \to \infty$.

In general, a is I-exceptional of mass δ_a if

$$\limsup |y|^{-1} \log |F(iy) - a| = -\delta_a < 0 \qquad \text{for } y \to \infty \text{ or } y \to -\infty.$$

We set $\delta_a = 0$ if a is not I-exceptional.

As another illustration, let $F(z) = z^{-1} \sin z$. Here $N_0 = 2\infty - 1$ for the roots of $F(z) = 0$, but $N_a = 2\infty + 0$ for the roots of $F(z) = a \neq 0$. In general, a is *E-exceptional* if $F(x) - a \in L^2$ on the real axis. Again it is seen that a is well approximated; in fact, $\lim F(x) = a$ (by the Paley–Wiener theorem) if $F(z)$ is of finite type. In the latter case there can be *at most one exceptional value* of either kind, as the reader will verify.

The following results were presented in (Redheffer, 1961) but not published until 1967. There is some overlap of the methods with those in (Levin, 1956).

THEOREM 25. *Let $F(z)$ be an entire function of exponential type and let the sets of a-points and b-points of $F(z)$ have the respective numbers*

$$N_a = I_a\infty + E_a, \qquad N_b = I_b\infty + E_b,$$

with $|E_a|$ and $|E_b|$ finite. Then:

(i) $I_a + \delta_a = I_b + \delta_b$,

(ii) *$E_a = E_b$ unless a or b is E-exceptional,*

(iii) *$E_a = 0$ and $E_b < 0$ if b is E-exceptional.*

The relation implied by (ii) and (iii) is depicted graphically in Fig. 2. It turns out that all possibilities allowed by the figure and by (i) actually occur:

THEOREM 26. *Let a and b be two different complex numbers, let E_a and E_b be integers such that (E_a, E_b) lies on the graph of Fig. 2, and let I_a and I_b be positive* *numbers. Then there exists an entire function of exponential type such that its a-points and b-points, respectively, satisfy*

$$N_a = I_a\infty + E_a, \qquad N_b = I_b\infty + E_b.$$

The cases $E_a = E_b = \infty$, or $E_a = 0$, $E_b = -\infty$, or $E_a = -\infty$, $E_b = 0$ are also realizable, though they are off scale on the graph.

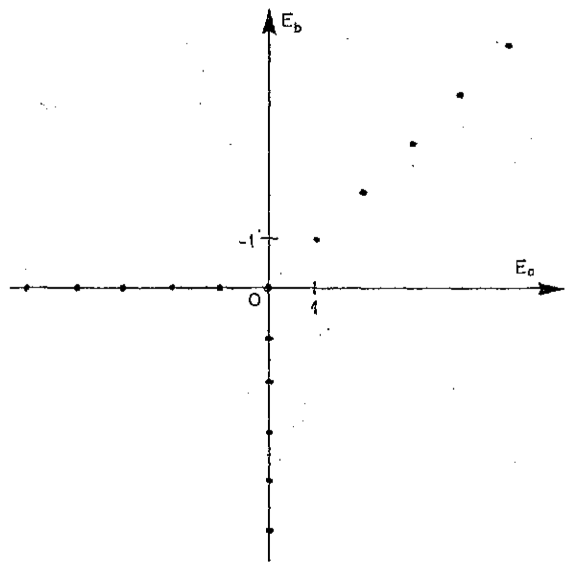

FIGURE 2

THEOREM 27. *Let $F(z)$ be an entire function of order less than 2 such that $|F(x)| = O(e^{\delta|x|})$ for each positive δ, $-\infty < x < \infty$. Let $I_a\infty + E_a$ be the number of a-points of F, where a runs through all complex values. Then E_a assumes at most two values, even if $E_a = \infty$ and $E_a = -\infty$ are allowed.*

Part (i) of Theorem 25 follows from the following:

THEOREM 28. *Let λ be the zeros of an entire function P of finite type which satisfies $\log^+|P(x)| \in B^+$. Then*

$$I(\lambda) = h(-\pi/2) + h(\pi/2),$$

where $h(\theta)$ is the Phragmén–Lindelöf function for P.

This holds if $\log^+|P(x)| \in B$ instead of B^+, the only difference being that we use the Beurling–Malliavin multiplier from Theorem D instead of the trivial multiplier given by Theorem 38 below. Although the weaker result suffices for the ends in view, the stronger result will be established here.

Multiplying $P(z)$ by e^{icz} for a suitable real constant c, we can ensure $h(-\pi/2) = h(\pi/2)$. Theorem D gives an entire function M of arbitrarily small type such that $M(x)\,P(x) \in L^2$, and the conclusion $I(\lambda) \leqslant 2h(\pi/2)$ follows from Theorem A.

On the other hand if the set is not complete in $[-a, a]$ there would be a function (1) such that $F(z) = P(z)\,M(z)$ for suitable M. The hypothesis $\log^+ |P(x)| \in B$ gives $\log |P(x)| \in B$ by Theorem B, the same holds for F, and hence for M. By Theorem C the types add, and this leads to a contradiction if a is too small.

Proof of Theorem 25(i) from Theorem 28 is easy and is omitted. To get the rest of Theorem 25, we compare the rates of growth of $F(x) - a$ and $F(x) - b$ on the real axis and use Theorem 22. Suppose, for example, that b is E-exceptional. Then $F - b \in L^2$ and Theorem 22 gives $E_b < 0$. But then $F(x) - a \sim b - a \neq 0$, and Theorem 22 gives $E_a = 0$. Discussion of other cases is similar.

Theorem 27 is a consequence of Theorem 25.

Examples for Theorem 26 are of form $e^{i\alpha z}F(\beta z + \gamma)$ where

$$F(z) = P(z)\sin z,\ (\sin z)/P(z),\ \sin z\cos(z)^{1/2},$$

$P(z)$ being a suitable polynomial. This accounts for all cases except $E_b = -\infty$, $E_a = 0$. The latter is included in Theorem 56.

An unsolved problem in this theory is the following: If $F(z)$ is an entire function of finite type, let λ_n be the a-points and μ_n the b-points. In what circumstances do we have $I(\lambda) = I(\mu)$? It follows from Theorem D that $I(\lambda) = I(\mu)$ when $\log |F(x)| \in B$, but the general case seems difficult.

11. Two More Theorems of Levinson

If $\{\lambda_n\}$ in some sense has density D, and $\lambda > 0$, most completeness theorems establish completeness only on every interval of length $<2\pi D$. In this respect the following is unusual:

Theorem 29 (Levinson, 1940). *Let $\lambda > 0$. Then the sequence $\{\lambda_n\}$ is complete on an interval of length $2\pi D$ if*

$$\limsup_{r\to\infty} \int_1^r \frac{A(u) - Du}{u^{1/2}} \left(\frac{1}{u} + \frac{1}{r}\right) du = \infty.$$

For proof, let $a = \pi D$, $p = 1$, and $G(z) = F(z^2)$ in (1). By inspection,

$$\log |G(re^{i\theta})| \leqslant \pi D r^2 |\sin 2\theta|.$$

A contradiction is now obtained by use of Carleman's theorem in the right half plane, much as a contradiction was obtained by Jensen's formula in Theorem 8.

As an illustration, if $\lambda_n \leqslant n - \delta n^{1/2}$ for some positive δ, the set is complete $L[-\pi, \pi]$.

THEOREM 30 (Levinson, 1935). *Let* $\{\lambda_n\}$ *be a real sequence. Then*

$$I(\lambda) \geqslant 2\pi \lim_{\xi \to 1+} \lim_{u \to \infty} \frac{\Lambda(\xi u) - \Lambda(u)}{\xi u - u}. \tag{16}$$

Let the limit (16) be denoted by D. If the conclusion fails then the set is incomplete on an interval of length $< 2\pi D$ and λ_n would be among the zeros of a function $F(z)$ in (1) with $a < \pi D$. However, by Theorem B, the zeros of such a function have a density, and furthermore the densities in the right and left half planes are equal. Since the Pólya maximum density agrees with the density when the latter exists, Jensen's formula gives an estimate which contradicts the hypothesis $a < \pi D$.

Historically, the first theorem of this general type is:

THEOREM 31 (Paley and Wiener, 1934). *If* $\lambda_n > 0$, *the completeness interval for the set* $\{e^{\pm i\lambda_n x}\}$ *satisfies* $I(\lambda) \geqslant 2\pi \limsup_{u \to \infty} \Lambda(u)/u$.

The essential difference between this result and that of Levinson is not in the use of the upper density, but in the fact that here an equal density is required on the negative real axis. Theorem 30 applies if there are no negative λ's at all.

For both theorems, by far the deepest part of the analysis is the proof that the zeros have a density; the completeness is a simple corollary. In the following section Theorem 30 is obtained without reference to the density, as a consequence of the easy part of Levinson's Theorem B. The main idea in this development is due to Koosis.

12. Lower Bounds for the Completeness Interval

If $R > 0$ and $\{\lambda_n\}$ is a real sequence with signed counting function Λ, we define

$$\Lambda^*(x, R) = \int_{x-R}^{x+R} \frac{\Lambda(u)}{u - x}\, du = \int_0^R \frac{\Lambda(x + t) - \Lambda(x - t)}{t}\, dt. \tag{17}$$

The same definition is used later when $\Lambda = \Lambda_\lambda - \Lambda_\mu$ is the difference of two counting functions, or when $\Lambda(u) = \Lambda_\lambda(u) - Du$. For the case considered here, $\Lambda(u)$ is increasing, and the right side of (17) shows that $\Lambda^*(x, R) \geqslant 0$.

THEOREM 32 (Koosis, 1960). *If $\{\lambda\}$ is real and incomplete on an interval of length $2\pi D$, there exists a positive function $K(x) \in B$ such that $\Lambda^*(x, R) \leqslant 2DR + K(x)$.*

If the conclusion fails then λ_n are among the zeros of a function $F(z)$ in (1) with $a = \pi D$. Jensen's theorem applied with x as origin gives

$$\Lambda^*(x, R) \leqslant (1/2\pi) \int_{-\pi}^{\pi} \log |F(x + Re^{i\theta})| \, d\theta - \log |F(x)|. \tag{18}$$

By inspection of the integral defining F,

$$|F(x + Re^{i\theta})| \leqslant e^{\pi DR |\sin\theta|}(\text{const}),$$

and hence the first term (18) does not exceed $2DR + \text{const}$. That $-\log |F(x)|$ belongs to B is a well-known consequence of Carleman's formula; see Theorem B.

If $x > 0$ and $y > 0$, as assumed henceforth, we have

$$\int_{x-y}^{x+y} \frac{\Lambda^*(t, R)}{t^2} \, dt \geqslant \frac{1}{(x + y)^2} \int_{x-y}^{x+y} \Lambda^*(t, R) \, dt.$$

Since Λ is nondecreasing a change of variable gives

$$(1/s) \int_{x-y}^{x+y} [\Lambda(t + s) - \Lambda(t - s)] \, dt \geqslant 2\Lambda(x + y - s) - 2\Lambda(x - y + s),$$

together with a similar inequality in the opposite direction. Combining these remarks with Theorem 32 gives:

THEOREM 33 (Redheffer, 1961). *Under the hypothesis of Theorem 32,*

$$\int_{x-y}^{x+y} \frac{K(t)}{t^2} \, dt \geqslant \frac{4Ry}{(x + y)^2} \left(\frac{\Lambda(x + y - R) - \Lambda(x - y + R)}{2y} - D \frac{x + y}{x - y} \right).$$

We now give a simple proof of Theorem 30. Let the lim sup in the theorem be denoted by D_1, so that the desired conclusion is $I \geqslant 2\pi D_1$, and let

$$D_1 > D_2 > D_3 > D_4 > 0.$$

Then $\Lambda(x + \eta x) - \Lambda(x) > D_2 \eta x$ holds for some small positive η and for a sequence $\{x_n\}$, $x_n \to \infty$. Without loss of generality we can assume $x_{n+1} > 2x_n$ and $\eta < \frac{1}{4}$. Then if $y_n = \eta x_n$ the intervals $(x_n - y_n, x_n + 2y_n)$ are nonoverlapping and furthermore

$$\Lambda(x_n + y_n) - \Lambda(x_n) > D_2 y_n .$$

If it is not the case that $I \geqslant 2\pi D_4$ the set is incomplete on an interval of length $2\pi D_3$, and therefore Theorem 33 can be applied with $D = D_3$. Since $D_2 > D_3$ we can find a small positive constant δ such that

$$\epsilon = \frac{D_2}{1 + 2\delta} - D_3 \frac{1 + \delta + \delta^2}{1 - \delta^2} > 0.$$

The choice

$$x + y - R = x_n + y_n, \qquad x - y + R = x_n, \qquad R = \delta y_n$$

leads to the inequality

$$\int_{x_n - \delta y_n}^{x_n + y_n + \delta y_n} \frac{K(t)}{t^2}\, dt \geqslant \epsilon\delta \frac{y_n{}^2}{x_n{}^2}.$$

Since the sum on n is divergent, this contradicts the fact that $K \in B$.

The above argument applies whenever the sum of $(y_n/x_n)^2$ diverges, and suggests the following:

THEOREM 34 (Beurling and Malliavin, 1961). *Let S denote the class of sequences $\{x_n, y_n\}$ such that $(x_n, x_n + y_n)$ are nonoverlapping intervals, $x_n > 0$, and $\sum (y_n/x_n)^2 = \infty$. Then if $\{\lambda\}$ is real*

$$I(\lambda) \geqslant 2\pi \sup_S \liminf_{n\to\infty} \frac{\Lambda(x_n + y_n) - \Lambda(x_n)}{y_n}.$$

When $\lambda_{n+1} - \lambda_n \geqslant c > 0$, Theorem 34 is implicit in (Koosis, 1960) and is also a simple consequence of Theorem 33. Indeed, within the context of the above proof of Levinson's theorem, the separation condition gives

$$\frac{\Lambda(x_n + y_n - \delta y_n) - \Lambda(x_n + \delta y_n)}{y_n} \geqslant \frac{\Lambda(x_n + y_n) - \Lambda(x_n)}{y_n} - \frac{2\delta}{c}.$$

Accordingly, we can ensure that this quantity exceeds D_2. If $y_n \geqslant \delta x_n$ holds for infinitely many n, the desired conclusion follows as in the above discussion of Levinson's theorem. Hence, we assume $y_n < \delta x_n$. The choice

$$x + y - R = x_n + y_n - \delta y_n, \qquad x - y + R = x_n + \delta y_n, \qquad R = \delta y_n$$

in Theorem 33 leads to a contradiction.

