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BMO from dyadic BMO on the bidisc

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Abstract

We generalize to the bidisc a theorem of Garnett and Jones relating the space BMO of functions of bounded mean oscillation to its martingale counterpart, dyadic BMO. Namely, translation-averages of suitable families of dyadic BMO functions belong to BMO. As a corollary, we deduce a biparameter version of a theorem of Burgess Davis connecting the Hardy space H^1 to martingale H^1. We also prove the analogs of the theorem of Garnett and Jones in the one-parameter and biparameter VMO spaces of functions of vanishing mean oscillation.

1. *Introduction*

Garnett and Jones [9] introduced a method for obtaining decomposition theorems for the space BMO of functions of bounded mean oscillation by a reduction to the dyadic space BMO_d, involving averaging over the translations of a family of functions. Specifically, they concluded the following theorem.

Theorem 1 (Garnett and Jones). *Suppose that $\alpha \mapsto \varphi^\alpha$ is a measurable mapping from $\mathbb{R}^m$ to the space $\mathrm{BMO}_\mathrm{d}(\mathbb{R}^m)$ of functions of dyadic bounded mean oscillation such that all $\varphi^\alpha(x)$ have support in a fixed dyadic cube, that $\|\varphi^\alpha\|_d \leqslant 1$ and that*

$$\int \varphi^\alpha(x)\, dx = 0.$$

Then

$$\varphi_N(x) := \frac{1}{(2N)^m} \int_{|\alpha_j| \leqslant N} \varphi^\alpha(x+\alpha)\, d\alpha$$

is in $\mathrm{BMO}(\mathbb{R}^m)$ and $\|\varphi_N\|_* \leqslant C$.

In this paper we work in the setting of the circle $\mathbb{T}$, and later the bidisc $\mathbb{T} \otimes \mathbb{T}$, rather than $\mathbb{R}^m$. For instance, in the circle setting the object of interest is the *translation-average*

$$\varphi(x) := \int_0^1 \varphi^\alpha(x+\alpha)\, d\alpha$$

of a family of $\mathrm{BMO}_\mathrm{d}(\mathbb{T})$ functions. Here $x+\alpha$ is to be understood as $x + \alpha \mod 1$.

Theorem 1 (unnumbered in [9]) follows implicitly from a stopping-time argument in their proof of a theorem of Carleson. We present in Section 3 a proof, for the circle, which does not require a stopping-time argument. Our method, together with Journé's lemma, allows us to prove a biparameter version of Theorem 1 for the Chang–Fefferman space of BMO functions on the bidisc (Theorem 2). We also prove similar results for the VMO spaces of functions of

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vanishing mean oscillation on the circle and on the bidisc. As a corollary of Theorem 2, we obtain a biparameter version (Theorem 6) of a theorem of Davis [7], namely that almost every translate of an H^1 function belongs to dyadic H^1.

The inherent difficulty in working with the multiparameter BMO and VMO spaces is the structure (or rather, the lack of structure) of the open sets. In the one-parameter setting open sets reduce to unions of disjoint intervals, but an open set in $\mathbb{R}^2$ has no canonical decomposition in terms of collections of disjoint rectangles. However, the geometric decomposition in Journé's lemma can permit a reduction to rectangles for certain estimates, and for ours in particular.

The paper is organized as follows. We recall some definitions (Section 2) and give a proof of the BMO result on the circle (Theorem 2 in Section 3). We prove the analogous result for VMO in Section 4. In Section 5 we prove the averaging result in the setting of BMO of the bidisc, as well as the generalization of Davis's theorem to H^1 functions on the bidisc. Section 6 contains our proof of the averaging theorem for VMO of the bidisc.

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2. *Definitions*

A real-valued function $f \in L^1(\mathbb{T})$ is in the space $\mathrm{BMO}(\mathbb{T})$ of functions of bounded mean oscillation on the circle if its BMO norm is finite:

$$\|f\|_* := \sup_{I \subset \mathbb{T}} \frac{1}{|I|} \int_I |f(x) - (f)_I| \, dx < \infty.$$

Here $(f)_I := (1/|I|) \int_I f(x)\, dx$ is the average value of f on the interval I, and the circle $\mathbb{T}$ is the interval $[0,1]$ with endpoints identified. Dyadic BMO of the circle, written $\mathrm{BMO_d}(\mathbb{T})$, is the space of functions which satisfy the corresponding estimate where the supremum is taken over all $I \in \mathcal{D}$, where $\mathcal{D} = \mathcal{D}[0,1]$ is the collection of dyadic subintervals of $[0,1]$. The dyadic BMO norm of f is denoted by $\|f\|_{\mathrm{d}}$.

We use a characterization of the dyadic BMO functions on the circle in terms of the size of Haar coefficients. The Haar function associated with the dyadic interval I is

$$h_I(x) := \begin{cases} |I|^{-1/2} & \text{if } x \in I_{\mathrm{l}}, \\ -|I|^{-1/2} & \text{if } x \in I_{\mathrm{r}}, \\ 0 & \text{otherwise.} \end{cases}$$

As usual I_{l} and I_{r} are the left and right halves, respectively, of the interval I. The Haar coefficient over I of f is

$$f_I = (f, h_I) := \int_I f(x) h_I(x)\, dx,$$

the Haar series for f is

$$f(x) := \sum_{I \in \mathcal{D}} (f, h_I)\, h_I(x),$$

and the L^2-norm of f is

$$\|f\|_{\mathrm{d},2} = \sum_{J \in \mathcal{D}} (f, h_J)^2.$$

It follows from the John–Nirenberg theorem [8, p. 230] that for each $p \geqslant 1$, for $f \in L^1(\mathbb{T})$ the expression

$$\|f\|_{\mathrm{d},p} := \sup_{I \in \mathcal{D}} \left(\frac{1}{|I|} \int_I |f(x) - (f)_I|^p \, dx \right)^{1/p}$$

is comparable to the dyadic BMO norm $\|f\|_{\mathrm{d}}$.

In particular, a function $f \in L^1(\mathbb{T})$ of mean value zero is in $\mathrm{BMO_d}(\mathbb{T})$ if and only if there is a constant C such that for all $I \in \mathcal{D}$,

$$\sum_{J \subset I,\, J \in \mathcal{D}} (f, h_J)^2 \leqslant C|I|. \tag{1}$$

Moreover, the smallest such constant C is equal to $\|f\|_{\mathrm{d},2}^2$.

Note that since the sum in (1) ranges over dyadic intervals only, there is no need to restrict the interval I itself to be dyadic. Here the notation $J \subset I$ includes the case $J = I$ if I is dyadic; we will also use the notation $\sum_{J \subseteq I}$ and $\sum_{J \supset I}$ for clarity.

A function is in BMO if and only if it satisfies (1) with a continuous wavelet expansion replacing the Haar series. When we define BMO on the bidisc, we will make use of the particular representation employed in Chang–Fefferman [**5**].

On the bidisc $\mathbb{T} \otimes \mathbb{T}$, we have an expansion of functions in terms of a double Haar series

$$f(x) = \sum_{R \in \mathcal{D} \otimes \mathcal{D}} (f, h_R) h_R(x),$$

where R denotes a dyadic rectangle $R = I \times J$ and $h_R = h_I \otimes h_J$.

Definition 1 (Dyadic product BMO). A function $f \in L^1(\mathbb{T} \otimes \mathbb{T})$ belongs to $\mathrm{BMO_d}(\mathbb{T} \otimes \mathbb{T})$ if there exists a constant C such that for every open set Ω,

$$\sum_{R \subset \Omega,\, R \in \mathcal{D} \otimes \mathcal{D}} (f, h_R)^2 \leqslant C|\Omega|. \tag{2}$$

See [**1**], and also [**2**], for equivalent definitions.

We now define BMO on the bidisc, recalling first the concept of the Carleson region associated to an open set. For an interval I, the associated Carleson box in the upper half-plane is $T(I) := I \times (0, \mathrm{length}(I))$. For a rectangle $R = I \times J$, the associated Carleson box in the product upper half-plane is $T(R) := T(I) \times T(J)$. For an open set Ω in the bidisc, define $T(\Omega) := \bigcup_{R \subset \Omega} T(R)$.

Let $\psi(x)$ be a smooth function supported on $[-1, 1]$ with mean value zero, and define the usual dilation $\psi_y(x) := y^{-1}\psi(x/y)$ for $y > 0$. In what follows we write $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $t = (t_1, t_2)$, and abbreviate the product $\psi_{y_1}(x_1)\psi_{y_2}(x_2)$ by $\psi_y(x)$. Thus for f defined on the bidisc, the expression $f * \psi_y(x)$ denotes the iterated convolution

$$f * \psi_{y_1}(x_1)\psi_{y_2}(x_2) = \iint f(x_1 - t_1, x_2 - t_2)\psi_{y_1}(t_1)\psi_{y_2}(t_2)\, dt_1\, dt_2.$$

When the function ψ is radial and satisfies the additional property

$$\int_0^\infty |\widehat{\psi}(t)|^2 \frac{dt}{t} = 1,$$

one has the Calderón–Torchinsky representation for $f \in L^2$:

$$f(x) = \iint f * \psi_y(t)\psi_y(x - t) \frac{dt_1\, dt_2\, dy_1\, dy_2}{y_1 y_2}.$$

See [**5**]. This representation in turn leads to a wavelet expansion of f, by decomposing the product upper half-plane into disjoint dyadic regions corresponding to top halves of Carleson boxes. Specifically, if for I dyadic of length $|I|$ we set $I^+ = I \times (|I|/2, |I|)$, and for $R = I \times J$ we set $R^+ = I^+ \times J^+$, then

$$f(x) = \sum_{R \in \mathcal{D} \otimes \mathcal{D}} \iint_{R^+} f * \psi_y(t)\psi_y(x - t) \frac{dt_1\, dt_2\, dy_1\, dy_2}{y_1 y_2}.$$

The following definition, from [**4**], therefore gives the (continuous) wavelet analog BMO of $\mathrm{BMO_d}$.

Definition 2 (Product BMO). A function f belongs to $\mathrm{BMO}(\mathbb{T}\otimes\mathbb{T})$ if there exists a constant C such that, for all open sets Ω, the Carleson-measure condition holds:

$$\iint_{T(\Omega)} |f * \psi_y(t)|^2 \,\frac{dt_1\,dt_2\,dy_1\,dy_2}{y_1 y_2} \leqslant C|\Omega|. \tag{3}$$

We defer the definitions of $\mathrm{VMO}(\mathbb{T})$, $\mathrm{VMO_d}(\mathbb{T})$, $\mathrm{VMO}(\mathbb{T}\otimes\mathbb{T})$, and $\mathrm{VMO_d}(\mathbb{T}\otimes\mathbb{T})$ to Sections 4 and 6.

3. $\mathrm{BMO}(\mathbb{T})$ *from averaging* $\mathrm{BMO_d}(\mathbb{T})$

We give a proof of the Garnett–Jones theorem on the circle $\mathbb{T}$.

Theorem 2. *Suppose that $\varphi^\alpha \in \mathrm{BMO_d}(\mathbb{T})$ for each $\alpha \in [0,1]$, that $\alpha \mapsto \varphi^\alpha$ is measurable, and that the $\mathrm{BMO_d}$ norms of the functions φ^α are uniformly bounded: there is a constant C_{d} such that*

$$\|\varphi^\alpha\|_{\mathrm{d}} \leqslant C_{\mathrm{d}}$$

for all $\alpha \in [0,1]$. Suppose also that

$$\int_{\mathbb{T}} \varphi^\alpha(x)\,dx = 0 \quad \text{for all } \alpha \in [0,1].$$

Then the translation-average

$$\varphi(x) := \int_0^1 \varphi^\alpha(x+\alpha)\,d\alpha$$

is in $\mathrm{BMO}(\mathbb{T})$.

Proof of Theorem 2. Using the Haar expansions of the functions φ^α, we write the translation-average $\varphi(x)$ as

$$\begin{aligned}
\varphi(x) = \int_0^1 \varphi^\alpha(x+\alpha)\,d\alpha &= \int_0^1 \sum_{I\in\mathcal{D}} (\varphi^\alpha, h_I)\, h_I(x+\alpha)\,d\alpha \\
&= \int_0^1 \sum_{n\in\mathbb{N}} \sum_{I\in\mathcal{D}_n} (\varphi^\alpha, h_I)\, h_I(x+\alpha)\,d\alpha \\
&= \sum_{n\in\mathbb{N}} \int_0^1 \sum_{I\in\mathcal{D}_n} (\varphi^\alpha, h_I)\, h_I(x+\alpha)\,d\alpha \\
&= \sum_{n\in\mathbb{N}} \int_0^1 \varphi^\alpha_n(x+\alpha)\,d\alpha \\
&= \sum_{n\in\mathbb{N}} \varphi_n(x).
\end{aligned}$$

Here $\mathcal{D}_n := \{I \in \mathcal{D} \mid |I| = 2^{-n}\}$ for $n \in \mathbb{N}$, and we have set

$$\varphi^\alpha_n(x) := \sum_{I\in\mathcal{D}_n} (\varphi^\alpha, h_I)\, h_I(x)$$

and

$$\varphi_n(x) := \int_0^1 \varphi^\alpha_n(x+\alpha)\,d\alpha = \int_0^1 \sum_{I\in\mathcal{D}_n} (\varphi^\alpha, h_I)\, h_I(x+\alpha)\,d\alpha.$$

Fix an interval $Q \subset \mathbb{T}$, not necessarily dyadic. Let N be the unique non-negative integer such that

$$2^{-N-1} < |Q| \leqslant 2^{-N}.$$

We split the sum, at the scale of $|Q|$, into two parts φ_A and φ_B in which the dyadic intervals I are, respectively, small and large compared with Q:

$$\varphi = \varphi_A + \varphi_B, \quad \varphi_A(x) := \sum_{n:2^{-n}<|Q|} \varphi_n(x), \quad \varphi_B(x) := \sum_{n:2^{-n}\geqslant|Q|} \varphi_n(x).$$

To prove that φ belongs to BMO, it suffices to show that there are constants C_A and C_B independent of Q, and a constant c_Q depending on Q, such that

$$\frac{1}{|Q|} \int_Q |\varphi_A(x)|^2 \, dx \leqslant C_A, \tag{4}$$

$$\frac{1}{|Q|} \int_Q |\varphi_B(x) - c_Q| \, dx \leqslant C_B. \tag{5}$$

We begin with inequality (4). The left-hand side is

$$\begin{aligned} ⨍_Q |\varphi_A(x)|^2 \, dx &= ⨍_Q \left| \sum_{n:2^{-n}<|Q|} \int_0^1 \varphi_n^\alpha(x+\alpha) \, d\alpha \right|^2 dx \\ &\leqslant \int_0^1 ⨍_Q \left| \sum_{n:2^{-n}<|Q|} \varphi_n^\alpha(x+\alpha) \right|^2 dx \, d\alpha. \end{aligned} \tag{6}$$

Fix an $\alpha \in [0,1]$. We shall provide a uniform estimate of the α-integrand in the last line of inequality (6). Let $Q_\alpha := Q - \alpha$ be the translate of Q to the left by α. Therefore $2^{-N-1} < |Q_\alpha| \leqslant 2^{-N}$. Now Q_α may be covered by at most two adjacent dyadic intervals Q_1, Q_2 of length $|Q_1| = |Q_2| = 2^{-N}$, such that $|Q_1 \cup Q_2| \leqslant 4|Q_\alpha|$. We obtain

$$\begin{aligned} \frac{1}{|Q|} \int_Q \left| \sum_{n:2^{-n}<|Q|} \varphi_n^\alpha(x+\alpha) \right|^2 dx &= \frac{1}{|Q_\alpha|} \int_{Q_\alpha} \left| \sum_{n:2^{-n}<|Q|} \varphi_n^\alpha(x) \right|^2 dx \\ &\leqslant \frac{1}{|Q_\alpha|} \int_{Q_1\cup Q_2} \left| \sum_{n:2^{-n}<|Q|} \varphi_n^\alpha(x) \right|^2 dx \\ &= \frac{|Q_1 \cup Q_2|}{|Q_\alpha|} ⨍_{Q_1\cup Q_2} \left| \sum_{n:2^{-n}<|Q|} \varphi_n^\alpha(x) \right|^2 dx \\ &\leqslant 4 ⨍_{Q_1} \left| \sum_{n:2^{-n}<|Q|} \varphi_n^\alpha(x) \right|^2 dx + 4 ⨍_{Q_2} \left| \sum_{n:2^{-n}<|Q|} \varphi_n^\alpha(x) \right|^2 dx. \end{aligned}$$

The interval Q_1 is dyadic and the functions φ^α are uniformly bounded in $\mathrm{BMO_d}$, so as in our discussion of equation (1) there is a constant C_d independent of Q_1 such that for all $\alpha \in \mathbb{T}$

$$\sum_{I\subset Q_1,\, I\in\mathcal{D}} (\varphi^\alpha, h_I)^2 \leqslant C_\mathrm{d} \, |Q_1|. \tag{7}$$

Therefore

$$\fint_{Q_1} \Big| \sum_{n:2^{-n}<|Q|} \varphi_n^\alpha(x) \Big|^2 \, dx = \frac{1}{|Q_1|} \Big\| \sum_{I\in\mathcal{D}(Q_1)} (\varphi^\alpha, h_I) h_I \Big\|^2$$
$$= \frac{1}{|Q_1|} \sum_{I\in\mathcal{D}(Q_1)} (\varphi^\alpha, h_I)^2$$
$$\leqslant C_{\mathrm{d}}.$$

Applying the same argument to Q_2 and integrating over $\alpha \in [0,1]$, we obtain inequality (4).

We turn to inequality (5). Recall that Q is a fixed interval in the circle $\mathbb{T}$, not necessarily dyadic. Also

$$\varphi_B(x) = \sum_{n:2^{-n}\geqslant|Q|} \varphi_n(x).$$

Fix a point $x_0 \in Q$. For instance, let x_0 be the left endpoint of Q. Let

$$c_Q := \varphi_B(x_0) = \sum_{n:2^{-n}\geqslant|Q|} \varphi_n(x_0).$$

Then, writing $I_\alpha := I - \alpha$ when $I \in \mathcal{D}_n$, we have

$$\fint_Q |\varphi_B(x) - c_Q| \, dx = \fint_Q \Big| \sum_{n:2^{-n}\geqslant|Q|} \int_0^1 \sum_{I\in\mathcal{D}_n} (\varphi^\alpha, h_I) \Big[h_I(x+\alpha) - h_I(x_0+\alpha) \Big] \, d\alpha \Big| \, dx$$
$$\leqslant \sum_{n:2^{-n}\geqslant|Q|} \fint_Q \Big| \int_0^1 \sum_{I\in\mathcal{D}_n} (\varphi^\alpha, h_I) \Big[h_{I_\alpha}(x) - h_{I_\alpha}(x_0) \Big] \, d\alpha \Big| \, dx. \tag{8}$$

We must show that this last expression is bounded by some C_B, independent of Q. Let

$$g_n(x,x_0) := \int_0^1 \sum_{I\in\mathcal{D}_n} (\varphi^\alpha, h_I) \Big[h_{I_\alpha}(x) - h_{I_\alpha}(x_0) \Big] \, d\alpha.$$