The only new problem posed by Theorem 34 results from the fact that the intervals $(x_n - \delta y_n, x_n + y_n + \delta y_n)$ might be overlapping. Given a sequence $\{x_n, y_n\}$ in S, denote the lim inf by D_1 (so that the desired conclusion is $I \geqslant 2\pi D_1$) and construct D_i and δ as in the above proof of Theorem 30. Again

$y_n \leqslant \delta x_n$ can be assumed, the sum on n is divergent by hypothesis, and if the intervals were nonoverlapping, a contradiction would be obtained as before.

The problem of overlapping was solved by Beurling and Malliavin and independently (though a little later) by the author. The latter solution is given here.

Let $0 < a < b$, let S be a finite set of nonoverlapping intervals $(x_n, x_n + y_n)$ each contained in (a, b), and let $s = \sum (y_n/x_n)^2$. If $\theta \geqslant 1$ there exists a subset $S(\theta)$ of expanded intervals $(x_n, x_n + \theta y_n)$ such that no point of (a, b) is in more than two intervals of $S(\theta)$ and such that the corresponding sum satisfies $s(\theta) \geqslant s$. To see this, assume $x_i < x_{i+1}$, let $(x_1, x_1 + y_1)$ be replaced by $(x_1, x_1 + \theta y_1)$ and let all other intervals of S that are wholly contained in the new interval be dropped. The first step increases s by the amount

$$\left(\frac{\theta y_1}{x_1}\right)^2 - \left(\frac{y_1}{x_1}\right)^2 = \left(\frac{y_1}{x_1}\right)^2 (\theta^2 - 1).$$

As for the second step, all other intervals wholly contained in $(x_1, x_1 + \theta y_1)$ must be contained in $(x_1 + y_1, x_1 + \theta y_1)$ since the original intervals are nonoverlapping. The sum of the squares of their lengths is therefore at most $(\theta y_1 - y_1)^2$, and dropping these intervals reduces s by at most

$$(y_1/x_1)^2 (\theta - 1)^2.$$

The net effect of both operations is to increase s. We now repeat the operation on the first interval not removed, and so on.

By expanding first to the left and then to the right we get a set of doubly expanded intervals such that no point of (a, b) is in more than three of them. After the intervals are thus expanded, Theorem 34 follows much as in the proof of Theorem 30.

13. Complete Sets of Zero Density

As already pointed out in (Levinson, 1940) one can have a set $\{\lambda_n\}$ of integers with arbitrarily small upper density and yet with $I(\lambda) = 2\pi$. For nearly 20 years it was conjectured that a set of zero density must have $I(\lambda) = 0$, but this was finally refuted by Kahane:

Theorem 35 (Kahane, 1959). *There exists a real set $\{\lambda_n\}$ of zero density such that $I(\lambda) = \infty$.*

Kahane's example has zeros of high multiplicity and leaves open the question whether a set of integers of zero density could have $I(\lambda) > 0$. This question was answered by Koosis.

THEOREM 36 (Koosis, 1960). *There exists a set of distinct positive integers* $\{\lambda_n\}$ *having zero density and* $I(\lambda) = 2\pi$.

Koosis' proof is based upon Theorem 32 and was, in fact, the motivation for Theorem 32.

Carrying out the details of Kahane's example shows that $I(\lambda) = \infty$ is compatible with

$$|A(u)| \leqslant \frac{|u|}{\log\log\log|u|} \qquad (|u| \to \infty).$$

Later it was seen that one can have $|A(u)| \leqslant |u|/(\log|u|)^{1/2}$; in fact, the following is true:

THEOREM 37 (Redheffer, 1961). *For* $u > 0$ *let* $H(u)$ *be a positive increasing function such that* $u^{-1}H(u)$ *is decreasing and*

$$\int_1^\infty (H^2(u)/u^3)\, du = \infty. \tag{19}$$

Then there exists a set of distinct positive integers λ_n *such that* $A(u) \leqslant H(u)$ *and* $I(\lambda) = 2\pi$.

This follows from Theorem 69, as seen in (Redheffer, 1968). If the λ_n are not required to be integers then one can have $A(u) \leqslant H(u)$ and $I(\lambda) = \infty$. Although the regularity conditions on H could be replaced by a requirement that $u^m H(u)$ be increasing and $u^{-m}H(u)$ decreasing for some constant m, it is not known whether (19) can be replaced by the weaker condition $H \in B^+$.

14. SIMPLE MULTIPLIERS

We shall presently obtain upper bounds for the completeness interval. The idea is to form a canonical product $G(z)$ with zeros λ and estimate G from above. If it is possible to construct a multiplier $M(z)$ of small type such that $M(x)\, G(x) \in L^2$ on the real axis, the Paley–Wiener theorem gives a representation for $M(z)\, G(z)$ of form (0) which shows that $\{\lambda\}$ is not complete.

For elementary arguments of this sort, the following suffices:

THEOREM 38 (Paley and Wiener, 1934; Ingham, 1934). *Let* $\phi \in B^+$. *Then there exists an entire function* $M(z) \not\equiv 0$, *of arbitrarily small type, such that* $|M(x)| \leqslant e^{-\phi(x)}$ *on the real axis.*

As pointed out in (Levinson, 1940), where an alternative proof is given, Theorem 38 is implicit in the work of Paley and Wiener. The theorem is rediscovered in (Boas, 1954), where reference is also made to (Ronkin, 1953).

However, a much simpler approach due to Ingham not only gives Theorem 38 as it stands, but allows a more general class of growth conditions. We shall establish the following:

THEOREM 39 (after Ingham, 1934). *For $x > 0$ let $\phi(x)$ be a positive, continuous increasing function such that $\phi(x)/x^2$ is integrable at $0+$ and $\phi(x)/x^3$ is integrable at ∞. Then there exists an even canonical product $M(z)$, with real zeros only, such that*

$$\log | M(x)| \leqslant 1 - \phi(| x |),$$

$$\log | M(iy)| \leqslant e\left(| y | \int_0^{|y|} \frac{\phi(u)}{u^2}\, du + y^2 \int_{|y|}^{\infty} \frac{\phi(u)}{u^3}\, du\right).$$

The basic construction

$$M(z) = \prod \frac{\sin(z/a_i)}{z/a_i} \tag{20}$$

is introduced in the reference cited, and the basic form of the inequality for $M(iy)$ is stated in (Redheffer, 1957). The latter has the improved constant 1 instead of e, but requires a regularity condition on ϕ which has been omitted here.

For proof, let $A(u) = [\phi(eu)]$ and form a sequence of positive real numbers $a_i \leqslant a_{i+1}$ with counting function $A(u)$. The hypothesis on ϕ ensures $A(u) = o(u^2)$. Since $| \sin u | \leqslant \min(1, u)$ for $u > 0$, $\log | M(x)|$ for $x > 0$ does not exceed

$$\log \prod_{a_n \leqslant x} \frac{a_n}{x} = \int_0^x \log \frac{u}{x}\, dA = -\int_0^x \frac{A(u)}{u}\, du < -\int_{x/e}^{x} \frac{\phi(eu) - 1}{u}\, du.$$

Since ϕ is monotone, the result is at most $1 - \phi(x)$.

An integration by parts gives

$$\log | M(iy)| = \int_0^{\infty} \frac{A(u)}{u} \left(\frac{y}{u} \coth \frac{y}{u} - 1\right) du \leqslant \int_0^{\infty} \frac{\phi(t)}{t} \left(\frac{ey}{t} \coth \frac{ey}{t} - 1\right) dt.$$

The desired estimate follows from $s \coth s - 1 \leqslant \min(s, s^2/3)$, which holds for $s > 0$ by use of the Taylor series for e^s and e^{-s}.

To get Theorem 38, redefine ϕ so as to be 0 on a long interval $[0, c]$ and multiply the resulting function M by a small constant.

Another approach to theorems of this type is to consider the canonical product associated with a zero distribution $A(u) = [L(u)]$, where $L(u)$ has the form

$$L(u) = u \int_0^u A(t)\, dt.$$

This method is used in (Koosis, 1958) to get an independent proof of Theorem 38.

Although Mandelbrojt's form of Theorem 38 is not needed here, it is presented because of its collateral interest:

THEOREM 40 (Mandelbrojt, 1963). *The function $M(z)$ in Theorem 39 can be constructed so as to satisfy*

$$\log|M(z)| \leqslant \frac{4}{\log 2}\left(|y|\int_0^{|y|}\frac{\phi(u)}{u^2}\,du + y^2\int_{|y|}^{\infty}\frac{\phi(u)}{u^3}\,du\right) - \phi(|z|) + 1.$$

The following exposition is due to Koosis (private communication). Start from the elementary inequalities

$$|\sin z| \leqslant \min(e^{|y|}, e^{y^2/2}); \qquad |(\sin z)/z| \leqslant \min(e^{|y|}, e^{y^2/2})$$

to deduce, for $t > 0$, that

$$\log\left|\frac{\sin(z/t)}{z/t}\right| \leqslant \min\left(\frac{|y|}{t}, \frac{y^2}{2t^2}\right) - \log^+\frac{|z|}{t}.$$

By partial integration, if $A(t)$ is increasing,

$$\int_0^{\infty}\log\left|\frac{\sin(z/t)}{z/t}\right| dA(t) \leqslant |y|\int_0^{|y|/2}\frac{A(t)}{t^2}\,dt + y^2\int_{|y|/2}^{\infty}\frac{A(t)}{t^3}\,dt - \int_0^{|z|}\frac{A(t)}{t}\,dt.$$

The choice $A(x) = [\phi(2x)/\log 2]$ gives the conclusion.

The relevance of Theorem 38 to completeness is illustrated by the following:

THEOREM 41 (Schwartz, 1943; Redheffer, 1961). *If $\{\lambda_n\}$ is a sequence of complex numbers such that $\sum 1/|\lambda_n|$ converges, then $I(\lambda) = 0$.*

The proof for real λ_n by Schwartz is based on the product

$$S(z) = R(z)\prod_{|n|\geqslant m}\frac{\sin(\pi z/\lambda_n)}{(\pi z/\lambda_n)},$$

where R is a suitable rational function. This proof does not generalize to the complex case; in fact, there are complex sequences satisfying the hypothesis of Theorem 41 such that $S(z)$ cannot be a factor of any function of form (1). Another proof in (Koosis, 1958) uses canonical products instead of sines, but also applies only if $\{\lambda_n\}$ is real.

The result for complex λ_n follows by consideration of

$$Q(z) = \prod\left(1 - \frac{z^2}{\lambda_n^2}\right), \qquad Q^+(r) = \prod\left(1 + \frac{r^2}{|\lambda_n|^2}\right).$$

Integration by parts gives a formula for Q^+ which shows that

$$\int_0^\infty \frac{\log Q^+(r)}{r^2}\, dr = \pi \int_0^\infty \frac{\Lambda(u)}{u^2}\, du.$$

Here $\Lambda(u)$ is the number of λ_n satisfying $|\lambda_n| \leqslant u$ and the latter integral converges by hypothesis. Since $|Q(z)| \leqslant Q^+(r)$ for $|z| = r$, and since $Q^+(r)$ is increasing, we conclude that $\log |Q(x)| \in B^+$. The same calculation shows that Q^+, and hence Q, has zero type. Theorem 38 gives a function M of arbitrarily small type such that $M(x)\, Q(x) \in L^2$, and the conclusion follows from the Paley–Wiener theorem.

15. The Graphical Estimation of Canonical Products

Let λ_n be a sequence of positive numbers, $\Lambda(u)$ the number of λ_n on $(0, u)$, and

$$Q(z) = \prod_{n=1}^\infty \left(1 - \frac{z^2}{\lambda_n{}^2}\right), \qquad \sum_1^\infty \frac{1}{|\lambda_n|^2} < \infty.$$

Multiplying $Q(z)$ by a rational function which has limit 1 at ∞ enables us to move any finite number of λ's. Hence, the assumption that $\lambda_n > 0$ will involve no loss of generality.

If $D = 0$ and

$$K(x, u) = \frac{2x^2}{u(x^2 - u^2)} = \frac{d}{du} \log \frac{u^2}{|x^2 - u^2|},$$

the equation

$$\log |Q(x)| = \int_0^\infty K(x, u)[\Lambda(u) - Du]\, du \tag{21}$$

follows by partial integration; the integral is a principal value at $u = x$ and we assume, of course, that x does not coincide with any λ. The truth of (21) for arbitrary D now follows from the fact that the integral of $K(x, u)\, u$ is zero. Since Q is even, we take $x > 0$.

Clearly $K(x, u) > 0$ for $0 \leqslant u < x$ and $K(x, u) < 0$ for $u > x$. This makes it possible to estimate $\log |Q(x)|$ *by inspection*, when suitable bounds for $\Lambda(u) - Du$ are known. Such bounds are readily obtained from bounds for $|\lambda_n - n|$; in fact, if h is constant,

$$|\lambda_n - n| \leqslant h \Rightarrow -h - 1 < \Lambda(u) - u \leqslant h. \tag{22}$$

We introduce constant bounds L^-, L^+, R^-, R^+, where L means "left of x" and R means "right of x," as follows:

$$-L^- \leqslant \Lambda(u) - Du \leqslant L^+, \qquad -R^- \leqslant \Lambda(u) - Du \leqslant R^+$$

for $0 < u < x$ and for $u > x$, respectively. The value of $\log |Q(x)|$ is maximized if $\Lambda(u) - Du$ is as large as possible at the left of x, and as small as possible at the right; cf. Fig. 3.

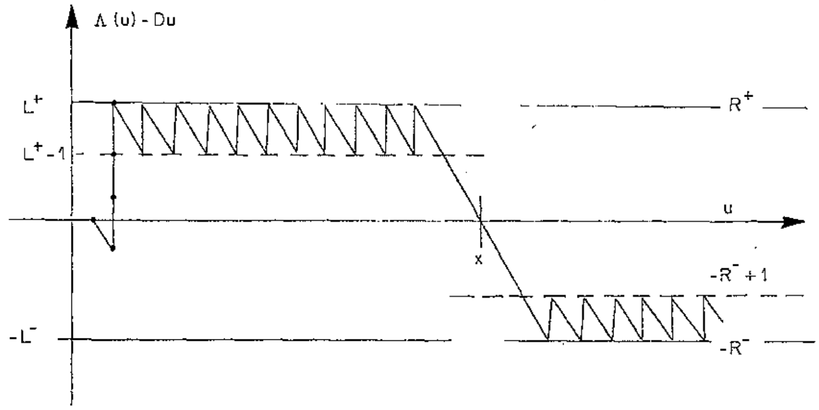

FIGURE 3

For any positive constant ϵ,

$$\int_1^{x-\epsilon} K(x, u)\, du = 3 \log x - \log \epsilon, \qquad \int_{x+\epsilon}^{\infty} K(x, u)\, du = -\log x + \log \epsilon,$$

aside from a term which remains bounded as $x \to \infty$. If the graph of $\Lambda(u) - Du$ crosses the line $u = x$ at (x, u_0), a short calculation shows that $u_0 = 0$ can be assumed. The extremizing function for the part of the integral near x is a line of slope $-D$, and the value of the integral is bounded as $x \to \infty$.

Because of the sawtooth effect shown in the figure, L and R behave in our calculations like $L - \frac{1}{2}$ and $R - \frac{1}{2}$, respectively. Accordingly,

$$\log |Q(x)| \leqslant [3(L^+ - \tfrac{1}{2}) + (R^- - \tfrac{1}{2})] \log x = (3L^+ + R^- - 2) \log x \quad (23)$$

aside from a constant.

In a like manner, if $|x - \lambda_n| \geqslant \epsilon$ for all n,

$$\log |Q(x)| \geqslant [-3(L^- - \tfrac{1}{2}) - (R^+ - \tfrac{1}{2})] \log x = -(3L^- + R^+ - 2) \log x \quad (24)$$

aside from a term of order $\log \epsilon$ + constant.

Setting $L^+ = R^+ = h$, $L^- = R^- = h + 1$ in agreement with (22), we get the following (recall that λ is real):

THEOREM 42 (after Paley and Wiener, 1934). *Let* $|\lambda_n - n| \leqslant h$ *where* $h > 0$ *is constant. Then the above function* $Q(x)$ *satisfies*

$$|xQ(x)| \leqslant (\text{const})\,|x|^{4h}, \qquad |xQ(x)| \geqslant (\text{const})\,|x|^{-4h}$$

for $|x| \geqslant 1$, *provided in the second case* $|x - \lambda| \geqslant \epsilon$ *for all* λ.

The first result is in the reference cited but the second is replaced by a lower bound for $|Q(x + i\epsilon)|$ which serves the same purpose. The Paley–Wiener proof is based on calculations involving the gamma function; the elementary proof given here follows (Redheffer, 1954).

For almost 20 years it was thought that the result of Theorem 42 is sharp. However, the theorem is sharp only if the whole class of functions $Q(x)$ is considered, the choice of Q for given x being allowed to depend on x. The growth for a single function $Q(x)$ is given by the following:

THEOREM 43 (Redheffer, 1954). *Any given function* $Q(x)$ *in Theorem* 42 *satisfies* $xQ(x) = o(x^{4h})$ *as* $x \to \infty$, *and if* $\delta(x) \to 0$, *there exists a function* $Q(x)$ *satisfying the hypothesis and also satisfying* $xQ(x) > \delta(x)\, x^{4h}$ *for a sequence* $x = x_i \to \infty$.

For proof, if $Q(x)$ is about as large as x^{4h-1}, it turns out that the graph of $\Lambda(u) - Du$ cannot differ a great deal from the graph shown in Fig. 3. This behavior for early values of x prevents the desired behavior for later values and leads to Theorem 43. An example for $\delta(x)$ is obtained in the course of the calculation.

In these results, a constant bound for $|\Lambda(u) - Du|$ was needed to get a polynomial majorant for $|Q(x)|$. No weaker bound will do, as shown by the following:

THEOREM 44 (Koosis, 1958). *If* $\lim H(u) = \infty$ *as* $u \to \infty$, *there exists a real zero distribution such that* $|\Lambda(u) - Du| \leqslant H(u)$ *for large* u, *but* $\limsup_{x\to\infty} x^{-m}Q(x) = \infty$ *for every constant* m.

Since we can find a minorant $h \leqslant H$ such that h increases, $h(u)/u$ decreases, and $\lim h(u) = \infty$, the result follows from Theorem 51(ii) below.

The real sequence $\{\lambda_n\}$ is said to be *convex* if the graph of the polygonal line joining the points (λ_n, n) is a convex curve; this line is called the *graph* of the sequence.

THEOREM 45 (Koosis, 1958). *If the graph of* $\{\lambda\}$ *is convex from some point on, the above function* $Q(x)$ *is dominated by a polynomial.*

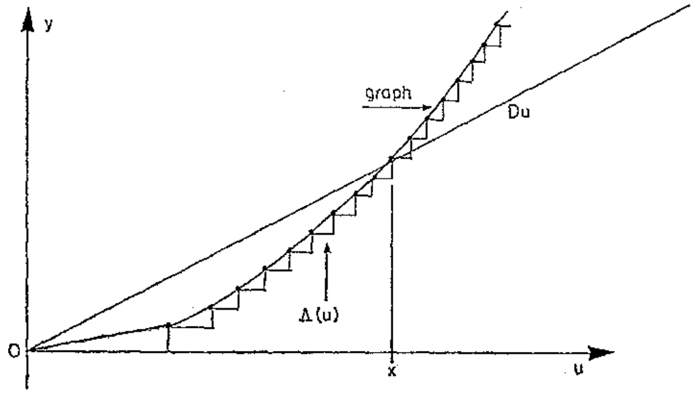

FIGURE 4

For proof, let the graph be convex from λ_m on, and remove all preceding λ's, thus dividing $Q(x)$ by a polynomial. The new graph is convex on $[0, \infty)$ and goes through the origin as shown in Fig. 4.

At any given x the line $y = Du$ through the point of the graph with ordinate x lies above the graph on $(0, x)$, but below the graph on (x, ∞). Accordingly, the new Λ satisfies $\Lambda(u) \leqslant Du$ on $(0, x)$ and $\Lambda(u) \geqslant Du - 1$ on (x, ∞). The part of the integral from $x - 1$ to $x + 1$ is bounded above by 2, approximately, and the term -1 on $(x + 1, \infty)$ contributes $\log x$. Hence,

$$\int_0^\infty \Lambda(u)\, K(x, u)\, du \leqslant \int_0^\infty Du\, K(x, u)\, du + 2 + \log x.$$

Since the integral of Du is 0, Theorem 45 follows. The proof in (Koosis, 1958) is different from that given here.

The above remarks have been developed for an even function $Q(z)$, in part for simplicity, and in part to respect the early history of the subject. We now discuss the extension to arbitrary real λ.

Let $\{\lambda_n\}$ be a real sequence satisfying

$$|\lambda_n - n| \leqslant h, \qquad -\infty < n < \infty$$

and form the product

$$P(z) = \lim_{r\to\infty} \prod_{|\lambda_n| \leqslant R} (1 - (z/\lambda_n)).$$

Just as in the previous discussion, if D is constant

$$\log |P(x)| = \int_{-\infty}^\infty [\Lambda(u) - Du]\, K(x, u)\, du,$$

where the integral is a principal value at 0, x, and ∞ and where

$$K(x, u) = \frac{x}{u(x-u)} = \frac{d}{du}\log\left|\frac{u}{x-u}\right|.$$

By moving a finite number of λ's we can ensure that $\lambda_n > 0$ for $n \geqslant 0$ and $\lambda_n < 0$ for $n < 0$. When this is done, $|\lambda_n - n| \leqslant h$ gives

$$-h \leqslant \Lambda(u) - u \leqslant h + 1.$$

Since the integral of $K(x, u)\,du$ is 0 we can add a constant to Λ; this is also obvious from the initial form of the expression as a Stieltjes integral. If $\frac{1}{2}$ is added, the new Λ satisfies $|\Lambda(u)| \leqslant h + \frac{1}{2}$. However, because of the sawtooth effect noted above, $\Lambda(u)$ will behave in our calculations as if $|\Lambda(u)| \leqslant h$.

The points x and 0 divide the axis into three intervals. The contribution to the integral of $K(x, u)\,du$ from the interval $(0, x)$ is essentially $2\log x$, and the contributions from the two unbounded intervals are each, essentially, $-\log x$. Hence if $\Lambda(u) - Du$ is chosen so as to maximize the integral at given x, we get $4h\log|x|$ or $-4h\log|x|$ for the principal term, just as before.

We summarize as follows:

THEOREM 46. *Let $\{\lambda_n\}$ be a real sequence satisfying $|\lambda_n - n| \leqslant h$ where $h > 0$ is constant and let $P(z)$ be the corresponding canonical product introduced above. Then there exists a positive function $\delta(x)$ such that*

$$|P(x)| \leqslant \delta(x)\,|x|^{4h}, \qquad |P(x)| \geqslant \frac{1}{\delta(x)\,|x|^{4h}}, \qquad \lim_{|x|\to\infty}\delta(x) = 0,$$

provided in the second case $|x - \lambda| \geqslant \epsilon$ for all λ.

16. APPLICATIONS

To see the connection of Theorem 46 with completeness, let m terms λ be removed. The new canonical product has growth $|x|^{4h-m}$, hence belongs to L^2 if $m > 4h + \frac{1}{2}$. A brief calculation shows that $P(z)$ has the same type as $\sin \pi z$, namely, π. It follows from the Paley–Wiener theorem that the new set is not complete on $[-\pi, \pi]$.

Hence, the excess cannot exceed $4h + \frac{1}{2}$.

We now estimate the deficiency. If the set is not complete $L^2[-\pi, \pi]$, there is an entire function $G(z)$ such that

$$P(z)\,G(z) = \int_{-\pi}^{\pi} f(t)\,e^{izt}\,dt, \qquad f \in L^2, \qquad f \not\equiv 0.$$

It is not difficult to show that the type of G is 0, and G has only finitely many zeros by Theorem 9 or by Theorem 63 below. Accordingly, G is a polynomial.

By the Plancherel theorem, $PG \in L^2$. If the lower bound of Theorem 46 were valid on the whole real axis, the degree m of G would have to satisfy $m < 4h - \frac{1}{2}$. The same result follows from Theorem 46 as it stands, because each interval $[n, n+1)$ contains an interval of fixed, positive length 3ϵ, free of zeros of P. The integral estimate applies to the middle third of such intervals and gives the desired conclusion.

Hence, the deficiency $m + 1$ is less than $4h + \frac{1}{2}$.

Upon using Theorem 17 to extend the result to complex λ we get the following:

THEOREM 47 (after Paley and Wiener, 1934). *For $-\infty < n < \infty$ let $\{\lambda_n\}$ be a sequence of complex numbers satisfying $|\lambda_n - n| \leqslant h$ where h is constant. Then $I(\lambda) = 2\pi$ and the L^2 excess on $[-\pi, \pi]$ satisfies*

$$-(4h + \tfrac{1}{2}) < E(\lambda) \leqslant 4h + \tfrac{1}{2}.$$

The result in the reference cited is given only for real sequences satisfying $\lambda_0 = 0$, $\lambda_{-n} = -\lambda_n$. Paley and Wiener do not construct an example to show that the theorem is sharp, but such an example is given by Theorem 43:

THEOREM 48. *If m is an integer satisfying $m < 4h$, there exists a real sequence $\{\lambda\}$ satisfying the hypothesis of Theorem 47 and having $E(\lambda) \geqslant m$.*

The discrepancy of $\frac{1}{2}$ results from the fact that $x^{-m}P(x) \in L^2$ was used in Theorem 47, whereas only $\lim x^{-m}P(x) = 0$ is used in the proof of Theorem 48.

Two more applications are mentioned because they are relevant to closure problems, as seen by the accompanying references. The following, in particular, is used in (Kahane, 1957).

THEOREM 49 (after Levin, 1949). *Under the hypothesis of Theorem 46*

$$|P(z)| \leqslant (\text{const})(1 + |z|)^{4h} e^{\pi |y|}.$$

This follows by an easy Phragmén–Lindelöf argument, or also by applying Theorem A to a high power of P in Theorem 46. Although the result in (Levin, 1949) seems to assume a separation condition $\lambda_{n+1} - \lambda_n > \epsilon$ no such condition is assumed here.

If the zeros are in fact separated, Levin and Mandelbrojt obtain a lower bound of form

$$|P'(\lambda)| \geqslant (\text{const}) |\lambda|^{-4h}, \quad \lambda = \lambda_n, \tag{25}$$

or a corresponding bound for $Q'(\lambda)$. The graphical method used here leads to a simple proof, and suggests a stronger result. Since we do not wish to assume that the zeros are separated, let us agree that a sequence of *isolated zeros* is a subsequence $\{\lambda_{n_j}\}$ such that

$$|\lambda_{n_j} - \lambda_m| > \epsilon \quad \text{for} \quad m \neq n_j .$$

THEOREM 50 (after Levin, 1949; Mandelbrojt, 1953). *Under the hypothesis of Theorem* 46 *we have* $\liminf |\lambda|^{4h} |P'(\lambda)| = \infty$, *whenever* $|\lambda| \to \infty$ *through a sequence of isolated zeros.*

It will suffice to derive (25) with $n = n_j$, since the stronger conclusion then follows as in the passage from O to o in Theorem 43.

We give the proof first for $xQ(x)$, using the fact that

$$\lambda Q'(\lambda) = \lambda \lim_{x\to\lambda} \frac{Q(x) - Q(\lambda)}{x - \lambda} = \lim_{x\to\lambda} \frac{Q(x)}{1 - (x/\lambda)^2} \frac{\lambda + x}{-\lambda} .$$

Clearly $\lim(\lambda + x)/\lambda = 2$, and the preceding product is exactly the same as $Q(x)$ would be if the factor associated with $\lambda = \lambda_n$ were removed. Accordingly, it satisfies the same inequalities for $\Lambda(u) - Du$ as before, when $0 < u < x = \lambda$, but the bounds are diminished by 1 when $u > x$. Setting

$$L^- = h + 1, \qquad L^+ = h, \qquad R^- = h + 2, \qquad R^+ = h - 1$$

in (23) and (24), we get Theorem 50 for $xQ(x)$ and also an upper bound.