For fixed $x \in Q$, $x_0 \in Q$, the expression $h_{I_\alpha}(x) - h_{I_\alpha}(x_0)$ will be zero for many values of α. We have $|x - x_0| \leqslant |Q| \leqslant |I|$. We consider two cases: (i) when $|x - x_0| \leqslant |I|/2$, and (ii) when $|I|/2 < |x - x_0| \leqslant |I|$. In case (i), the expression $h_{I_\alpha}(x) - h_{I_\alpha}(x_0)$ can be non-zero only in two situations. First, $h_{I_\alpha}(x) - h_{I_\alpha}(x_0)$ is non-zero when α is such that the midpoint of I_α falls between x and x_0. This happens exactly when α lies in a particular interval, call it $A_{x,x_0,I}$, of length $|x - x_0|$. Second, $h_{I_\alpha}(x) - h_{I_\alpha}(x_0)$ is non-zero when one of the endpoints of I_α falls between x and x_0. This happens exactly when α lies in a set, call it $B_{x,x_0,I}$, consisting of the union of two intervals, each of length $|x - x_0|$. In the first situation, the value of $|h_{I_\alpha}(x) - h_{I_\alpha}(x_0)|$ is $2|I|^{-1/2}$, and in the second situation it is $|I|^{-1/2}$. In short,

$$|h_{I_\alpha}(x) - h_{I_\alpha}(x_0)| \leqslant 2|I|^{-1/2}$$

when $\alpha \in E_{x,x_0,I} := A_{x,x_0,I} \cup B_{x,x_0,I}$, and $|h_{I_\alpha}(x) - h_{I_\alpha}(x_0)| = 0$ for all other α. Here $|E_{x,x_0,I}| \leqslant 3|x - x_0|$.

In case (ii), $|I|/2 < |x - x_0| \leqslant |I|$, and so x and x_0 never fall in the same half of I_α. Then $h_{I_\alpha}(x) - h_{I_\alpha}(x_0)$ can be non-zero only when α lies in one single interval, call it $E_{x,x_0,I}$, of length $|E_{x,x_0,I}| = |I| + |x - x_0| \leqslant 3|x - x_0|$. When $\alpha \in E_{x,x_0,I}$, we have $|h_{I_\alpha}(x) - h_{I_\alpha}(x_0)| \leqslant 2|I|^{-1/2}$ as in case (i).

We also note the following estimate on Haar coefficients of $\mathrm{BMO_d}$ functions: for each $\alpha \in \mathbb{T}$ and for each $I \in \mathcal{D}$,

$$|(\varphi^\alpha, h_I)| \, |I|^{-1/2} \leqslant \fint_I |\varphi^\alpha(x) - (\varphi^\alpha)_I| \, dx \leqslant \|\varphi^\alpha\|_{\mathrm{d}} \leqslant C_{\mathrm{d}}, \tag{9}$$

where C_{d} is the uniform bound on the dyadic BMO norms of the functions φ^α.

Now we can estimate $|g_n(x,x_0)|$, using inequality (9) in the last line:

$$\begin{aligned}
|g_n(x,x_0)| &= \left| \int_0^1 \sum_{I\in\mathcal{D}_n} (\varphi^\alpha, h_I)\Big[h_{I_\alpha}(x) - h_{I_\alpha}(x_0)\Big]\, d\alpha \right| \\
&= \left| \sum_{I\in\mathcal{D}_n} \int_0^1 (\varphi^\alpha, h_I)\Big[h_{I_\alpha}(x) - h_{I_\alpha}(x_0)\Big]\, d\alpha \right| \\
&= \left| \sum_{I\in\mathcal{D}_n} \int_{E_{x,x_0,I}} (\varphi^\alpha, h_I)\Big[h_{I_\alpha}(x) - h_{I_\alpha}(x_0)\Big]\, d\alpha \right| \\
&\leqslant \sum_{I\in\mathcal{D}_n} \int_{E_{x,x_0,I}} |(\varphi^\alpha, h_I)| \Big|\Big[h_{I_\alpha}(x) - h_{I_\alpha}(x_0)\Big]\Big|\, d\alpha \\
&\leqslant \sum_{I\in\mathcal{D}_n} \int_{E_{x,x_0,I}} |(\varphi^\alpha, h_I)|\, 2\, |I|^{-1/2}\, d\alpha \\
&\leqslant 2^n \cdot 2 \cdot C_{\mathrm{d}} \cdot 3|x - x_0|.
\end{aligned} \tag{10}$$

Therefore, using inequalities (8) and (10), we obtain

$$\begin{aligned}
⨍_Q |\varphi_B(x) - c_Q|\, dx &\leqslant \sum_{n:2^{-n}\geqslant|Q|} ⨍_Q |g_n(x,x_0)|\, dx \\
&\leqslant \sum_{n:2^{-n}\geqslant|Q|} ⨍_Q 6\cdot 2^n \cdot C_{\mathrm{d}} \cdot |x - x_0|\, dx \\
&= 6\, C_{\mathrm{d}} \sum_{n:2^{-n}\geqslant|Q|} 2^n ⨍_Q |x - x_0|\, dx \\
&\leqslant 6\, C_{\mathrm{d}} \sum_{n:2^{-n}\geqslant|Q|} 2^n \frac{|Q|}{2} \\
&\leqslant 6\, C_{\mathrm{d}}.
\end{aligned}$$

This proves inequality (5), and hence Theorem 2. $\square$

4. VMO($\mathbb{T}$) *from averaging* VMO$_{\mathrm{d}}$($\mathbb{T}$)

In this section we define the space VMO($\mathbb{T}$) of functions of vanishing mean oscillation on the circle, and the corresponding dyadic space VMO$_{\mathrm{d}}$($\mathbb{T}$). Then we state and prove the averaging theorem for VMO, namely that translation-averages of suitable VMO$_{\mathrm{d}}$($\mathbb{T}$) functions belong to VMO($\mathbb{T}$).

The space VMO was introduced by Sarason in [**14**]. A function belongs to VMO if its BMO norm goes to zero uniformly as the intervals shrink to zero, or equivalently if the function belongs to the closure of the continuous functions C_0^∞ in BMO.

Definition 3 (VMO). A function $f \in$ BMO($\mathbb{T}$) belongs to VMO($\mathbb{T}$) if for each $\varepsilon > 0$ there exists a δ such that for all intervals I with $|I| < \delta$,

$$\frac{1}{|I|} \int_I |f(x) - (f)_I|\, dx \leqslant \varepsilon |I|.$$

Definition 4 (Dyadic VMO). A function f belongs to $\mathrm{VMO_d}(\mathbb{T})$ if for each $\varepsilon > 0$ there exists a δ such that the BMO norm of

$$\sum_{\substack{J\in\mathcal{D},\\ |J|<\delta}} (f, h_J)h_J(x)$$

is at most ε.

Theorem 3. *Suppose that the functions φ^α satisfy the hypotheses of Theorem 2 and, in addition, belong to* $\mathrm{VMO_d}(\mathbb{T})$ *uniformly: for each $\varepsilon > 0$ there is a δ such that for all $\alpha \in [0, 1]$,*

$$\left\| \sum_{\substack{|J|<\delta,\\ J\in\mathcal{D}}} (\varphi^\alpha, h_J)h_J(x) \right\|_* \leqslant \varepsilon.$$

Then the translation-average

$$\varphi(x) := \int_0^1 \varphi^\alpha(x + \alpha)\, d\alpha$$

is in $\mathrm{VMO}(\mathbb{T})$.

Proof. The proof follows the same lines as that of the BMO result: we split φ into two functions, one corresponding to the part of the expansion over small intervals (this part has small BMO norm), and the remaining function which is controlled by averaging. Fix an $\varepsilon > 0$. For this ε, we have on hand a δ that is guaranteed by the uniform VMO condition on the functions φ^α. Choose a large N equal to $N(\varepsilon, \delta)$ satisfying $2^{-N} < \delta$. We aim to find a K such that if $|Q| < 2^{-K}$ then

$$\frac{1}{|Q|} \int_Q |\varphi(x) - (\varphi)_Q|\, dx \leqslant \varepsilon.$$

Split $\varphi = \varphi_1 + \varphi_2$, where

$$\varphi_1(x) := \int_0^1 \sum_{\substack{I\in\mathcal{D},\\ |I|<2^{-N}}} (\varphi^\alpha, h_I)h_I(x + \alpha)\, d\alpha$$

and

$$\varphi_2(x) := \int_0^1 \sum_{\substack{I\in\mathcal{D},\\ |I|\geqslant 2^{-N}}} (\varphi^\alpha, h_I)h_I(x + \alpha)\, d\alpha.$$

We claim that for $|Q| < 2^{-K}$ and K sufficiently large, there is a constant c_Q such that

$$\frac{1}{|Q|} \int_Q |\varphi_1(x) - c_Q|\, dx \leqslant \varepsilon. \tag{11}$$

To see this, fix such a Q and make a further split of φ_1 as in the proof of Theorem 2: $\varphi_1 = \varphi_{1,A} + \varphi_{1,B}$, where

$$\varphi_{1,A}(x) = \int_0^1 \sum_{\substack{I\in\mathcal{D},\\ |I|\leqslant 2^{-K}}} (\varphi^\alpha, h_I)h_I(x + \alpha)\, d\alpha$$

and

$$\varphi_{1,B}(x) = \int_0^1 \sum_{\substack{I\in\mathcal{D},\\ 2^{-K}<|I|<2^{-N}}} (\varphi^\alpha, h_I) h_I(x+\alpha)\, d\alpha.$$

Then exactly the same argument as in the BMO situation proves that

$$\frac{1}{|Q|}\int_Q |\varphi_{1,A}(x)|^2\, dx \leqslant 2\varepsilon$$

as long as $2^{-K} < \delta$.