For $P(x)$, if a $\lambda > 0$ is removed, this decreases $\Lambda(u)$ by 1 for $u > \lambda$, and if a $\lambda < 0$ is removed, this increases $\Lambda(u)$ by 1 for $u < \lambda$. In both cases the effect is to increase the exponent on x by 1, and the conclusion follows from

$$\lambda P'(\lambda) = \lambda \lim \frac{P(x) - P(\lambda)}{x - \lambda} = -\lim \frac{P(x)}{1 - x/\lambda} .$$

17. FURTHER REMARKS ON CANONICAL PRODUCTS

In the following theorem $\lambda > 0$, $\Lambda(u)$ is the counting function, and Q is the even canonical product considered in Section 15.

THEOREM 51 (Redheffer, 1954). *Let* $|\Lambda(u) - Du| \leqslant H(u)$ *where* $H(r)$ *is a positive increasing function such that* $H(r)/r$ *is decreasing and* $H(r) = o(r)$ *as* $r \to \infty$. *Then:*

(i) $\log |Q(x)| \leqslant 4H(x) \log[x/H(x)] + o(\text{same})$ *as* $x \to \infty$.

(ii) $\log |Q(x)| \geqslant 2H(x) \log[x/H(x)] + o(\text{same})$ *for some function* Q *and some sequence* $x = x_i \to \infty$.

Although the factors 2 and 4 leave a gap between (i) and (ii), it turns out that there is a function H for which (i) is best possible, and another function H for which (ii) is best possible. If H is constant, the bound $4H \log x$ is substantially in agreement with the bound $4h \log x$ obtained in Theorem 42. Thus, in several respects, Theorem 51 is sharp.

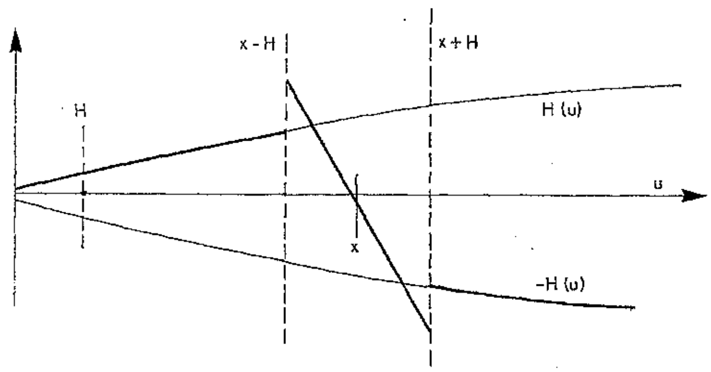

FIGURE 5

The proof can be read off from Fig. 5. As explained in Section 15, the maximizing function $\Lambda(u) - Du$ should be as large as possible on $(0, x)$ and as small as possible on (x, ∞). We write H for $H(x)$ and divide the range of integration at $H, x - H, x + H$. Separate study of the interval $(x - H, x + H)$ shows that the worst choice of $\Lambda(u) - Du$ associated with $(x - H, x + H)$ is $D(x - u)$, and this gives a term H which can be neglected in comparison with $H \log(x/H)$. Likewise, the integral on $(0, H)$ can be neglected. For the remaining integrals, the monotony of H gives

$$\int_H^{x-H} H(u)\, K(x, u)\, du \leqslant H(x) \int_H^{x-H} K(x, u)\, du \leqslant 3H \log(x/H),$$

$$\int_{x+H}^{\infty} \frac{H(u)}{u} \,|\, uK(x, u)|\, du \leqslant \frac{H(x)}{x} \int_{x+H}^{\infty} |\, uK(x, u)|\, du \leqslant H \log(x/H).$$

This gives Part (i). Part (ii) is obtained similarly.

The condition $H(x) \log x/H(x) \in B$ was introduced in (Levinson, 1940) in connection with a problem of interpolation. It also occurs in (Boas, 1954), where an earlier announcement is corrected so as to agree with Theorem 51. A third use of this condition is in the following:

THEOREM 52. *With H as in Theorem 51, let $\{\lambda_n\}$ be a positive sequence whose counting function satisfies*

$$|\, \Lambda(u) - Du \,| \leqslant H(u).$$

Suppose further that $H(x) \log[x/H(x)] \in B$. *Then the completeness intervals for the sets* $\{e^{i\lambda_n x}\}$ *and* $\{e^{\pm i\lambda_n x}\}$ *have length* $I = 2\pi D$.

Since $\Lambda(u) \sim Du$ the fact that $I \geqslant 2\pi D$ for the first set (and hence for the second) follows from Theorem 30. To show that $I \leqslant 2\pi D$ for the second set, note that $\log^+ | Q(x)| \in B^+$ by Theorem 51, and hence Theorem 38 gives an entire function $M \not\equiv 0$ of arbitrarily small type such that $Q(x) M(x) \in L^2$. Since $\Lambda(u) \sim Du$ the type of Q is πD, and the conclusion $I \leqslant 2\pi D + \epsilon$ follows from the Paley–Wiener representation theorem.

For example, $I = 2\pi D$ follows if

$$| \Lambda(u) - Du | \leqslant \frac{u}{(\log u)(\log \log u)^{2+\epsilon}},$$

where $\epsilon > 0$. It will be seen later that the exponent $2 + \epsilon$ can be replaced by $1 + \epsilon$, but the only proof known at present uses Theorem D.

Most estimation theorems require a bound for $| \Lambda(u) - Du |$ or for its integral. In this respect the following is unusual:

THEOREM 53 (Koosis, 1958). *If* λ *is positive and*

$$\left| \int_0^x ((\Lambda(u) - Du)/u) \, du \right| = O(\log x) \qquad (x \to \infty),$$

then $\log^+ | Q(x)| \in B^+$, *and hence* $\Lambda(u) \sim Du$ *entails* $I(\lambda) = 2\pi D$.

The first statement is established in the reference cited, and the second follows as in the proof of Theorem 52.

The same method as that used in the proof of Theorem 51 gives counter-examples to certain rather natural conjectures, even when the zeros are separated. Since these examples shed some light on the completeness problem they are recorded here:

THEOREM 54. *Let* $0 < D < 1$, *let* Λ *be the counting function for* λ, *and let* $Q(x)$ *be the corresponding even canonical product. Then there exist sets of distinct positive integers* λ *such that:*

(i) $\Lambda(u) - Du \in B^+$ *but* $\log^+ | Q(x)| \notin B^+$;

(ii) $\Lambda(u) - Du \in B$ *but* $\log | Q(x)| \notin B$;

(iii) $\log | Q(x)| \in B$ *but* $\log^+ | Q(x)| \notin B^+$.

An example with $\Lambda(u) - Du \in B$ but $\log^+ | Q(x)|$ not in B^+ is given in (Koosis, 1958); the results above are from (Redheffer, 1968). It is shown also that Theorem 51 remains sharp after integration; that is, if $H(x) \log[x/H(x)] \notin B^+$, we can find a Q such that $\log^+ | Q | \notin B^+$.

The technique of constructing counterexamples leads to an interpolation theorem which is mentioned here because of its collateral interest. For $D > 0$, let (D) represent the class of even entire functions of order less than 2, with real zeros only, whose counting function satisfies $\Lambda(u) - Du \in B$. Let $\{x_n\}$ be a sequence of numbers satisfying $x_0 = 0$ and $x_{n+1} > \theta x_n$, where $\theta > 1$ is constant, and let $\{y_n\}$ be any sequence of real numbers. We then have:

THEOREM 55 (Redheffer, 1967). *The equations* $F(x_n) = y_n$ $(n = 0, 1, 2,...)$ *have a solution* $F \in (D)$ *if, and only if,*

$$\sum_{n=1}^{\infty} \left(\frac{\log^+ | y_n |}{x_n} \right)^2 < \infty.$$

For proof, the necessity follows from estimates of $\log | Q(x)|$ similar to those above. The same calculation shows that if the series converges, $| F(x_n)|$ can be made much larger than $| y_n |$ at each n. We now add an extra zero λ_n near x_n. As λ_n moves from left to right through the value x_n, the corresponding $F(x_n)$ changes sign, and a suitable choice of λ_n gives $F(x_n) = y_n$. By judicious arrangement of details this can be managed for all x_n simultaneously, and Theorem 55 follows.

18. Extensions and Ramifications

We indicate additional results related to Theorem 51. It was already mentioned that the function $H(x) \log[x/H(x)]$ occurring there was introduced in connection with another theorem in (Levinson, 1940). Combining these two theorems gives the following, which supplements the result of Section 10:

THEOREM 56 (Redheffer, 1961). *Let* Q, H, *and* λ *be as in Theorem* 51 *and suppose further that* $\lambda_{n+1} - \lambda_n > \epsilon$. *For any complex constant* $a \neq 0$ *let* μ_n *be the roots of* $Q(z) = a$. *Then*:

$$H(x) \log(x/H(x)) \in B \quad \textit{implies} \quad I(\mu) = 2\pi D, \quad E(\mu) \geqslant 0.$$

If the set is not complete we get $F = (Q - a) M$ as usual. Theorem 51 gives $Q - a \in B^+$ and Theorem C shows that the type of M is 0. Since F is bounded, M is bounded on the set λ, and the conclusion follows from the Levinson theorem on entire functions of zero type. Thus the condition $H \log(x/H) \in B$ is used twice.

Theorem 56 is actually true if only $H \in B$ instead of $H \log(x/H) \in B$. This deep result follows from Theorem D together with the improvement of Levinson's theorem implicit in (de Branges, 1968, Theorems 61, 66, 68).

In the presence of a separation condition $\lambda_{n+1} - \lambda_n > \epsilon$, Theorem 51 was extended in (Boas, 1954) to allow estimation of both Q and its reciprocal in the complex plane, as well as estimation of $1/Q'(\lambda)$. These results are derived in somewhat sharper form in (Redheffer, 1968), where it is shown, in particular, that

$$\log |Q(z)| \leqslant \pi D |y| + 4H(|x|) \log(|x|/H(|x|)) + o(\text{same}).$$

From another point of view, Theorem 51 can be regarded as a quantitative form of a theorem of (Pfluger, 1943) or (Titchmarsh, 1927) to the effect that $\Lambda(u) \sim Du$ implies $\log^+ |Q(x)| = o(x)$. It was noticed in (Redheffer, 1954) that $\log^+ |Q(x)| = o(x)$ follows from the weaker condition

$$\lim_{r\to\infty} [D(rt) - D(r)] = 0 \qquad (t > 0),$$

where $D(u) = \Lambda(u)/u$, and this result was extended to the complex plane in (Kahane and Rubel, 1959) to get the conclusion

$$\log |Q(re^{i\theta})| = \pi\Lambda(r) |\sin\theta| + o(r) \tag{26}$$

for $\theta \neq 0$ or π. Using (26) and a lemma from (Koosis, 1958), they establish the following:

THEOREM 57 (Kahane and Rubel, 1959). *Let $H(r)$ be positive and increasing for $r > 0$ with $H(r)/r$ decreasing and $H(r)/\log r$ increasing. Suppose further that H does not belong to B. Then there exist even canonical products Q_1 and Q_2, of preassigned types $T_1 > 0$ and $T_2 > 0$, such that $\log^+ |Q_i(x)| \leqslant H(x)$ for large x but type $(Q_1Q_2) = \max(T_1, T_2)$.*

It is easily checked that the error in (26) is $o(r) |\csc\theta|$ uniformly in $0 < \theta < \pi$. However, the correct error term should have a convergent integral with respect to θ, as seen next:

THEOREM 58. *With $D(u) = \Lambda(u)/u$ and $Q(z)$ as above, the following conditions are equivalent:*

(i) $\lim_{r\to\infty} [D(rt) - D(r)] = 0$ *for* $t > 0$.

(ii) $\log |Q(re^{i\theta})| = \pi\Lambda(r) |\sin\theta| + o(r) \log |2 \csc\theta|$.

(iii) $\log |Q(re^{i\theta})| = \pi\Lambda(r) |\sin\theta| + \eta(r, \theta)$, *where the integral of $\eta(r, \theta)\, d\theta$ is $o(r)$.*

(iv) $\Lambda(r) = \int_0^r (\Lambda(t)/t)\, dt + o(r)$.

It was shown in (Redheffer, 1968) that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). All that remains is to get (iv) $\Rightarrow$ (i), which was not done there.

In terms of $D(r)$ we have

$$D(r) = (1/r)\int_0^r D(t)\,dt + \epsilon(r),$$

where $\lim \epsilon(r) = 0$. If $-\epsilon \leqslant \epsilon(r) \leqslant \epsilon$ for $r > r_0$, where ϵ is constant, replacing $\epsilon(r)$ by $-\epsilon$ and ϵ gives a minorant and a majorant for $D(r)$ beyond r_0 which leads to the condition (i).

19. A Meromorphic Function

In Sections 19, 20, and 21, $\{\lambda_n\}$ and $\{\mu_n\}$ denote real sequences for $-\infty < n < \infty$, with $\lambda_n \mu_n \neq 0$. The signed counting functions for μ and λ are respectively $\Lambda_\lambda(u)$ and $\Lambda_\mu(u)$, and we assume

$$\Lambda(u) = \Lambda_\lambda(u) - \Lambda_\mu(u) \in B.$$

Hence the meromorphic function

$$G(x) = \prod_{-\infty}^{\infty} \frac{(1 - x/\lambda_n)\, e^{x/\lambda_n}}{(1 - x/\mu_n)\, e^{x/\mu_n}}$$

exists and, by partial integration,

$$\log |\, G(x)| = \int_{-\infty}^{\infty} \frac{x^2}{u^2(x-u)} \Lambda(u)\, du,$$

where the integral is a Cauchy principal value at $u = x$. The function $G(x)$ is formed by taking the product over all factors for which λ_k and μ_k lie on the interval $(-R, R)$ and then letting $R \to \infty$. Convergence of the products forming numerator and denominator separately is not required; on the contrary, both λ and μ could belong to products of infinite genus without invalidating the conclusions.