Now, following the argument of equation (8) and with the same notation, we have

$$\frac{1}{|Q|}\int_Q |\varphi_{1,B}(x) - c_Q|\, dx \leqslant \sum_{n:2^{-K}<2^{-n}<2^{-N}} \frac{1}{|Q|}\int_Q |g_n(x,x_0)|\, dx,$$

where $c_Q = \sum_{n:2^{-K}<2^{-n}<2^{-N}} \varphi_n(x_0)$.

As before, $|h_{I_\alpha}(x) - h_{I_\alpha}(x_0)| \leqslant 2|I|^{-1/2}$, while the difference is only non-zero for $\alpha \in E_{x,x_0,I}$, and $|E_{x,x_0,I}|$ is approximately $|x - x_0|$ and therefore bounded by 2^{-K}. Note that

$$|g_n(x,x_0)| \leqslant \int_{E_{x,x_0,I}} \sum_{I\in D_n} |(\varphi^\alpha, h_I)| 2|I|^{-1/2}\, d\alpha,$$

and this expression is bounded by $C2^n\varepsilon|x - x_0|$, since $2^{-n} < \delta$. Thus

$$\begin{aligned}\frac{1}{|Q|}\int_Q |\varphi_{1,B}(x) - c_Q|\ dx &\leqslant \sum_{n:2^{-K}<2^{-n}<2^{-N}} C2^n\varepsilon\frac{1}{|Q|}\int_Q |x - x_0|\ dx\\ &= C\varepsilon \sum_{n:2^{-K}<2^{-n}<2^{-N}} 2^n\frac{|Q|}{2}\\ &\leqslant C\varepsilon|Q| \sum_{n=N}^{K} 2^n\\ &= C\varepsilon.\end{aligned}$$

This completes the proof of (11).

To estimate $\varphi_2(x)$, for $c_Q = \sum_{n\leqslant N} \varphi_n(x_0)$ we have

$$\frac{1}{|Q|}\int_Q |\varphi_2(x) - c_Q|\, dx \leqslant \sum_{n\leqslant N} \frac{1}{|Q|}\int_Q |g_n(x,x_0)|\, dx$$

and $|g_n(x,x_0)| \leqslant C2^n|x - x_0|$. Thus

$$\begin{aligned}\frac{1}{|Q|}\int_Q |\varphi_2(x) - c_Q|\, dx &\leqslant C\sum_{n\leqslant N} 2^n|Q|\\ &\leqslant C2^{-K}\sum_{n\leqslant N} 2^n\\ &\leqslant C2^{-K}2^N \leqslant \varepsilon,\end{aligned}$$

if K is chosen sufficiently large. □

5. $\mathrm{BMO}(\mathbb{T}\otimes\mathbb{T})$ *from averaging* $\mathrm{BMO_d}(\mathbb{T}\otimes\mathbb{T})$

We work on the bidisc $\mathbb{T}\otimes\mathbb{T}$; in other words, on $[0,1]\times[0,1]$ with appropriate faces identified.

THEOREM 4. *Suppose that* $\varphi^\alpha \in \mathrm{BMO_d}(\mathbb{T}\otimes\mathbb{T})$ *for each* $\alpha = (\alpha_1, \alpha_2) \in [0,1]\times[0,1]$, *that* $\alpha \mapsto \varphi^\alpha$ *is measurable, and that the* $\mathrm{BMO_d}$ *norms of the functions* φ^α *are uniformly bounded: there is a constant* C_d *such that*

$$\|\varphi^\alpha\|_\mathrm{d} \leqslant C_\mathrm{d}$$

for all $\alpha \in [0,1]\times[0,1]$. *Let* $x = (x_1, x_2)$. *Suppose also that*

$$\int \varphi^\alpha(x)\,dx = 0 \quad \text{for all } \alpha \in [0,1]\times[0,1].$$

Then the translation-average

$$\varphi(x) := \int_0^1\int_0^1 \varphi^\alpha(x+\alpha)\,d\alpha$$

is in $\mathrm{BMO}(\mathbb{T}\otimes\mathbb{T})$.

In [10], Journé defined a wide class of multiparameter Calderón–Zygmund singular integrals, and proved a $T(1)$ theorem characterizing boundedness of these operators. His geometric observations were synthesized into a covering lemma for open sets in $\mathbb{R}^2$ (see [11]), which was extended to open sets in $\mathbb{R}^n$, $n > 2$, in [13]. For several recent variants of Journé's lemma, see [3] and the references therein.

We begin with some definitions.

DEFINITION 5 (Dyadic rectangles in Ω). Let Ω be an open set in $\mathbb{T}\otimes\mathbb{T}$. From now on, let $\mathcal{D}$ (rather than $\mathcal{D}\otimes\mathcal{D}$ as used earlier) denote the collection of dyadic rectangles $R = I \times J$ in $\mathbb{T}\otimes\mathbb{T}$, where I and J are dyadic intervals in $\mathbb{T}$. For a dyadic interval I, let $2I$ denote the dyadic parent of I. Define the subcollections $\mathcal{M}_1(\Omega)$ and $\mathcal{M}_2(\Omega)$ of $\mathcal{D}$ to be the collections of dyadic rectangles in Ω which are maximal in the first and second components, respectively:

$$\mathcal{M}_1(\Omega) := \{R = I\times J \in \mathcal{D} \mid I\times J \subset \Omega \text{ but } 2I\times J \not\subset \Omega\},$$
$$\mathcal{M}_2(\Omega) := \{R = I\times J \in \mathcal{D} \mid I\times J \subset \Omega \text{ but } I\times 2J \not\subset \Omega\}.$$

We use the notation M to denote the strong maximal operator:

$$Mf(x) := \sup\left\{\frac{1}{|R|}\int_R f(x)\,dx \;\middle|\; R\in\mathcal{D},\ x\in R\right\}.$$

If Ω is an open set in $\mathbb{T}\otimes\mathbb{T}$, then $\widetilde{\Omega}$ denotes the following enlargement of Ω:

$$\widetilde{\Omega} := \left\{M\chi_\Omega > \tfrac{1}{2}\right\}.$$

Thus $\Omega \subset \widetilde{\Omega}$, and there is a constant C such that $|\widetilde{\Omega}| \leqslant C\,|\Omega|$ for all open $\Omega \subset \mathbb{T}\otimes\mathbb{T}$.

Later we will also consider enlargements of enlargements:

$$\widetilde{\widetilde{\Omega}} := \left\{M\chi_{\widetilde{\Omega}} > \tfrac{1}{2}\right\}.$$

DEFINITION 6 ($\mathcal{F}_k$). To each rectangle $R = I\times J$ in $\mathcal{M}_2(\Omega)$ we associate a natural number $k = k(R) \in \mathbb{N}\cup\{0\}$ as follows. Let 2^kI denote the unique dyadic interval of length $2^k|I|$ that contains I, and set

$$k(R) := \text{the largest non-negative integer such that } 2^kI\times J \subset \widetilde{\Omega}$$

and

$$\mathcal{F}_k = \mathcal{F}_k(\Omega) := \{R = I\times J \in \mathcal{M}_2(\Omega) \mid k(R) = k\}.$$

In other words, $R = I \times J \subset \Omega$ is in $\mathcal{F}_k$ if $I \times 2J \not\subset \Omega$ and k is the unique integer such that $2^k I \times J \in \mathcal{M}_1(\widetilde{\Omega})$. Each $R \in \mathcal{M}_2(\Omega)$ lies in exactly one $\mathcal{F}_k$, and so $\mathcal{M}_2(\Omega)$ can be written as the disjoint union

$$\mathcal{M}_2(\Omega) = \bigcup_{k=0}^{\infty} \mathcal{F}_k.$$

THEOREM 5 (Journé's lemma). *Let Ω be an open set in $\mathbb{T} \otimes \mathbb{T}$. Then there is a constant C such that*

$$\sum_{\substack{R:R\in\mathcal{M}_2(\Omega),\\ R\in\mathcal{F}_k}} |R| \leqslant Ck|\Omega|.$$

Let

$$\mathcal{M}(\Omega) := \mathcal{M}_1(\Omega) \cap \mathcal{M}_2(\Omega)$$

denote the dyadic rectangles in Ω which are maximal in both directions.

DEFINITION 7 ($\mathcal{G}_l$). For $l \in \mathbb{N}$, define

$$\mathcal{G}_l = \mathcal{G}_l(\Omega) := \Big\{ R = I \times J \in \mathcal{M}_2(\Omega) \;\Big|\; \text{for the unique } k \text{ such that } R \in \mathcal{F}_k,$$
$$l \text{ is the largest non-negative integer such that } 2^k I \times 2^l J \subset \widetilde{\widetilde{\Omega}} \Big\}.$$

Then $\mathcal{M}_2(\Omega)$ can also be written as the disjoint union

$$\mathcal{M}_2(\Omega) = \bigcup_{l=0}^{\infty} \mathcal{G}_l.$$

As a corollary of Journé's lemma, we have an analogous result for the sets $\mathcal{G}_l$.