We define

$$\Lambda^*(x, R, S) = \int_{x-R}^{x+S} \frac{\Lambda(u)}{u - x}\, du, \qquad \Lambda^0(x) = x \int_{-|x|}^{|x|} \frac{\Lambda(u)}{u^2}\, du,$$

where R and S are any positive functions of x. It is said that R and S have *the order of* x if the ratios R/x, S/x, x/R, x/S are bounded as $|\, x \,| \to \infty$. Note that $\Lambda^*(x, R, R)$ agrees with $\Lambda^*(x, R)$ in (17).

Theorem 59 (Redheffer, 1961). *If R and S have the order of x then $\Lambda(u) \in B$ implies* $\log |\, G(x)| - \Lambda^0(x) + \Lambda^*(x, R, S) \in B^+$.

The expression in the conclusion is an integral of $k(x, u)\,A(u)/u^2$ where the kernel k has the values

$$\frac{x^2}{x-u}, \qquad \frac{xu}{x-u}, \qquad u, \qquad x+u, \qquad \frac{x^2}{x-u}$$

for $x > 0$, on the intervals

$$(-\infty, -x), \qquad (-x, x-R), \qquad (x-R, x), \qquad (x, x+S), \qquad (x+S, \infty),$$

respectively. These functions admit the respective bounds

$$\frac{x^2}{|u|}, \qquad \frac{x\,|u|}{R}, \qquad |u|, \qquad \left(1+\frac{S}{x}\right)\left(2+\frac{S}{x}\right)\frac{x^2}{u}, \qquad \left(1+\frac{x}{S}\right)\frac{x^2}{u}$$

on their intervals, and thus we get a bound for the integral of the form

$$(\text{const})\left(\int_0^{|x|}\frac{H(u)}{u}\,du + x^2\int_{|x|}^{\infty}\frac{H(u)}{u^3}\,du\right),$$

where $H(u) = |A(u)| + |A(-u)|$. Replacing x by $-x$ amounts to replacing $A(u)$ by $-A(-u)$ and interchanging the roles of R and S. Hence, a similar estimate holds for $x < 0$. Since each term of the foregoing expression belongs to B, and since the expression as a whole has a positive derivative, Theorem 59 follows.

In Theorem 60, $\{\mu_n\}$ is empty, $\lambda_{-n} = -\lambda_n$, and $Q(z)$ is the even canonical product introduced in Section 15.

THEOREM 60 (Matzayev, 1966). *If* $A_\lambda(u) - u \in B^+$ *then* $\log|Q(x)| \in B$.

Theorem 60 was presented at the Moscow congress in 1966 and was communicated to the author by Koosis. The result is unusual in that the Hilbert transform is not a bounded operator from L^1 to L^1.

Matzayev's ingenious proof depends on the Hardy–Littlewood rearrangement theorem and is not easy. However, as stated in (Redheffer, 1968) the following more general result is an immediate consequence of Theorem 59:

THEOREM 61. *If* $A(u) \in B^+$ *then* $\log|G(x)| - A^0(x) \in B$.

By Theorem 59, it suffices to show that $A \in B^+$ implies $A^*(x, R, S) \in B$ when R and S have the order of x. Without loss of generality take $x > 0$ and let $x_n = 2^n$. If $x_n \leqslant x < x_{n+1}$ we define $R = x - x_{n-1}$, $S = x_{n+2} - x$, and note that R and S have the order of x.

Also

$$\int_{x_n}^{x_{n+1}} \frac{|\Lambda^*(v, R, S)|}{v^2}\, dv = \int_{x_n}^{x_{n+1}} \frac{1}{v^2} \left| \int_{x_{n-1}}^{x_{n+2}} \frac{\Lambda(u)}{v-u}\, du \right| dv$$

$$\leqslant \pi \left(\int_{x_n}^{x_{n+1}} \frac{dv}{v^4} \right)^{1/2} \left(\int_{x_{n-1}}^{x_{n+2}} |\Lambda(u)|^2\, du \right)^{1/2} \leqslant 5\pi \frac{H(x_{n+2})}{x_{n+2}},$$

where the first inequality follows from the Schwarz inequality together with the familiar fact that $\|\tilde{f}\|_2 \leqslant \pi \|f\|_2$, and the second inequality holds if $|\Lambda(u)| \leqslant H(u)$ where H is increasing. If, in addition, $H \in B$, then the series converges and Theorem 61 follows. More general results of this kind are given in (Redheffer, 1971).

In conclusion, we mention that if $|\Lambda(u)| \leqslant H$, where H is constant, the term

$$\Lambda^1(x) = \int_{-|x|}^{|x|} \frac{\Lambda(u)}{u}\, du$$

should be considered as a principal term rather than as an error term. Subtracting off this term changes $k(x, u)$ on $(-x, x)$ in such a way as to give the following, when R and S have the order of x:

THEOREM 62. *If* $|\Lambda(u)|$ *is bounded, then* $\log |G(x)| - \Lambda^1(x) - \Lambda^0(x) + \Lambda^*(x, R, S)$ *is bounded.*

20. FURTHER COMPARISON THEOREMS

Continuing in the notation of the foregoing section, we define

$$c = \int_{-\infty}^{\infty} \frac{\Lambda(u)}{u^2}\, du, \qquad \Lambda^\infty(x) = x \int_{|x|}^{\infty} \frac{\Lambda(u) + \Lambda(-u)}{u^2}\, du.$$

Thus, $\Lambda^0(x) + \Lambda^\infty(x) = cx$ and Theorems 61, 59, and 62 give

$$\begin{aligned} &\log |e^{-cx} G(x)| + \Lambda^\infty(x) \in B, \\ &\log |e^{-cx} G(x)| + \Lambda^\infty(x) + \Lambda^*(x, R, S) \in B^+, \\ &\log |e^{-cx} G(x)| - \Lambda^1(x) + \Lambda^\infty(x) + \Lambda^*(x, R, S) = O(1), \end{aligned} \tag{27}$$

under the respective hypotheses $\Lambda \in B^+$, $\Lambda \in B$, $\Lambda = O(1)$. The reader is reminded that

$$\Lambda^*(x, R, S) = \int_{x-R}^{x+S} \frac{\Lambda(u)}{u-x}\, du, \qquad \Lambda^*(x, R) = \int_{x-R}^{x+R} \frac{\Lambda(u)}{u-x}\, du.$$

The function $e^{-cx}G(x)$ agrees with

$$\lim \prod \frac{1 - x/\lambda_n}{1 - x/\mu_n},$$

where the product is obtained by taking factors with λ and μ on $(-R, R)$ and letting $R \to \infty$. The more elaborate form with exponential factors is preferred here, because it represents $G(x)$ as a quotient of entire functions of exponential type. Indeed, by Jensen's theorem, all the following theorems give $I(\lambda) = I(\mu) = \infty$ unless $\Lambda_\lambda(u)/u$ and $\Lambda_\mu(u)/u$ are bounded. Thus familiar properties of entire functions will be available.

If $\{\mu\}$ is not complete there is a function $F(z)$ in (1) with $F(\mu_n) = 0$. It is a remarkable fact that $e^{-cx}F(x)\, G(x)$ behaves much as $e^{-cx}G(x)$ would behave if $\Lambda(u) = \Lambda_\lambda(u) - Du$. (What is surprising about this is that it requires *no regularity* of $\{\lambda\}$ or $\{\mu\}$.) The basis of the behavior is the inequality

$$-\Lambda^*(x, S) + \log |F(x)| \leqslant \Lambda_\mu^*(x, S) + \log |F(x)| \leqslant (I/\pi)\, S, \qquad (28)$$

which follows from (18) and the estimate below (18). Weaker results based on (2) or on Rolle's theorem were used previously. However, the loss of precision is important only in Theorem 64, which was obtained in (Alexander and Redheffer, 1967) with I instead of I/π.

When (28) holds, substitution in (27) gives

$$\log |e^{-cx}F(x)\, G(x)| \leqslant U - \Lambda^*(x, R) + \Lambda^*(x, S) + (I/\pi)\, S - \Lambda^\infty(x)$$

for $0 < S \leqslant R$, where the unknown error U satisfies $U \in B^+$ or $U = \Lambda^1(x) + O(1)$ as the case may be. The three terms following U can be estimated by making a sensible choice of S in the inequalities

$$|\Lambda^*(x, R) - \Lambda^*(x, S)| + \frac{I}{\pi} S \leqslant 2 \sup_{|x-u| \leqslant R} |\Lambda(u)| \log \frac{R}{S} + \frac{I}{\pi} S, \qquad (29)$$

$$|\Lambda^*(x, R) - \Lambda^*(x, S)| + \frac{I}{\pi} S \leqslant \frac{1}{S} \int_{x-R}^{x+R} |\Lambda(u)|\, du + \frac{I}{\pi} S. \qquad (30)$$

Equation (29) is the basis for Theorems 63, 65, and 67, and (30) is the basis for Theorems 64 and 66.

The reader is reminded that throughout this section λ and μ are real, and $\Lambda = \Lambda_\lambda - \Lambda_\mu$. We use m for a positive integer, here and below.

THEOREM 63 (Alexander and Redheffer, 1967). *Let* $|\Lambda(u)| \leqslant H$ *where* H *is constant, let* $p = 2$, *and let* $m > 4H + \frac{1}{2}$. *Then* $I(\lambda) = I(\mu)$ *and* $|E(\lambda) - E(\mu)| \leqslant m$.

The method explained above gives

$$\log |e^{-cx}F(x)\, G(x)| \leqslant 4H \log |x| + O(1),$$

and the conclusion follows as in the proof of Theorem 47. However, the sawtooth effect leading to the sharper result is not available here, since $\Lambda(u)$ is a step function.

THEOREM 64 (after Alexander and Redheffer, 1967). *Let $I(\lambda) = I$, let $p = 2$, and for some $\delta > 0$ let*

$$\left(\frac{m}{2} - \frac{1}{4}\right)^2 > \frac{I}{\pi} \limsup_{|x|\to\infty} \frac{1}{(\log |x|)^2} \int_{x-\delta|x|}^{x+\delta|x|} |\Lambda(u)|\, du.$$

Then $I(\mu) = I$ and $|E(\lambda) - E(\mu)| \leqslant m$.

The hypothesis implies that Λ^1 and Λ^∞ are bounded, and Theorem 64 follows as explained above. Note that Theorem 64 gives $|E(\lambda) - E(\mu)| < \infty$ under conditions which allow $\sup |\Lambda(u)| = \infty$.

In the following theorem we depart from the convention introduced in Section 1, and denote the L^p excess by $E(\lambda, p)$. It is assumed that $|\lambda_n|$ or $|\mu_n|$ is an increasing function of $|n|$:

THEOREM 65. *Let $|\lambda_n - \mu_n| \leqslant \epsilon(|n|)$ where $\sum_{n=1}^{\infty} \epsilon(n)/n < \infty$ and $\epsilon(n) = O(1/\log|n|)$. Then $|E(\lambda, p_1) - E(\mu, p_2)| \leqslant 1$ for $1 \leqslant p_i \leqslant 2$.*

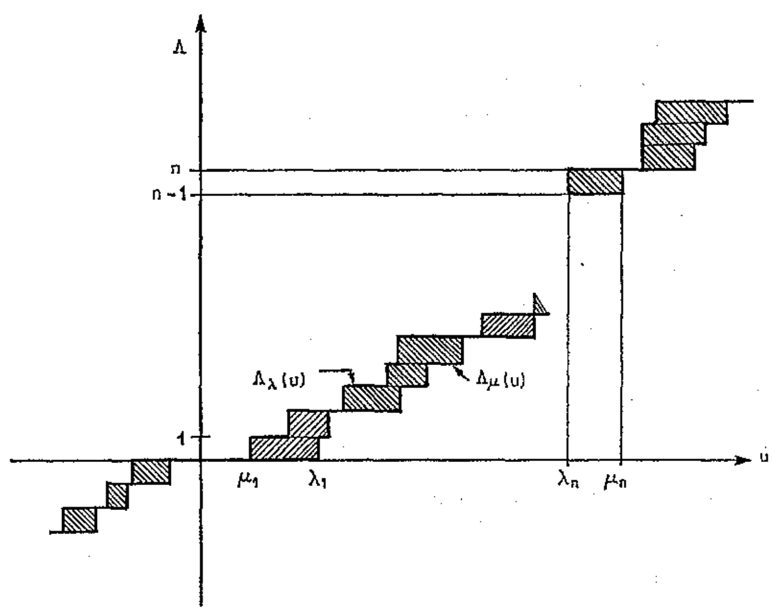

FIGURE 6

Without loss of generality $|\mu_n| \geqslant \epsilon |n|$. By this and by Fig. 6, integrals involving $|\Lambda(u)|$ are readily estimated in terms of sums involving $|\lambda_n - \mu_n|$, and the above methods show that the transformation $\mu \to \lambda$ changes the bounded function F_μ into a corresponding bounded function F_λ. Removing one zero, we get $F_\lambda \in L^2$, and the result follows.

An interesting complement to Theorem 65 is

$$|\lambda_n| \leqslant |n| + (1/2q) + \psi(|n|) \Rightarrow E(\lambda) \geqslant 0,$$

where $\sum_1^\infty |\psi(n)|/n < \infty$. This follows from Theorem 8.

We now discuss the completeness interval. The use of a general function G leads to a complication which does not arise when G is even, and which turns out to be harmless in the cases considered above. This complication comes from the term Λ^∞.

It is easily checked that $|\Lambda(u)| \log |u| \in B$ implies $\Lambda^\infty(u) \in B^+$ and hence if $|\Lambda(u)| \leqslant H(|u|)$ with H as in Theorem 51,

$$H(r) \log r \in B \Rightarrow I(\lambda) = I(\mu). \tag{31}$$

A similar method gives the following:

THEOREM 66 (after Koosis, 1958). *If* $|u|^\epsilon \Lambda(u) \in B$ *then* $I(\lambda) = I(\mu)$.

The result in the reference cited is for an even function and for $\Lambda(u) = \Lambda_\lambda(u) - Du$. To get Theorem 66 as it stands, the above remark concerning Λ^∞, and (30) with $R = x$, give $e^{-\sigma x}F(x)\, G(x) \leqslant e^{\phi(x)}$ where $\phi \in B^+$. Theorem 38 gives $I(\lambda) \leqslant I(\mu)$, and equality follows by symmetry. The method applies if $|u|^\epsilon$ is replaced by $(\log |u|)^{2+\epsilon}$ but not by $(\log |u|)^2$.

The interest of Theorem 66 is that if $|u|^\epsilon$ could be dropped altogether, the result would be Theorem 75 below. The latter is equivalent to the Beurling–Malliavin solution of the completeness problem and is obtained only at the end of a difficult investigation. By contrast, Theorem 66 is entirely elementary.

The condition $H(r) \log r \in B$ in (31) is necessary in the sense that, if it fails, there is an example for which Λ^∞ is not in B. In the next section it will be seen, nevertheless, that λ and μ can be modified so as to ensure $\Lambda^\infty \in B^+$, without changing $I(\lambda)$ or $I(\mu)$ and without losing the hypothesis of the following theorems. We take this for granted here.

THEOREM 67. *Let* $|\Lambda(u)| \leqslant H(|u|)$ *where* $H(r)$ *is increasing,* $H(r) = o(r)$, *and* $H(r) \log[r/H(r)] \in B$. *Then* $I(\lambda) = I(\mu)$.

The hypothesis shows that the function (29) is in B^+ when $S = H(|2x|)$. If also $\Lambda^\infty \in B^+$ as assumed here, Theorem 67 follows in the same way as Theorem 66.

These results would appear to be about the best that can be done without Theorem D. However, if the latter is used we get a stronger result, which not only sharpens Theorem 67, but gives a very short proof, and generalization, of the following theorem of Binmore:

THEOREM 68 (Binmore, 1970). *Let C denote the class of positive functions y such that $y(x)$ increases, $y(x)/x$ decreases, and $y \in B$. Let $\{\lambda_n\}$ be distinct positive integers with counting function $\Lambda_\lambda(u)$. Then*

$$I(\lambda) \leqslant 2\pi \inf_{y\in C} \limsup_{x\to\infty} \frac{\Lambda_\lambda[x + y(x)] - \Lambda_\lambda(x)}{y(x)}.$$

The fact that Binmore's condition $y \in B$ is sharp is shown by the following:

THEOREM 69. *Let y satisfy the conditions of Theorem 68, except that $y \notin B$. Then there exists a set of distinct positive integers λ such that $I(\lambda) = 2\pi$ and*

$$\limsup_{x\to\infty} \frac{\Lambda_\lambda[x + y(x)] - \Lambda_\lambda(x)}{y(x)} = 0.$$

For proof, see (Redheffer, 1968). Here we prove Theorem 68.

Denoting the lim sup in the hypothesis by D_1, choose $D > D_1$, then choose $y(x)$. With $x_{n+1} = x_n + y_n$ and $y_n = y(x_n)$ the hypothesis gives

$$\Lambda_\lambda(x_n + y_n) - \Lambda_\lambda(x_n) < Dy_n,$$

aside from finitely many values at the beginning. Hence $\Lambda(u) = \Lambda_\lambda(u) - Du$ satisfies $\Lambda(x_n + y_n) - \Lambda(x_n) < 0$. Avoiding cumulative error, add enough λ's so that within ± 1 equality holds. The extreme values for the new function Λ are approximately as shown in Fig. 7, for all practical purposes $|\Lambda(x)| \leqslant y(2x)$, and the hypothesis $y \in B^+$ gives $\Lambda(u) \in B^+$. This does not use the separation condition or the fact that $y(x)/x$ is decreasing.

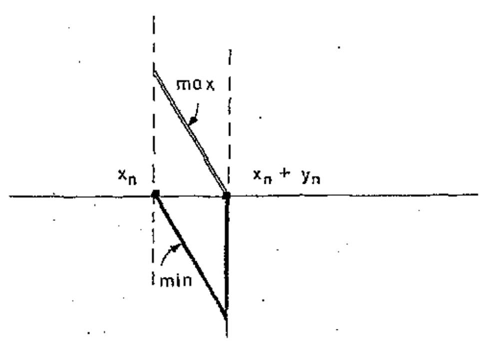

FIGURE 7

Accordingly, with $\Lambda(u) = \Lambda_\lambda(u) - \Lambda_\mu(u)$ as before, Theorem 69 is a special case of the following:

THEOREM 70. *If* $\Lambda(u) \in B^+$, *then* $I(\lambda) = I(\mu)$.

It is important that Theorem 70 assumes *no regularity* of λ or μ, whereas Theorem 68 has a separation condition for both λ and μ; note that $\Lambda_\mu(u) = [Du]$, essentially, for this application.

If $\Lambda^\infty \in B^+$, as permitted by the results of Section 21, then Theorem 61 ensures $\log | e^{-cx}F(x)\, G(x)| \in B$. Theorem D gives an entire function $M(x)$ of arbitrarily small type such that $e^{-cx}F(x)\, G(x)\, M(x) \in L^2$, and Theorem 70 follows at once.

The simplicity of this proof as compared to the proof of Theorem 68 in the reference cited may serve to show the power of the Beurling–Malliavin theory. Later we shall find that B^+ can be changed to B in Theorem 70; this is Theorem 75 again.

21. Matching of Areas

If H is positive and increasing, and belongs to B^+, we can find a majorant $H_1 \geqslant H$ such that $H_1 \in B^+$ and $r^{-1/2}H_1(r)$ increases. To see this, assume without loss of generality that H is a step function with jumps at the integers. Let $x_1 = 1$ and define

$$H_1(x) = H(x_1{}^+)(x/x_1)^{1/2}, \qquad x_1 \leqslant x < x_2\,,$$

where x_2 is the first point beyond x_1 where the graph of H_1 crosses the graph of H; the latter graph is assumed to contain vertical segments at the jumps. Clearly, $x_1 + 1 \leqslant x_2 \leqslant \infty$. Similarly we go from x_2 to x_3 and so on. Since

$$\int_{x_n}^{\infty} H(x_n) \left(\frac{x}{x_n}\right)^{1/2} \frac{dx}{x^2} \leqslant \frac{2H(x_n)}{x_n} \leqslant 2 \int_{x_n}^{\infty} \frac{H(x)}{x^2}\, dx,$$

the function H_1 so obtained belongs to B^+.

Hence, we can assume $r^{-1/2}H(r)$ increasing in Theorem 70. Under the hypothesis of Theorem 67 the function $H(r) \log[r/H(r)]$ is increasing for large r, and a similar argument shows that we can assume $r^{-1/2}H(r)$ increasing in Theorem 67. These assumptions are made here.

If we add to $\{\lambda_n\}$ a zero distribution whose counting function Λ^+ satisfies $\Lambda^+ \in B^+$ then the completeness interval $I(\mu)$ is not affected. (More generally, we could add any complex zeros ν_n for which $\sum 1/|\nu_n|$ converges. This follows from the proof of Theorem 41.)

We shall add to $\{\lambda_n\}$ and $\{\mu_n\}$ two zero distributions whose counting functions Λ^+ and Λ^- satisfy

$$| \Lambda^-(u)| \leqslant 4H(9u), \qquad | \Lambda^+(u)| \leqslant 4H(9u) \qquad (u > 0).$$

Since $H \in B^+$, the above remarks show that the completeness intervals $I(\lambda)$ and $I(\mu)$ are not affected by the additional terms. The new distribution has the counting function

$$L(u) = \Lambda(u) + \Lambda^+(u) - \Lambda^-(u).$$

Considering the case $x > 0$, we set $x_n = 9^n$ and show that Λ^- and Λ^+ can be chosen so that

$$\int_{x_n}^{x_{n+1}} \frac{L(u)}{u^2}\, du = 0.$$

The idea of making the integral 0 in selected intervals is suggested by the work of Beurling and Malliavin.

To see that this area-matching is possible, let us show that the integral of $\Lambda^+ - \Lambda^-$ can be made large enough to exceed the integral of Λ over (x_n, x_{n+1}). In the worst case $\Lambda^-(x_n)$ has the value $-4H(9x_n)$. We keep Λ^- constant on $[x_n, x_{n+1})$ and assign to $\Lambda^+(x)$ its largest admissible value, $4H(9x)$. Then

$$\int_{x_n}^{x_{n+1}} \frac{\Lambda^+(u) - \Lambda^-(u)}{u^2}\, du \doteq 4 \int_{x_n}^{9x_n} \left[\frac{x^{-1/2}H(9x)}{x^{3/2}}\, dx - \frac{H(9x_n)}{x^2}\right] dx$$

$$> \frac{H(9x_n)}{x_n} > \int_{x_n}^{9x_n} \frac{H(x)}{x^2}\, dx \geqslant \int_{x_n}^{x_{n+1}} \frac{\Lambda(u)}{u^2}\, du$$

as desired. Considering the minimum value similarly, we see that the areas can be effectively matched. (Note that an error of 1 in $L(u)$ on the whole interval (x_n, x_{n+1}) produces an error of order $1/x_n$ in the integral, which is entirely negligible here.)

Accordingly, if x_n is the first x_n beyond x,

$$\left| \int_x^\infty \frac{L(x)}{x^2}\, dx \right| \doteq \left| \int_x^{x_n} \frac{L(x)}{x^2}\, dx \right| < \int_x^{x_n} \frac{|L(x)|}{x^2}\, dx < \frac{\alpha H(\beta x)}{x},$$

for constants α and β. A similar discussion applies for $x < 0$ and $u < 0$ and shows that, with the new distribution of zeros, $L^\infty \in B^+$.

This completes the proof of Theorems 67 and 70.

22. Matching of Areas, Continued

The entire discussion of Sections 22–25 is based on the work of Beurling and Malliavin. The details of exposition are different, however, and should be compared with (Beurling and Malliavin, 1961, 1967) and with (Kahane, 1962, 1966).

Let $\{\nu_n\}$ be a real sequence with counting function $\Lambda_\nu(u)$. The function $\sin \pi\epsilon z$ vanishes on the arithmetic progression $\{n/\epsilon\}$, is bounded on the real axis, and has type $\pi\epsilon$. This shows that adjoining a subset of $\{n/\epsilon\}$ to $\{\nu_n\}$ increases $I(\nu)$ by $2\pi\epsilon$ at most.

A similar remark applies to the removal of terms $\{n/\epsilon\}$, provided they were present to begin with. That is, the removal reduces $I(\nu)$ by $2\pi\epsilon$ at most. To ensure that these terms were present, let us first adjoin $\{n/\epsilon\}$ to $\{\nu_n\}$. The augmented set so obtained is denoted by $\{\mu_n\}$ and its counting function by $\Lambda_\mu(u)$. Starting with $\{\mu_n\}$, we can adjoin *or remove* any terms of the given sparse arithmetic sequence, $\{n/\epsilon\}$, to get a new set $\{\lambda_n\}$. The inequality

$$I(\nu) \leqslant I(\lambda) \leqslant I(\nu) + 4\pi\epsilon$$

shows that bounds $I_1 \leqslant I(\lambda) \leqslant I_2$ would give the same bounds for $I(\nu)$ upon letting $\epsilon \to 0$.

In terms of counting functions, this means that we can add to $\Lambda_\mu(u)$ any counting function $\Lambda_\epsilon(u)$ which has jumps 0, +1, or −1 at the points n/ϵ. Assuming $\Lambda_\mu(u) - Du \in B$, where D is a positive constant, we want to choose Λ_ϵ so that the counting function $\Lambda_\lambda(u) = \Lambda_\mu(u) + \Lambda_\epsilon(u)$ for $\{\lambda_n\}$ satisfies $\Lambda_\lambda(u) - Du \in B$ and also

$$\int_{x_n}^{x_n+y_n} \frac{\Lambda_\lambda(u) - Du}{u^2}\, du = 0, \qquad \sum_{n=1}^{\infty} \left(\frac{y_n}{x_n}\right)^2 < \infty. \tag{32}$$

Here $\{x_n\}$, with $x_{n+1} = x_n + y_n$, is a sequence which arises in the course of the analysis. We assume $x > 0$; the discussion of the case $x < 0$ is similar.

It is convenient to require $\Lambda_\epsilon(x_n) = 0$. With this understanding, the maximum function Λ_ϵ on (x_n, x_{n+1}) is as shown in Fig. 8, and the corresponding integral is

$$\int_{x_n}^{x_n+y_n} \frac{\Lambda_\epsilon(u)}{u^2}\, du \doteq \frac{\epsilon}{2}\left(\frac{y_n}{x_n}\right)^2. \tag{33}$$

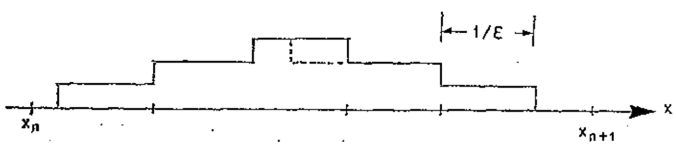

Figure 8

On the other hand if we define

$$\alpha_n = \int_{x_n}^{\infty} \frac{| \Lambda_\mu(u) - Du |}{u^2} \, du, \tag{34}$$

then α_n approaches 0, and the integral (33) exceeds the corresponding integral for $\Lambda_\mu(u) - Du$ if y_n is large enough. The approximate inequality

$$\frac{\epsilon}{2}\left(\frac{y_n}{x_n}\right)^2 \leqslant \int_{x_n}^{x_{n+1}} \frac{| \Lambda_\mu(u) - Du |}{u^2} \, du$$

is allowed by the analysis, and leads to the second condition (32). It also gives $\Lambda_\epsilon(u) \in B$, which ensures $\Lambda_\lambda(u) - Du \in B$.

In a like manner, the minimum Λ_ϵ is obtained by reflecting the graph of Fig. 8 in the x axis. Since the area of $\Lambda_\epsilon(u)$ can be changed in units of $1/\epsilon$, as shown by the figure, we can almost match the areas exactly. To get a perfect match, let us move one of the points j/ϵ through a distance at most $1/\epsilon$ as suggested by the dotted line in Fig. 8. Only one such move is made on any interval $(x_n, x_n + y_n)$. Accordingly, the counting functions Λ_1 and Λ_2 before and after any number of such moves satisfy $| \Lambda_1 - \Lambda_2 | \leqslant 1$, and several results in the preceding discussion show that this hardly changes the excess, much less the completeness interval. Hence, (32) is justified.