PROPOSITION 1 (Journé's lemma for $\mathcal{G}_l$). *Let Ω be an open set in $\mathbb{T} \otimes \mathbb{T}$. Then there is a constant C such that*

$$\sum_{R:R\in\mathcal{G}_l} |R| \leqslant Cl|\Omega|.$$

Proof of Proposition 1. Writing $R = I \times J$, we see that

$$\sum_{R:R\in\mathcal{G}_l} |R| = \sum_k \sum_{R:R\in\mathcal{G}_l\cap\mathcal{F}_k} |R|$$
$$= \sum_k \sum_{R:R=I\times J\in\mathcal{G}_l\cap\mathcal{F}_k} 2^{-k} |2^k I \times J|.$$

The inner sum is over a collection of distinct rectangles R, and the rectangle $R' = 2^k I \times J$ belongs to $\mathcal{M}_1(\widetilde{\Omega})$. However, more than one R can lead to the same rectangle $2^k I \times J \in \mathcal{M}_1(\widetilde{\Omega})$. Specifically, fix $R' = 2^k I \times J$. For each dyadic subinterval $\widehat{I}$ of $2^k I$ of length I, if $\widehat{I} \times J \in \mathcal{M}_2(\Omega)$, then the rectangle $R = \widehat{I} \times J$ gives rise to R' again. These are the only rectangles R that can lead to R', and so there are at most 2^k rectangles R in $\mathcal{F}_k$ that can give rise to a given $R' = 2^k I \times J$. Now, letting

$$M_{l,k}(\widetilde{\Omega}) := \{R' \mid R' \in \mathcal{M}_1(\widetilde{\Omega}),\ R' \in \mathcal{G}_l(\widetilde{\Omega}),\ R' = 2^k I \times J \text{ and } I \times J \in \mathcal{F}_k\},$$

we obtain

$$\begin{aligned}\sum_k \sum_{R:R=I\times J\in\mathcal{G}_l\cap\mathcal{F}_k} 2^{-k}|2^k I\times J| &\leqslant \sum_k \sum_{R':R'\in M_{l,k}(\widetilde{\Omega})} 2^{-k}2^k|R'| \\ &\leqslant Cl|\widetilde{\Omega}| \qquad \text{by Journé's lemma} \\ &\leqslant C'l|\Omega|,\end{aligned}$$

as required. □

Proof of Theorem 4. To show that the translation-average φ of the $\mathrm{BMO_d}$ functions φ^α is in BMO, it suffices to show that there is a constant C such that for all open sets Ω in the bidisc $\mathbb{T}\otimes\mathbb{T}$

$$\iint_{T(\Omega)} |\varphi * \psi_y(t)|^2 \,\frac{dt_1\,dt_2\,dy_1\,dy_2}{y_1y_2} \leqslant C|\Omega|, \tag{12}$$

where $t=(t_1,t_2)$, $y=(y_1,y_2)$, $\psi_y(t)=\psi_{y_1}(t_1)\psi_{y_2}(t_2)$, $\widehat{\psi}$ has sufficient decay at the origin, and $T(\Omega)$ is the union of those regions $T(R_0)$ such that $R_0\in\mathcal{M}(\Omega)$.

For $\alpha=(\alpha_1,\alpha_2)\in[0,1]\times[0,1]$, let

$$R_\alpha = I_{\alpha_1}\times J_{\alpha_2} := (I-\alpha_1)\times(J-\alpha_2)$$

be the α-translation of the dyadic rectangle $R=I\times J$.

Note first that

$$\varphi * \psi_y(t) = \int_0^1\int_0^1 \sum_{R:R\in\mathcal{D}} (\varphi^\alpha, h_R)\, h_{R_\alpha} * \psi_y(t)\, d\alpha.$$

Now $h_{R_\alpha} * \psi_y(t) = \big[h_{I_{\alpha_1}} * \psi_{y_1}(t_1)\big]\,\big[h_{J_{\alpha_2}} * \psi_{y_2}(t_2)\big]$ is non-zero only if

$$R_\alpha \cap \big(I_{y_1}(t_1)\times I_{y_2}(t_2)\big) \neq \emptyset,$$

since $I_{y_1}(t_1) := [t_1-y_1, t_1+y_1] = \operatorname{supp}\psi_{y_1}(t_1-\cdot)$.

We split the integral over the Haar series into two parts: the part involving $\varphi^{(1)}$ that sums over those rectangles R_α contained in $\widetilde{\widetilde{\Omega}}$ and the part involving $\varphi^{(2)}$ that sums over the remaining rectangles. Set

$$\varphi^{(1)} * \psi_y(t) := \int_0^1\int_0^1 \sum_{R:R_\alpha\subset\widetilde{\widetilde{\Omega}}} (\varphi^\alpha, h_R)\, h_{R_\alpha} * \psi_y(t)\, d\alpha.$$

Then equation (12) with φ replaced by $\varphi^{(1)}$ holds by L^2-theory. That is, because $\|\varphi^\alpha\|_{\mathrm{d}} \leqslant C_{\mathrm{d}}$ for all α, we obtain the estimate

$$\iint_{T(\Omega)} \Big|\sum_{R:R_\alpha\subset\widetilde{\widetilde{\Omega}}} (\varphi^\alpha, h_R)\, h_{R_\alpha} * \psi_y(t)\Big|^2 \,\frac{dt_1\,dt_2\,dy_1\,dy_2}{y_1y_2} \leqslant \sum_{R:R_\alpha\subset\widetilde{\widetilde{\Omega}}} (\varphi^\alpha, h_R)^2 \leqslant C|\Omega|,$$

and the bound is unchanged when we integrate in α.

Set

$$\varphi^{(2)} := \varphi - \varphi^{(1)}.$$

Since $T(\Omega) = \cup\{T(R_0) \mid R_0\in\mathcal{M}(\Omega)\}$, to show that equation (12) holds for $\varphi^{(2)}$ it suffices to show that

$$\sum_{R_0:R_0\in\mathcal{M}(\Omega)} \iint_{T(R_0)} |\varphi^{(2)} * \psi_y(t)|^2 \,\frac{dt_1\,dt_2\,dy_1\,dy_2}{y_1y_2} \leqslant C|\Omega|.$$

We use Journé's lemma for this.

Fix k and l and a rectangle $R_0 \in \mathcal{F}_k \cap \mathcal{G}_l$, such that $2^k I \times 2^l J \subset \tilde{\tilde{\Omega}}$. Consider the quantity

$$\int_0^1 \int_0^1 \sum_{\substack{R_\alpha : R_\alpha \not\subset \tilde{\tilde{\Omega}}, \\ R_\alpha \cap 3R_0 \neq 0}} (\varphi^\alpha, h_R)\, h_{R_\alpha} * \psi_y(t)\, d\alpha.$$

(Note that for each R_α in this sum, we have $R_\alpha \cap 3R_0 \neq \emptyset$, since $I_{y_1}(t_1) \times I_{y_2}(t_2) \subset 3R_0$.) At this point, we would like to argue that if the sum in the above integral is non-zero, then either

$$|I_{\alpha_1}| > 2^k |I_0| \quad \text{or} \quad |J_{\alpha_2}| > 2^l |J_0|,$$

or both.

In fact this is only true if we are summing over those rectangles R_α not contained in a (further) enlargement of $\tilde{\tilde{\Omega}}$, obtained by doubling the size of rectangles contained in $\tilde{\tilde{\Omega}}$ about their centers. To avoid introducing more notation, we will assume that $\tilde{\tilde{\Omega}}$ has been so enlarged. Then, it suffices to estimate over each of the following four subcollections of rectangles:

Case (i): $|I_{\alpha_1}| > 2^k |I_0|$ but $|J_{\alpha_2}| \leqslant 2^k |J_0|$;
Case (ii): $|I_{\alpha_1}| > 2^k |I_0|$ and $|J_{\alpha_2}| > 2^k |J_0|$;
Case (iii): $|J_{\alpha_2}| > 2^l |J_0|$ but $|I_{\alpha_1}| \leqslant 2^l |I_0|$;
Case (iv): $|J_{\alpha_2}| > 2^l |J_0|$ and $|I_{\alpha_1}| > 2^l |I_0|$.

Case (i): For fixed $R_0 = I_0 \times J_0$ in $\mathcal{F}_k \cap \mathcal{G}_l$, we estimate

$$\iint_{T(R_0)} [C_{(\mathrm{i})}]^2 \,\frac{dt\,dy}{y},$$

where

$$C_{(\mathrm{i})} := \left| \int_0^1 \int_0^1 \sum_{I : |I_{\alpha_1}| > 2^k |I_0|} \; \sum_{J : |J_{\alpha_2}| \leqslant 2^k |J_0|} (\varphi^\alpha, h_R)\, h_{R_\alpha} * \psi_y(t)\, d\alpha \right|.$$

Let

$$c_J := \sum_{I : |I| > 2^k |I_0|} \int_0^1 (\varphi^{\alpha_1,\alpha_2}, h_R)\, h_{I_{\alpha_1}} * \psi_{y_1}(t_1)\, d\alpha_1.$$

Then

$$\iint_{T(J_0)} \left| \sum_{J : |J_{\alpha_2}| \leqslant 2^k |J_0|} c_J h_{J_{\alpha_2}} * \psi_{y_2}(t_2) \right|^2 \frac{dt_2\, dy_2}{y_2} \leqslant \sum_{J : J_{\alpha_2} \subset 3 \cdot 2^k J_0} c_J^2, \tag{13}$$

by L^2-theory. It remains to estimate the quantity

$$\iint_{T(I_0)} \sum_{J : J_{\alpha_2} \subset 3 \cdot 2^k J_0} c_J^2 \,\frac{dt_1\, dy_1}{y_1}.$$

For fixed $(t_1, y_1) \in T(I_0)$, and fixed I, we have

$$\int_0^1 (\varphi^{\alpha_1,\alpha_2}, h_R) h_{I_{\alpha_1}} * \psi_{y_1}(t_1)\, d\alpha_1 = \int_{E_{y_1,t_1,I}} (\varphi^{\alpha_1,\alpha_2}, h_R) h_{I_{\alpha_1}} * \psi_{y_1}(t_1)\, d\alpha_1,$$

where

$$E_{y_1,t_1,I} := \left\{ \alpha_1 \;\middle|\; h_{I_{\alpha_1}} * \psi_{y_1}(t_1) \neq 0 \right\}.$$