We shall discuss a special case, in which the proof is somewhat simpler when

$$\int_{x_n}^{x_n+y_n} [\Lambda_\lambda(u) - Du] \, du = 0, \qquad \sum_{n=1}^{\infty} \left(\frac{y_n}{x_n}\right)^2 < \infty \tag{35}$$

instead of (32). Derivation of (35) is similar to the derivation of (32).

In conclusion, we mention that an arithmetic sequence $\{n/\epsilon\}$ was used to make it obvious that the initial sequence $\{\nu_n\}$ and the final sequence $\{\lambda_n\}$ have nearly the same completeness interval. If, instead, we use any sequence $\{\epsilon_n\}$ of small density, Theorem 30 gives $I(\nu) \geqslant 2\pi D$ and the results of the next section will give $I(\lambda) \leqslant 2\pi D + \delta$, where δ is small. Hence, the complexity of the added sequence $\{\epsilon_n\}$ is really irrelevant.

23. The Completeness Interval for Real Sequences

We shall establish the following:

THEOREM 71 (Beurling and Malliavin, 1961, 1967). *Let $\{\lambda_n\}$ be a real sequence with counting function $\Lambda_\lambda(u)$. If $\Lambda_\lambda(u) - Du \in B$, then $I(\lambda) = 2\pi D$.*

As usual, we introduce the entire function

$$P(z) = \lim_{r\to\infty} \prod_{|\lambda_n|\leqslant r} \left(1 - \frac{z}{\lambda_n}\right) = e^{-cz} \prod \left(1 - \frac{z}{\lambda_n}\right) e^{z/\lambda_n}$$

and set $\Lambda(u) = \Lambda_\lambda(u) - Du$. Then

$$\log | P(x)| + \Lambda^*(x, R, S) + \Lambda^\infty(x) \in B^+, \tag{36}$$

where R and S have the order of x and where, as before,

$$\Lambda^*(x, R, S) = \int_{x-R}^{x+S} \frac{\Lambda(u)}{u - x}\, du, \qquad \Lambda^\infty(x) = x \int_{|x|}^{\infty} \frac{\Lambda(u) + \Lambda(-u)}{u^2}\, du.$$

This is a special case of (27); for a direct proof, cf. proof of Theorem 59.

Assuming that $\{\lambda\}$ has the properties described in the last section, we want to prove that $\Lambda^\infty \in B$ and $\Lambda^* \in B$. Since the type of P is πD, as seen by an easy evaluation of $P(iy)$, Theorem D will give $I(\lambda) \leqslant 2\pi D$. This is the main desideratum; $I(\lambda) \geqslant 2\pi D$ follows from Theorem 30.

In the earlier reference cited, and also in (Kahane, 1962), Theorem 71 is obtained only for $\lambda_{-n} = -\lambda_n$, so that $P(z)$ is the even canonical product $Q(z)$ introduced in Section 15. This case is somewhat easier and is discussed first, assuming (35) rather than (32).

Since $\Lambda^\infty = 0$ for P even, all we have to do is to show that $\Lambda^* \in B$ when R and S have the order of x. This is accomplished by using (35) in two ways. The first use of (35) is to estimate $\Lambda(u)$, and the second is to estimate the integral by observing that, for any K,

$$\int_{x_n}^{x_n+y_n} \Lambda(u)\, K(x, u)\, du = \int_{x_n}^{x_n+y_n} \Lambda(u)[K(x, u) - K(x, x_n)]\, du.$$

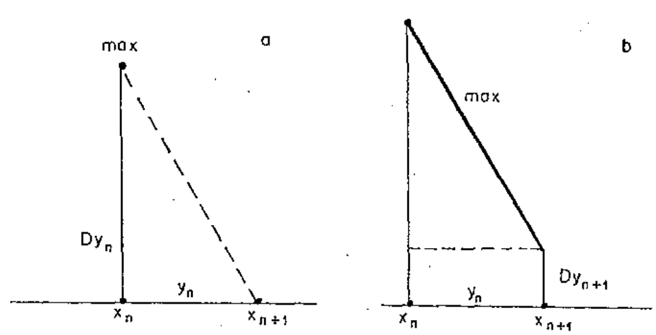

FIGURE 9

The graph of $\Lambda(u)$ has slope $\geqslant -D$ and, since the area is 0, it crosses the axis at some point of (x_n, x_{n+1}). Accordingly, the height of the graph at the left end, x_n, cannot exceed Dy_n as shown in Fig. 9a. Since x_{n+1} is the left end of the next interval, the height there cannot exceed Dy_{n+1}, and

$$\Lambda(u) \leqslant D(y_n + y_{n+1}), \qquad x_n \leqslant u \leqslant x_{n+1},$$

as shown by Fig. 9b. In a like manner $\Lambda(u) \geqslant -D(y_n + y_{n-1})$ on this interval, and we conclude that

$$|\Lambda(u)| \leqslant D\tilde{y}_n = D(y_{n-1} + y_n + y_{n+1}), \qquad x_n \leqslant u \leqslant x_{n+1}, \tag{37}$$

where $\tilde{y}_n$ is defined by the equation. This is the desired estimate of $|\Lambda|$.

To estimate the integral, choose a geometric sequence $\{2^m\}$ and let

$$I_m = (2^{m-2}, 2^{m+3}), \qquad J_m = (2^{m-1}, 2^{m+2}), \qquad K_m = (2^m, 2^{m+1}).$$

For $x \in J_m$ we define R and S in such a way that

$$x - R = 2^{m-2}, \qquad x + S = 2^{m+3}.$$

Then R and S have the order of x and

$$\log|P(x)| \doteq \int_{I_m} \frac{\Lambda(u)}{x-u}\,du, \qquad x \in J_m, \tag{38}$$

aside from a function of class B^+. If x_j and x_{j+1} are points in (35) on J_m, and if $x < x_j$, then

$$\left|\int_{x_j}^{x_{j+1}} \frac{\Lambda(u)}{x-u}\,du\right| = \left|\int_{x_j}^{x_{j+1}} \Lambda(u)\left(\frac{1}{x-u} - \frac{1}{x-x_j}\right)du\right| \leqslant \frac{(y_j)(D\tilde{y}_j)(y_j)}{(x-x_j)^2}. \tag{39}$$

In a like manner, when $x > x_{j+1}$ we get the denominator $(x - x_{j+1})^2$.

Now we want to integrate over K_m, using

$$\int_{K_m}\left|\int_{I_m} \frac{\Lambda(u)}{x-u}\,du\right| dx \leqslant \sum_j \int_{K_m}\left|\int_{x_j}^{x_j+y_j} \frac{\Lambda(u)}{x-u}\,du\right| dx, \tag{40}$$

where the sum is over the intervals (x_j, x_{j+1}) which overlap K_m. Since $y_j = o(x_j)$ we can assume that all the relevant points x_j are on J_m. Thus, (39) is available.

If x is not on the interval $(x_j - y_j, x_{j+1} + y_j)$ the two relations

$$\int_{-\infty}^{x_j-y_j} \frac{dx}{(x-x_j)^2} = \int_{x_{j+1}+y_j}^{\infty} \frac{dx}{(x-x_{j+1})^2} = \frac{1}{y_j}$$

give the estimate $2Dy_j\tilde{y}_j$ for the integral of (39) over the corresponding part of $(-\infty, \infty)$. There remains

$$\int_{x_j-y_j}^{x_{j+1}+y_j} \left| \int_{x_j}^{x_{j+1}} \frac{A(u)}{x-u}\, du \right| dx \leqslant \left(\int_{x_j-y_j}^{x_{j+1}+y_j} 1\, dx \right)^{1/2} \pi \left(\int_{x_j}^{x_{j+1}} | A(u)|^2\, du \right)^{1/2},$$

where we have used the Schwarz inequality and the familiar L^2 inequality $\|\hat{f}\| \leqslant \pi \|f\|$. This reduces to at most $(3^{1/2}\pi D y_j \tilde{y}_j)$. Combining with the previous estimate, we get

$$\int_{-\infty}^{\infty} \left| \int_{x_j}^{x_{j+1}} \frac{A(u)}{x-u}\, du \right| dx < \alpha y_j \tilde{y}_j ,$$

where $\alpha = 2D + (3^{1/2}\pi D)$ is constant. Naturally, the integral over K_m is dominated by the same expression, so that by (40)

$$\int_{K_m} \left| \int_{I_m} \frac{A(u)}{x-u}\, du \right| dx \leqslant \alpha \sum y_j \tilde{y}_j ,$$

the summation being over values y_j associated with some of the x_j on J_m.

If x_0 is the smallest value of x on J_m and x is on J_m, we have $x_0 \leqslant x \leqslant 8x_0$ and the above gives

$$\int_{K_m} \left| \int_{I_m} \frac{A(u)}{x-u}\, du \right| \frac{dx}{x^2} \leqslant \alpha \sum \frac{y_j \tilde{y}_j}{x_0^2} \leqslant 64\alpha \sum \left(\frac{y_j}{x_j} \right) \left(\frac{y_{j-1}}{x_{j-1}} + \frac{y_j}{x_j} + \frac{y_{j+1}}{x_{j+1}} \right).$$

By the Schwarz inequality this does not exceed

$$192\alpha \sum (y_j/x_j)^2,$$

where the summation is over x_j on J_m. Although the J_m overlap, the sum on m gives a convergent series by (35), and hence $A^* \in B$. This completes the proof for the special case.

The trouble with the argument in the general case is that it does not control the term A^∞. The condition $A^\infty \in B$ can be deduced if, and only if,

$$\sum_{n=1}^{\infty} \left(\frac{y_n}{x_n} \right) \left[\left(\frac{y_n}{x_n} \right)^3 + \left(\frac{y_{n+1}}{x_{n+1}} \right)^3 + \left(\frac{y_{n+2}}{x_{n+2}} \right)^3 + \cdots \right] < \infty.$$

This is not a consequence of the convergence of $\sum (y_n/x_n)^2$, as shown by examples.

We therefore use (32) instead of (35).

As in Section 21, the evaluation of A^∞ reduces to the integral from x to the first value x_n beyond x. Since $y_n = o(x_n)$ we have effectively

$$| A^\infty(x)| \leqslant x \int_x^{2x} \frac{H(u)}{u^2}\, du \leqslant \int_0^{2x} \frac{H(u)}{u}\, du \qquad (x > 0),$$

where $H(u) = | \Lambda(u) + \Lambda(-u)|$. In connection with Theorem 59, it was found that this function not only belongs to B, if H does, but it even belongs to B^+. Hence, when (32) holds, the term Λ^∞ is entirely harmless.

Since $\Lambda(u)/u^2$ has zero area on (x_j, x_{j+1}), it is still true that Λ must vanish at some point of this interval, and we get the same estimate $| \Lambda(u)| \leqslant D\tilde{y}_j$ as before. The integration from $x_j - y_j$ to $x_{j+1} + y_j$ is also unchanged. However, the integral from x_j to x_{j+1} must now be estimated by use of

$$\int_{x_j}^{x_{j+1}} \frac{\Lambda(u)}{x-u}\, du = \int_{x_j}^{x_{j+1}} \frac{\Lambda(u)}{u^2} \left(\frac{u^2}{x-u} - \frac{x_j^{\,2}}{x-x_j} \right) du.$$

By the mean value theorem, the term multiplying $\Lambda(u)$ has magnitude

$$\left| \frac{u - x_j}{(x-\xi)^2} \frac{\xi(2x-\xi)}{u^2} \right| \leqslant \frac{32 y_j}{(x-\xi)^2} \qquad (x_j < \xi < x_{j+1}),$$

where we have used $\xi/u \leqslant 2$ and $x/u \leqslant 8$, which holds for $x \in J_m$. If $x < x_j$ the denominator is assessed by $(x - x_j)^2$, and if $x > x_{j+1}$ it is assessed by $(x - x_{j+1})^2$. Aside from the constant factor 32 we get the same estimate (39), and this gives the result for the general case.

The essence of the analysis of this and the preceding section is as follows: If $| \Lambda_\nu(u) - D_\nu u | \in B$, the canonical product P_ν associated with $\{\nu\}$ can be multiplied by a function M_ϵ of arbitrarily small type so that the zeros λ of $P_\lambda = M_\epsilon P_\nu$ satisfy

$$| \Lambda_\lambda(u) - D_\lambda u | \in B, \qquad \log | P_\lambda(x)| \in B.$$

24. Complex Sequences

We recall the transformation

$$\frac{1}{\tilde{\lambda}} = \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\bar{\lambda}}\right) = \operatorname{Re}\left(\frac{1}{\lambda}\right),$$

which was used in connection with Theorem B, and state the following:

THEOREM 72 (Beurling and Malliavin, 1967). *If* $\sum | \operatorname{Im}(1/\lambda_n)| < \infty$ *then* $I(\lambda) = I(\tilde{\lambda})$.

In the proof, one of the two sets λ or μ will be augmented until it consists precisely of the zeros of a function F in (1), and the other set will be augmented correspondingly. If λ are the zeros of F, then

$$\sum \frac{1}{|\lambda_n|^2} < \infty, \qquad \sum \left| \operatorname{Im} \frac{1}{\lambda_n} \right| < \infty, \qquad \lim_{R\to\infty} \sum_{|\lambda_n| \leqslant R} \frac{1}{\lambda_n} \quad \text{exists.}$$

The first condition follows from Jensen's formula and the other two from Carleman's formula. Accordingly, the simplified canonical product

$$P(z) = \lim_{R\to\infty} \prod_{|\lambda_n|\leqslant R} \left(1 - \frac{z}{\lambda_n}\right)$$

exists, and need not be distinguished from $F(z)$. If

$$\sum \left| \frac{1}{\lambda_n} - \frac{1}{\mu_n} \right| < \infty,$$

then the above conditions for λ imply the same conditions for μ, except that it is by no means assured that μ are the zeros of a function (1).