By the argument we used in the one-parameter setting,

$$|E_{y_1,t_1,I}| \leqslant C y_1.$$

Then, using Cauchy–Schwarz in the second line,

$$\begin{aligned} c_J^2 &= \Big| \sum_{I:|I|>2^k|I_0|} \int_{E_{y_1,t_1,I}} (\varphi^{\alpha_1,\alpha_2}, h_R)\, h_{I_{\alpha_1}} * \psi_{y_1}(t_1)\, d\alpha_1 \Big|^2 \\ &\leqslant \Big(\sum_{I:|I|>2^k|I_0|} 1 \Big) \sum_{I:|I|>2^k|I_0|} \Big[\int_{E_{y_1,t_1,I}} (\varphi^{\alpha_1,\alpha_2}, h_R)\, h_{I_{\alpha_1}} * \psi_{y_1}(t_1)\, d\alpha_1 \Big]^2 \\ &\leqslant C \frac{1}{2^k|I_0|} \sum_{I:|I|>2^k|I_0|} \Big[\int_{E_{y_1,t_1,I}} |(\varphi^{\alpha_1,\alpha_2}, h_R)|\, |I_{\alpha_1}|^{-1/2}\, d\alpha_1 \Big]^2. \end{aligned}$$

In the last line we have used the observation that the number of dyadic intervals I in $\mathbb{T}$ at the k scales of length at least $2^{k-1}I_0$ is $1/(2^k|I_0|)$, and also that

$$|h_{I_{\alpha_1}} * \psi_{y_1}(t_1)| \leqslant |h_{I_{\alpha_1}}| \leqslant |I|^{-1/2}.$$

Therefore, using the Cauchy–Schwarz inequality again,

$$c_J^2 \leqslant \frac{C}{2^k|I_0|} \sum_{I:|I|>2^k|I_0|} y_1 \int_{E_{y_1,t_1,I}} |(\varphi^{\alpha_1,\alpha_2}, h_R)|^2\, |I|^{-1}\, d\alpha_1.$$

Returning to the sum in equation (13), we have

$$\sum_{J:J_{\alpha_2}\subset 3\cdot 2^k J_0} c_J^2 \leqslant \frac{C}{2^k|I_0|} \sum_{I:|I|>2^k|I_0|} y_1\, |I|^{-1} \int_{E_{y_1,t_1,I}} \sum_{J:J_{\alpha_2}\subset 3\cdot 2^k J_0} (\varphi^{\alpha_1,\alpha_2}, h_R)^2\, d\alpha_1.$$

The integrand is less than or equal to a constant times $2^k|I||J_0|$, by the BMO condition on the open set $I \times 2^k J_0$. Integrating over $E_{y_1,t_1,I}$, we obtain

$$\sum_{J:J_{\alpha_2}\subset 3\cdot 2^k J_0} c_J^2 \leqslant \frac{C}{2^k|I_0|} \sum_{I:|I|>2^k|I_0|} y_1\, |I|^{-1}\, \big(2^k|I||J_0|y_1\big).$$

It remains to integrate the right-hand side over $T(I_0)$. Then

$$\begin{aligned} \iint_{T(I_0)} \frac{1}{2^k|I_0|} \sum_{I:|I|>2^k|I_0|} y_1^2\, |J_0|\, 2^k\, \frac{dt_1\, dy_1}{y_1} &\leqslant \frac{1}{2^{2k}|I_0|^2}\, 2^k|J_0| \iint_{T(I_0)} y_1^2\, \frac{dt_1\, dy_1}{y_1} \\ &\leqslant \frac{1}{2^{2k}|I_0|^2}\, 2^k|J_0|\, |I_0|^3 \\ &\leqslant 2^{-k}\, |I_0 \times J_0|. \end{aligned}$$

Integrating over $\mathbb{T}$ in α_1 does not change this bound.

Now, summing over the rectangles R_0, we obtain

$$\begin{aligned} \sum_{k,l} \sum_{R_0:R_0\in\mathcal{F}_k\cap\mathcal{G}_l} \iint_{T(R_0)} [C_{(\mathrm{i})}]^2\, \frac{dt\, dy}{y} &\leqslant \sum_k \sum_{R_0:R_0\in\mathcal{F}_k} \iint_{T(R_0)} [C_{(\mathrm{i})}]^2\, \frac{dt\, dy}{y} \\ &\leqslant \sum_k \sum_{R_0:R_0\in\mathcal{F}_k} 2^{-k}|R_0| \\ &\leqslant C\, |\Omega|, \end{aligned}$$

by Journé's lemma. This controls the sum over the rectangles covered by case (i).

Case (ii): Here we consider those rectangles $R = I \times J$ for which I and J are both large. Fix a point $(t, y) = (t_1, t_2, y_1, y_2)$ in $T(R_0)$. We must estimate the quantity

$$\iint_{T(R_0)} [C_{(\mathrm{ii})}]^2\, \frac{dt\, dy}{y},$$

where

$$C_{(\mathrm{ii})} := \left| \int_0^1 \int_0^1 \sum_{I:|I_{\alpha_1}|>2^k|I_0|} \ \sum_{J:|J_{\alpha_2}|>2^k|J_0|} (\varphi^\alpha, h_R)\, h_{R_\alpha} * \psi_y(t)\, d\alpha \right|. \tag{14}$$

For fixed I and J in that sum, consider the expression

$$C_{(\mathrm{ii})}(R) := \left| \int_0^1 \int_0^1 (\varphi^\alpha, h_R) \left[h_{R_{\alpha_1}} * \psi_{y_1}(t_1)\right] \left[h_{R_{\alpha_2}} * \psi_{y_2}(t_2)\right] \, d\alpha \right|. \tag{15}$$

Again, the integrand can be non-zero only when $\alpha_1 \in E_{y_1,t_1,I}$ and $\alpha_2 \in E_{y_2,t_2,I}$, where $E_{y_1,t_1,I}$ and $E_{y_2,t_2,I}$ are of size y_1 and y_2, respectively. Also

$$\begin{gathered} |(\varphi^\alpha, h_R)| \leqslant C_{\mathrm{d}} |R|^{1/2}, \\ |h_{I_{\alpha_1}} * \psi_{y_1}(t_1)| \leqslant |I|^{-1/2}, \\ |h_{I_{\alpha_2}} * \psi_{y_2}(t_2)| \leqslant |J|^{-1/2}. \end{gathered}$$

Integrating over $E_{y_1,t_1,I}$ and $E_{y_2,t_2,I}$ gives

$$C_{(\mathrm{ii})}(R) \leqslant C_{\mathrm{d}} y_1 y_2. \tag{16}$$

Summing $C_{(\mathrm{ii})}(R)$ over I and J, we find that

$$C_{(\mathrm{ii})} \leqslant C_{\mathrm{d}} \left[\frac{1}{2^k|I_0|} \, \frac{1}{2^k|J_0|} \; y_1 y_2 \right]. \tag{17}$$

Therefore

$$\iint_{T(R_0)} [C_{(\mathrm{ii})}]^2 \, \frac{dt_1\, dt_2\, dy_1\, dy_2}{y_1\, y_2} \leqslant C_{\mathrm{d}}^2 2^{-4k} |R_0|. \tag{18}$$

As in the previous case, we sum over these rectangles R_0 in $\mathcal{F}_k$ and use Journé's lemma to conclude that the sum is bounded by a constant times $|\Omega|$.

Case (iii): We must estimate the quantity

$$\iint_{T(R_0)} [C_{(\mathrm{iii})}]^2 \, \frac{dt\, dy}{y},$$

where

$$C_{(\mathrm{iii})} := \left| \int_0^1 \int_0^1 \sum_{J:|J_{\alpha_2}|>2^l|J_0|} \ \sum_{I:|I_{\alpha_1}|\leqslant 2^l|I_0|} (\varphi^\alpha, h_R)\, h_{R_\alpha} * \psi_y(t)\, d\alpha \right|. \tag{19}$$

Move the integral in α_1 to the outside, by Cauchy–Schwarz. Let

$$c_I := \sum_{J:|J_{\alpha_2}|>2^l|J_0|} \int_0^1 (\varphi^\alpha, h_R)\, h_{J_{\alpha_2}} * \psi_{y_2}(t_2)\, d\alpha_2.$$

Fix α_1. If $h_{J_{\alpha_2}} * \psi_{y_2}(t_2) \neq 0$, then $I_{\alpha_1} \cap 3I_0 \neq \emptyset$, and so $I_{\alpha_1} \subset 3 \cdot 2^l I_0$. Therefore, by the L^2-theory again,

$$\begin{aligned} \iint_{T(I_0)} \left| \sum_{\substack{I:|I|<2^l|I_0| \\ I_{\alpha_1}\cap 3I_0 \neq \emptyset}} c_I\, h_{I_{\alpha_1}} * \psi_{y_1}(t_1) \right|^2 \frac{dt_1\, dy_1}{y_1} &\leqslant \iint_{T(I_0)} \left| \sum_{\substack{I:|I|<2^l|I_0| \\ I_{\alpha_1}\subset 3\cdot 2^l I_0}} c_I\, h_{I_{\alpha_1}} * \psi_{y_1}(t_1) \right|^2 \frac{dt_1\, dy_1}{y_1} \\ &\leqslant \sum_{I:I_{\alpha_1}\subset 3\cdot 2^l I_0} c_I^2. \end{aligned}$$

Following the argument laid out in case (i), we get

$$\sum_{I:I_{\alpha_1}\subset 3\cdot 2^l I_0} c_I^2 \leqslant \frac{1}{2^l|J_0|} \sum_{J:|J|>2^l|J_0|} y_2|J|^{-1}2^l|I_0||J|. \tag{20}$$

Summing over the rectangles R_0, we obtain

$$\begin{aligned}\sum_{k,l}\ \sum_{R_0:R_0\in\mathcal{F}_k\cap\mathcal{G}_l} \iint_{T(R_0)} [C_{(\mathrm{iii})}]^2\,\frac{dt\,dy}{y} &\leqslant \sum_{l}\ \sum_{R_0:R_0\in\mathcal{G}_l} \iint_{T(R_0)} [C_{(\mathrm{iii})}]^2\,\frac{dt\,dy}{y}\\ &\leqslant \sum_{l}\ \sum_{R_0:R_0\in\mathcal{G}_l} 2^{-2l}|J_0|\cdot 2^l|I_0|\\ &= \sum_{l}\ \sum_{R_0:R_0\in\mathcal{G}_l} 2^{-l}|R_0|\\ &\leqslant \left[\sum_l C\,l\,2^{-l}\right]|\Omega|,\end{aligned}$$

by the version in Proposition 1 of Journé's lemma for the sets $\mathcal{G}_l$. This controls the sum over the rectangles covered by case (iii).

Case (iv): We omit the argument for this case. The argument is similar to that for case (ii), and uses Proposition 1.

This completes the proof of Theorem 4. □

As a corollary of Theorem 4, by duality we can establish the product version on the bidisc of Davis's theorem connecting H^1 and dyadic H^1_{d} [**7**, Theorem 3.1, case $p=1$], just as Garnett and Jones noted for the one-parameter case in [**9**]. For complete information about the Hardy space H^1 on the bidisc, see [**6**] and the references therein. Product VMO on the bidisc is discussed in Section 6 below; here we use only that $\mathrm{VMO}_{\mathrm{d}}(\mathbb{T}\otimes\mathbb{T})\subset\mathrm{BMO}_{\mathrm{d}}(\mathbb{T}\otimes\mathbb{T})$ and that product H^1 is the dual of product VMO.

THEOREM 6 (Biparameter Davis theorem). *If $f\in H^1(\mathbb{T}\otimes\mathbb{T})$, then for almost every $\alpha\in[0,1]\times[0,1]$, the translation $T_\alpha f(\cdot):=f(\cdot-\alpha)$ belongs to $H^1_{\mathrm{d}}(\mathbb{T}\otimes\mathbb{T})$, and*

$$\int_0^1\int_0^1 \|T_\alpha f\|_{H^1_{\mathrm{d}}}\,d\alpha \leqslant C\|f\|_{H_1}.$$

Proof. We will use the following facts about the Hardy space $H^1(\mathbb{T}\otimes\mathbb{T})$:

$$\big(H^1(\mathbb{T}\otimes\mathbb{T})\big)^* = \mathrm{BMO}(\mathbb{T}\otimes\mathbb{T}),\quad H^1(\mathbb{T}\otimes\mathbb{T}) = \big(\mathrm{VMO}(\mathbb{T}\otimes\mathbb{T})\big)^*,$$

and their dyadic analogs.

Take $f\in H^1(\mathbb{T}\otimes\mathbb{T})$. If f is also continuous, then f and all its translates $T_\alpha f$ belong to $H^1_{\mathrm{d}}(\mathbb{T}\otimes\mathbb{T})$. To get the norm estimate, note that $\|T_\alpha f\|_{H^1_{\mathrm{d}}}$ varies continuously and uniformly in α. By duality the norm $\|T_\alpha f\|_{H^1_{\mathrm{d}}}$ is given by pairing with a $\mathrm{BMO}(\mathbb{T}\otimes\mathbb{T})$ function. If we approximate these norms, we can choose a family of φ^α which vary measurably in α. Indeed the map $\alpha\mapsto\varphi^\alpha$ will be piecewise constant.

By Theorem 4, the translation-average $\varphi(\cdot) := \int_0^1 \int_0^1 \varphi^\alpha(\cdot + \alpha)\, d\alpha$ is in $\mathrm{BMO}(\mathbb{T} \otimes \mathbb{T})$, and $\|\varphi\|_* \leqslant 1$. Then

$$\begin{aligned} \int_0^1 \int_0^1 \langle T_\alpha f, \varphi^\alpha \rangle \, d\alpha &= \int_0^1 \int_0^1 \int_{\mathbb{T}\otimes\mathbb{T}} T_\alpha f(x) \varphi^\alpha(x) \, dx \, d\alpha \\ &= \int_{\mathbb{T}\otimes\mathbb{T}} f(x) \int_0^1 \int_0^1 \varphi^\alpha(x + \alpha) \, d\alpha \, dx \\ &\leqslant \|f\|_{H^1}. \end{aligned}$$

In particular, $T_\alpha f$ is in $H^1_{\mathrm{d}}(\mathbb{T} \otimes \mathbb{T})$ for almost all α.

Now assume that $f \in H^1(\mathbb{T} \otimes \mathbb{T})$, $\|f\| = 1$. We can represent $f = \sum_n f_n$, where the f_n are continuous and $\sum_n \|f_n\|_{H^1} \leqslant (1 + \varepsilon)\|f\|_{H^1}$. Define

$$F(\alpha) := \sum_n \|T_\alpha f_n\|_{H^1_d}.$$

The estimate for the continuous functions implies that

$$\int_0^1 \int_0^1 F(\alpha) \, d\alpha = \sum_n \int_0^1 \int_0^1 \|T_\alpha f_n\|_{H^1_{\mathrm{d}}} \, d\alpha \leqslant \sum_n C \|f_n\|_{H^1} \leqslant C(1 + \varepsilon)\|f\|_{H^1}.$$

Since

$$\left| \int_I T_\alpha f(t) \, dt \right| \leqslant \sum_n \left| \int_I T_\alpha f_n(t) \, dt \right|$$

we have

$$(T_\alpha f)^*(x) \leqslant \sum_n (T_\alpha f_n)^*(x),$$

where $(T_\alpha f)^*$ denotes the the martingale maximal function of $T_\alpha f$.

Integrating with respect to x we obtain

$$\|T_\alpha f\|_{H^1_d} \leqslant F(\alpha). \qquad \square$$

6. VMO($\mathbb{T} \otimes \mathbb{T}$) *from averaging* VMO$_{\mathrm{d}}$($\mathbb{T} \otimes \mathbb{T}$)

The product VMO space $\mathrm{VMO}(\mathbb{T} \otimes \mathbb{T})$ was investigated in [**12**] where, among other things, the authors gave a definition of product VMO in terms of Carleson measures, and identified product VMO as the predual of product H^1. We recall their definition of product VMO. Let $\mathcal{D}_n$ denote the class of dyadic rectangles Q such that $|Q|$ is less than 2^{-n}.

Definition 8 (Product VMO). A function b belongs to $\mathrm{VMO}(\mathbb{T} \otimes \mathbb{T})$ if b belongs to $\mathrm{BMO}(\mathbb{T} \otimes \mathbb{T})$, and for each $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that for every open set Ω in the bidisc $\mathbb{T} \otimes \mathbb{T}$,

$$\sum_{Q : Q \subset \Omega,\, Q \in \mathcal{D}_n} \iint_{Q^+} |b * \psi_y(t)|^2 \, \frac{dt_1 \, dt_2 \, dy_1 \, dy_2}{y_1 y_2} \leqslant \varepsilon |\Omega|, \tag{21}$$

where

$$\mathcal{D}_n := \{ Q = Q_1 \times Q_2 \mid Q_1, Q_2 \text{ are dyadic intervals in } \mathbb{T} \text{ with } |Q| := |Q_1|\,|Q_2| < 2^{-n} \}.$$

Specializing equation (21) to one parameter, it can be seen that this definition of VMO is equivalent to Definition 3.

As in the one-parameter case, product VMO can also be characterized as the closure of C_0^∞ in BMO (see [**12**]).

DEFINITION 9 (Dyadic product VMO). A function b belongs to the space *dyadic product* VMO, denoted by $\mathrm{VMO_d}(\mathbb{T}\otimes\mathbb{T})$, if for each $\varepsilon>0$ there is an N such that for all open sets $\mathcal{A}\subset\mathbb{T}\otimes\mathbb{T}$, we have

$$\sum_{R:R\subset\mathcal{A},\, R\in\mathcal{D},\, |R|<2^{-N}} (\varphi^\alpha, h_R)^2 \leqslant \varepsilon|\mathcal{A}|.$$

We now prove the averaging theorem for product VMO, namely that translation-averages of suitable $\mathrm{VMO_d}(\mathbb{T}\otimes\mathbb{T})$ functions belong to $\mathrm{VMO}(\mathbb{T}\otimes\mathbb{T})$. The argument requires one essential modification from the product BMO averaging theorem. When specialized to one parameter, the argument gives another proof of Theorem 3.

THEOREM 7. *Suppose that $\varphi^\alpha\in\mathrm{BMO_d}(\mathbb{T}\otimes\mathbb{T})$ for each $\alpha=(\alpha_1,\alpha_2)\in[0,1]\times[0,1]$, that $\alpha\mapsto\varphi^\alpha$ is measurable, and that the $\mathrm{BMO_d}$ norms of the functions φ^α are uniformly bounded: there is a constant C_d such that*

$$\|\varphi^\alpha\|_\mathrm{d}\leqslant C_\mathrm{d}$$

for all $\alpha\in[0,1]\times[0,1]$. Let $x=(x_1,x_2)$. Suppose also that

$$\int \varphi^\alpha(x)\,dx = 0 \quad \textit{for all } \alpha\in[0,1]\times[0,1].$$

Suppose in addition that the functions φ^α belong to $\mathrm{VMO_d}(\mathbb{T}\otimes\mathbb{T})$ uniformly: for each $\varepsilon>0$ there is an N such that for all $\alpha\in[0,1]\times[0,1]$ and for all open sets $\mathcal{A}\subset\mathbb{T}\otimes\mathbb{T}$,

$$\sum_{R:R\subset\mathcal{A},\, R\in\mathcal{D},\, |R|<2^{-N}} (\varphi^\alpha, h_R)^2 \leqslant \varepsilon|\mathcal{A}|.$$

Then the translation-average

$$\varphi(x) := \int_0^1\int_0^1 \varphi^\alpha(x+\alpha)\,d\alpha$$

is in $\mathrm{VMO}(\mathbb{T}\otimes\mathbb{T})$.