To prove Theorem 72, let us first show $I(\tilde{\lambda}) \leqslant I(\lambda)$. If $2a > I(\lambda)$ there is a function F of form (1) vanishing on λ, and we enlarge λ so that it contains all the zeros of F. By the Hadamard factorization theorem F agrees with the normalized product P over the enlarged set. Theorem B gives $|\tilde{P}(x)| \leqslant |P(x)| = |F(x)|$ for the corresponding product $\tilde{P}$ over the enlarged set $\tilde{\lambda}$, as well as $\tilde{T} \leqslant T$. The Paley–Wiener theorem then gives a representation for $\tilde{P}$ which shows $I(\tilde{\lambda}) \leqslant I(\lambda)$.

The proof that $I(\lambda) \leqslant I(\tilde{\lambda})$ is more difficult. Suppose $I(\tilde{\lambda}) < 2a$ and enlarge $\tilde{\lambda}$ as before so as to contain all zeros of the corresponding function $\tilde{F}$ in (1). The function $|\tilde{F}(x)|$ would become smaller if any complex zero λ introduced by this process were replaced by the corresponding $\tilde{\lambda}$, and this change also does not increase the type. Hence, we can assume that the enlarged set $\tilde{\lambda}$ is real. Since $\lambda = \tilde{\lambda}$ for the new zeros, the enlarged set satisfies

$$\sum \left| \frac{1}{\lambda_n} - \frac{1}{\tilde{\lambda}_n} \right| = \sum \left| \operatorname{Im} \frac{1}{\lambda_n} \right| < \infty. \tag{41}$$

The function $|1 - x/\lambda|$ is clearly an even function of $\operatorname{Im}(1/\lambda)$, hence also of $\operatorname{Im}\lambda$, and thus we can assume $\operatorname{Im}\lambda < 0$ in the following evaluation:

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \log \left| \frac{1 - x/\lambda}{1 - x/\tilde{\lambda}} \right| \frac{dx}{1 + x^2}$$
$$= \log \left| \frac{1 - i/\lambda}{1 - i/\tilde{\lambda}} \right| \leqslant \left| \frac{1}{\lambda} - \frac{1}{\tilde{\lambda}} \right| + O\left(\frac{1}{|\lambda|^2} + \frac{1}{|\tilde{\lambda}|^2}\right).$$

Here the equality follows by easy contour integration, and the inequality is obvious. The integrand on the left is nonnegative, hence (41) gives $\log|P(x)| - \log|\tilde{P}(x)| \in B$ upon summation. Since $\tilde{P}$ agrees with $\tilde{F}$, we have $\log|\tilde{P}| \in B$, hence $\log|P| \in B$ also.

This shows that the type of P can be estimated by looking at $P(iy)$, and a short calculation gives $T \leqslant \tilde{T}$. By Theorem D, we multiply P down and conclude $I(\lambda) \leqslant I(\tilde{\lambda})$.

The above calculation indicates that

$$\sum \left| \frac{1}{\lambda_n} - \frac{1}{\mu_n} \right| < \infty \Rightarrow \sum \int_{-\infty}^{\infty} \log \left| \frac{1 - x/\mu_n}{1 - x/\lambda_n} \right| \frac{dx}{1 + x^2} \qquad \text{exists}$$

for complex λ and μ. If we could interchange the summation and integration it would follow much as before that $I(\lambda) = I(\mu)$.

Although the method is fallacious, the conclusion is correct, as seen later.

25. Measures of Density

We denote the signed counting function of the real sequence $\{\lambda_n\}$ by $\Lambda(u)$, and only in Theorem 75 by $\Lambda_\lambda(u)$ as in Sections 18–23.

For any positive constant a let M_a be the class of continuous, piecewise differentiable functions satisfying $0 \leqslant \phi'(x) \leqslant a$ at points where the derivative exists. According to Beurling and Malliavin, the *shadow function* $S_a(u)$ for the

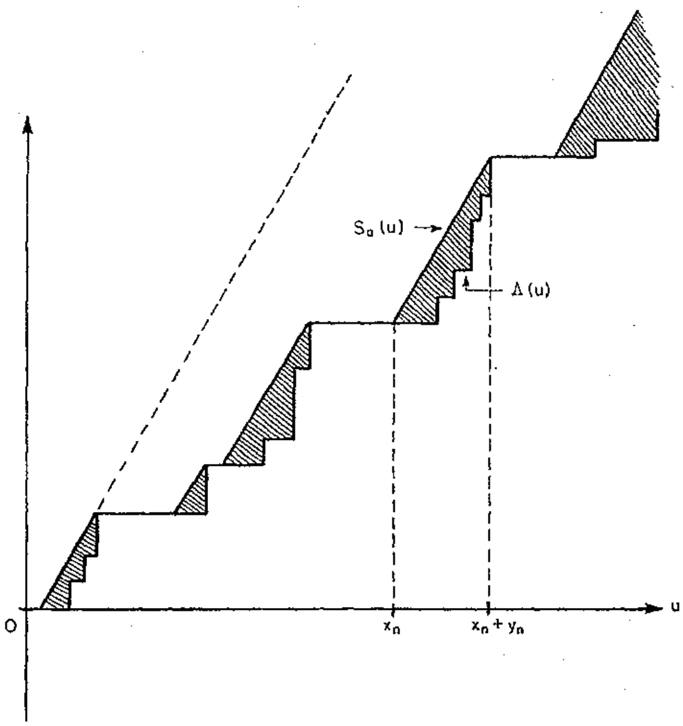

Figure 10

counting function $\Lambda(u)$ is the smallest majorant of Λ which is in the class M_a; see Fig. 10. If $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ is a given sequence of positive numbers, the *effective density* in the sense of Beurling and Malliavin is

$$A(\lambda) = \inf a \qquad \text{for which} \qquad S_a(u) - \Lambda(u) \in B.$$

THEOREM 73 (Beurling and Malliavin, 1961). *If* $\lambda_n \geqslant 0$ *then* $I(\lambda) \geqslant 2\pi A(\lambda)$ *and* $I(\pm\lambda) \leqslant 2\pi A(\lambda)$.

We give the proof in outline only, taking simple properties of the shadow function for granted. Let $a > A(\lambda)$, and add enough evenly spaced λ's on the intervals where $S_a(u)$ is horizontal to straighten out the graph of S_a, as suggested by the slanting dotted line in Fig. 10. Then augment the set $\{\lambda\}$ still more by adjoining for each λ its negative $-\lambda$. The counting function for the enlarged set satisfies $\Lambda(u) - au \in B$ and hence $I(\lambda) \leqslant 2\pi a$ by Theorem 71.

On the other hand if $a < A(\lambda)$ let the graph of S_a have slope a on intervals $(x_n, x_n + y_n)$ as shown in Fig. 10. The definition of $A(\lambda)$ as inf shows that $\sum (y_n/x_n)^2$ diverges; otherwise we would have $S_a(u) - \Lambda(u) \in B$. Since $\Lambda(x_n + y_n) - \Lambda(x_n) = ay_n$, Theorem 34 gives $I(\lambda) \geqslant 2\pi a$ and completes the proof.

The shadow function not only belongs to M_2, but has the additional properties that $S_a(u) - \Lambda(u) \geqslant 0$. Another density, also introduced by Beurling and Malliavin, is

$$B(\lambda) = \inf a \qquad \text{such that there exists} \qquad \phi \in M_a, \qquad \phi - \Lambda \in B.$$

Since Theorem 71 makes no assumption about the sign of $\Lambda(u) - Du$, we get $I(\pm\lambda) \leqslant 2\pi B(\lambda)$ just as for $A(\lambda)$. On the other hand $A(\lambda) \geqslant B(\lambda)$ since $A(\lambda)$ involves an extra restriction on the sign of $\phi - \Lambda$. Hence the above proof that $I(\lambda) \geqslant 2\pi A(\lambda)$ gives also $I(\lambda) \geqslant 2\pi B(\lambda)$, and it follows that $I(\lambda) = 2\pi B(\lambda)$.

So far we have taken $\lambda > 0$, in agreement with the reference cited. However, Theorem 34 gives $I(\lambda) \geqslant 2\pi D$ where E is the larger of the two densities obtained for $u \to \infty$ or $u \to -\infty$, and Theorem 71 gives $I(\lambda) \leqslant 2\pi D$ even if λ is distributed on the whole real axis. With a natural extension of $A(\lambda)$ and $B(\lambda)$ to allow $-\infty < \lambda < \infty$, we have thus established:

THEOREM 74 (Beurling and Malliavin, 1961, 1967). *If* $\{\lambda_n\}$ *is real then* $I(\lambda) = 2\pi A(\lambda) = 2\pi B(\lambda)$.

Because of the structure of $B(\lambda)$, Theorem 74 has the following corollary:

THEOREM 75. *If* $\Lambda_\lambda(u) - \Lambda_\mu(u) \in B$ *then* $I(\lambda) = I(\mu)$.

This differs from other theorems of the sort in that the proof involves separate consideration of λ and μ. It would be desirable to have a proof along the lines of $I(\mu) \leqslant I(\lambda)$ and symmetry, but no such proof is known.

A third measure of density is used in (Beurling and Malliavin, 1967) to describe a sharp density theorem for entire functions, and also to express the final form of their completeness criterion. Let a closed, open, or semiclosed interval ω be represented by a point $T\omega = x(\omega) + iy(\omega)$ in the upper half plane, where

$$x(\omega) = \text{center of } \omega, \qquad y(\omega) = \text{length of } \omega.$$

If Ω is a set of intervals such that the set of points $T\Omega$ is measurable, the measure assigned to Ω is

$$m(\Omega) = \int_{T\Omega} \frac{dx\, dy}{1 + x^2 + y^2}.$$

If ω is an interval, $\bar{\omega}$ denotes the collection of all its subintervals, and by definition $\bar{\Omega} = \bar{\omega}$ for $\omega \in \Omega$. Any set of intervals Ω is *negligible* in the sense of Beurling and Malliavin if $m(\bar{\Omega}) < \infty$.

A positive measure $d\mu$ on the real axis is regular and of density $A(d\mu) = a$ if the family of intervals

$$\Omega_\epsilon = \left\{\omega \text{ such that} \left| (1/|\,\omega\,|) \int_\omega d\mu - a \right| > \epsilon\right\}$$

is negligible for each $\epsilon > 0$; here $|\,\omega\,|$ denotes the length of ω. The exterior density $A_e(d\mu)$ is the lower bound of $A(d\nu)$ for regular $d\nu \geqslant d\mu$.

The extension to complex sets is made by Theorem 72. Since any finite number of λ can be changed without altering the completeness interval, we ignore a possible value $\lambda = 0$ of finite multiplicity and compute all expressions involving reciprocals over the λ satisfying $\lambda \neq 0$. Referring to Theorem 7, let us agree that $C(\lambda) = \infty$ if $\sum |\operatorname{Im}(1/\lambda_n)| = \infty$ and otherwise $C(\lambda) = A_e(d\tilde{\Lambda})$ where $\tilde{\Lambda}$ is the counting function for the set $\tilde{\lambda}$ introduced in Theorem 72. The final form of the Beurling–Malliavin formula is then as follows:

THEOREM 76 (Beurling and Malliavin, 1967). *If λ is complex, $I(\lambda) = 2\pi C(\lambda)$.*

We now describe a fourth measure of density which applies to complex λ without introduction of $\tilde{\lambda}$. The above convention regarding reciprocals is retained.

It is said that the positive number c *belongs to* λ if there exists a sequence $\{\nu_k\}$ of distinct integers such that

$$\sum \left| \frac{1}{\lambda_n} - \frac{c}{\nu_n} \right| < \infty.$$

The set of all c belonging to λ is either empty, or is a semiinfinite interval of the real axis. Let the left-hand end point of this interval be denoted by $D(\lambda)$, so that $D(\lambda) = \infty$ if no c belongs to λ, otherwise

$$D(\lambda) = \inf c \qquad \text{such that } c \text{ belongs to } \lambda.$$

THEOREM 77 (Redheffer, 1967). *If λ is complex, $I(\lambda) = 2\pi D(\lambda)$.*

By Theorem 72 it suffices to consider real sequences; note that $\sum |\operatorname{Im}(1/\lambda_n)| = \infty$ gives $D(\lambda) = \infty$ automatically since c/ν is real. Theorem 77 then follows from Theorem 74 as shown in (Redheffer, 1972). It would be desirable to have a more direct proof that $D(\lambda)$ agrees with the Beurling–Malliavin density $C(\lambda)$ but such a proof is not available.

An immediate consequence of Theorem 77 is:

THEOREM 78. *If λ and μ are complex, then*

$$\sum \left| \frac{1}{\lambda_n} - \frac{1}{\mu_n} \right| < \infty \Rightarrow I(\lambda) = I(\mu). \tag{42}$$

This forms a counterpart to the elementary implication

$$\sum | \lambda_n - \mu_n | < \infty \Rightarrow E(\lambda) = E(\mu),$$

which is a special case of Theorem 14.

Convergence of the series (42) is just what is needed for existence of the meromorphic function used elsewhere to show $I(\mu) \leqslant I(\lambda)$, but no proof along these lines is known. It is not hard to show that Theorem 78 implies Theorem 77; hence an elementary proof of Theorem 78 would solve the completeness problem with no advanced analysis at all. This makes it appear doubtful that an elementary proof of Theorem 78 will be found.

As pointed out in the early work of Beurling and Malliavin, the fact that various measures of density agree with $I(\lambda)/2\pi$ shows that they agree with each other. In particular, we have:

THEOREM 79. *If λ is real then $A(\lambda) = B(\lambda) = C(\lambda) = D(\lambda)$.*

One of the main applications of the exterior density associated with $C(\lambda)$ is to sharpen the Levinson density theorem, Theorem B. This aspect of the subject is not discussed here, but we conclude by mentioning a simpler result of similar nature. Let E_a denote the class of entire functions F of type a such that $\log |F(x)| \in B$. Then the following holds:

THEOREM 80 (Beurling and Malliavin, 1967). *If $a > \pi C(\lambda)$ then E_a contains a function vanishing on λ, and if $a < \pi C(\lambda)$ then E_a contains no function vanishing on λ.*

The first statement follows from Theorem 76 and from the fact that functions of form (1) belong to E_a. For the second statement, if there is a function in E_a vanishing on λ we could multiply it down by Theorem D and conclude that $I(\lambda) \leqslant 2a$. This contradicts Theorem 76.

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