Proof. By Theorem 4, φ is in $\mathrm{BMO}(\mathbb{T}\otimes\mathbb{T})$. Let Ω be an open set in the bidisc $\mathbb{T}\otimes\mathbb{T}$, and fix $\varepsilon>0$. Since the functions φ^α are uniformly in $\mathrm{VMO}(\mathbb{T}\otimes\mathbb{T})$, there is some N such that for all $\alpha\in[0,1]\times[0,1]$ and for all open sets $\mathcal{A}\subset\mathbb{T}\otimes\mathbb{T}$,

$$\sum_{R:R\subset\mathcal{A},\, R\in\mathcal{D},\, |R|<2^{-N}} (\varphi^\alpha, h_R)^2 \leqslant \varepsilon|\mathcal{A}|.$$

It suffices to show that for $K=K(\varepsilon,N)$ sufficiently large,

$$\sum_{Q:Q\subset\Omega,\, Q\in\mathcal{D},\, |Q|<2^{-K}} \iint_{Q^+} |\varphi*\psi_y(t)|^2\,\frac{dt_1\,dt_2\,dy_1\,dy_2}{y_1y_2} \leqslant \varepsilon|\Omega|. \tag{22}$$

We first split the sum in the integrand of φ at scale 2^{-N} such that $\varphi=\varphi_1+\varphi_2$, where

$$\varphi_1 := \int_0^1\int_0^1 \sum_{R:R\in\mathcal{D},\, |R|<2^{-N}} (\varphi^\alpha, h_R)\,h_R(x+\alpha)\,d\alpha,$$

$$\varphi_2 := \varphi-\varphi_1 = \int_0^1\int_0^1 \sum_{R:R\in\mathcal{D},\, |R|\geqslant 2^{-N}} (\varphi^\alpha, h_R)\,h_R(x+\alpha)\,d\alpha.$$

Thus $\varphi * \psi_y(t) = \varphi_1 * \psi_y(t) + \varphi_2 * \psi_y(t)$, where

$$\begin{aligned}\varphi_1 * \psi_y(t) &= \int_0^1 \int_0^1 \sum_{R:R\in\mathcal{D},\, |R|<2^{-N}} (\varphi^\alpha, h_R)\, h_{R_\alpha} * \psi_y(t)\, d\alpha,\\ \varphi_2 * \psi_y(t) &= \int_0^1 \int_0^1 \sum_{R:R\in\mathcal{D},\, |R|\geqslant 2^{-N}} (\varphi^\alpha, h_R)\, h_{R_\alpha} * \psi_y(t)\, d\alpha. \end{aligned} \tag{23}$$

Here, as usual,

$$R_\alpha = R_{(\alpha_1,\alpha_2)} := (I - \alpha_1) \times (J - \alpha_2)$$

is the translate of the rectangle $R \in \mathcal{D}$ by $\alpha = (\alpha_1, \alpha_2)$.

The estimate for φ_1 is straightforward. We apply the arguments of Section 5, including the splitting into four cases. The arguments go through without change, and we obtain a stronger inequality than (22), namely

$$\sum_{Q:Q\subset\Omega,\, Q\in\mathcal{D}} \iint_{Q^+} |\varphi_1 * \psi_y(t)|^2 \frac{dt_1\, dt_2\, dy_1\, dy_2}{y_1 y_2} \leqslant \varepsilon|\Omega|.$$

We turn to the estimate for φ_2. We must show that there is a K such that

$$\sum_{Q:Q\subset\Omega,\, Q\in\mathcal{D},\, |Q|<2^{-K}} \iint_{Q^+} |\varphi_2 * \psi_y(t)|^2 \frac{dt_1\, dt_2\, dy_1\, dy_2}{y_1 y_2} \leqslant \varepsilon|\Omega|, \tag{24}$$

where $\varphi_2 * \psi_y(t)$ is as defined in equation (23).

Fix δ with $0 < \delta < 2^{-N}$, and let $K = K(\varepsilon, N, \delta) \gg N$ be a positive integer to be determined later.

Write $Q = Q_1 \times Q_2$ and $Q^+ = Q_1^+ \times Q_2^+$. If $|Q| = |Q_1|\,|Q_2| < 2^{-K}$, then either $|Q_1| < 2^{-K/2}$ or $|Q_2| < 2^{-K/2}$, or both.

We consider two cases for inequality (22): one in which we sum over rectangles Q with $|Q_1| < 2^{-K/2}$, and one in which we sum over rectangles Q with $|Q_2| < 2^{-K/2}$. For notational convenience we relabel $K/2$ as K. By symmetry, we may assume that our sum is taken over rectangles Q for which $|Q_1| < 2^{-K}$. Then

$$\begin{aligned}\sum_{Q:Q\subset\Omega,\, Q\in\mathcal{D},|Q_1|<2^{-K}} &\iint_{Q^+} |\varphi_2 * \psi_y(t)|^2 \frac{dt_1\, dt_2\, dy_1\, dy_2}{y_1 y_2}\\ &\leqslant \int_{(t_1,t_2)\in\Omega} \int_{0<y_1<2^{-K}} \int_{0<y_2<1} |\varphi_2 * \psi_y(t)|^2 \frac{dt_1\, dt_2\, dy_1\, dy_2}{y_1 y_2}. \end{aligned} \tag{25}$$

Now we make use of our previously chosen δ, splitting the integral in y_2 into an integral over $0 < y_2 < \delta$ and another over $\delta < y_2 < 1$.

First, consider the part of the integral in the right-hand side of inequality (25) with $0 < y_2 < \delta$. Fix $R \in \mathcal{D}$ such that $|R| \geqslant 2^{-N}$. Because both $y_1 < |I|$ and $y_2 < |J|$, the same arguments used to establish inequality (16) apply, and we obtain the inequality

$$\int_0^1 \int_0^1 \big|(\varphi^\alpha, h_R)\, h_{R_\alpha} * \psi_y(t)\big|\, d\alpha \leqslant C y_1 y_2.$$

Since there are no more than $(N+1)2^{N+2}$ dyadic rectangles R of area $|R| \geqslant 2^{-N}$, we find that

$$\begin{aligned}
|\varphi_2 * \psi_y(t)| &= \Big| \int_0^1 \int_0^1 \sum_{R : R \in \mathcal{D},\, |R| \geqslant 2^{-N}} (\varphi^\alpha, h_R)\, h_{R_\alpha} * \psi_y(t)\, d\alpha \Big| \\
&\leqslant \sum_{R : R \in \mathcal{D},\, |R| \geqslant 2^{-N}} \Big| \int_0^1 \int_0^1 (\varphi^\alpha, h_R)\, h_{R_\alpha} * \psi_y(t)\, d\alpha \Big| \\
&\leqslant (N+1)2^{N+2} C C_{\mathrm{d}} y_1 y_2.
\end{aligned}$$

Hence

$$|\varphi_2 * \psi_y(t)|^2 \leqslant \left[(N+1)2^{N+2} C C_{\mathrm{d}} y_1 y_2\right]^2.$$

Therefore

$$\begin{aligned}
&\int_{(t_1,t_2)\in\Omega} \int_{0<y_1<2^{-K}} \int_{0<y_2<\delta} |\varphi_2 * \psi_y(t)|^2 \, \frac{dt_1\, dt_2\, dy_1\, dy_2}{y_1 y_2} \\
&\quad \leqslant \int_{(t_1,t_2)\in\Omega} \int_{0<y_1<2^{-K}} \int_{0<y_2<\delta} \left[(N+1)2^{N+2} C C_{\mathrm{d}} y_1 y_2\right]^2 \, \frac{dt_1\, dt_2\, dy_1\, dy_2}{y_1 y_2} \\
&\quad = \left[(N+1)2^{N+2} C C_{\mathrm{d}}\right]^2 |\Omega| \int_0^{2^{-K}} y_1^2 \, \frac{dy_1}{y_1} \int_0^\delta y_2^2 \, \frac{dy_2}{y_2} \\
&\quad = \left[(N+1)2^{N+2} C C_{\mathrm{d}}\right]^2 |\Omega| \, \frac{(2^{-K})^2}{2} \, \frac{\delta^2}{2} \\
&\quad \leqslant \varepsilon |\Omega|
\end{aligned}$$

as required, if $K = K(\varepsilon, N, \delta)$ is chosen sufficiently large.

Second, consider the part of the integral with $\delta < y_2 < 1$:

$$\int_{(t_1,t_2)\in\Omega} \int_{0<y_1<2^{-K}} \int_{\delta<y_2<1} |\varphi_2 * \psi_y(t)|^2 \, \frac{dt_1\, dt_2\, dy_1\, dy_2}{y_1 y_2},$$

where

$$\varphi_2 * \psi_y(t) = \int_0^1 \int_0^1 \sum_{R : R \in \mathcal{D},\, |R| \geqslant 2^{-N}} (\varphi^\alpha, h_R)\, h_{R_\alpha} * \psi_y(t)\, d\alpha.$$

As before, $|(\varphi^\alpha, h_R)| \leqslant C_{\mathrm{d}} |R|^{1/2}$. Also $|h_{I_{\alpha_1}} * \psi_{y_1}(t_1)| \leqslant |I|^{-1/2}$ and $|h_{J_{\alpha_2}} * \psi_{y_2}(t_2)| \leqslant |J|^{-1/2}$. Further, $h_{I_{\alpha_1}} * \psi_{y_1}(t_1) = 0$ except when α_1 lies in a specific set of total length at most $3|I_{t_1}(y_1)| = 6y_1$, because $y_1 < |I|$. We obtain

$$\begin{aligned}
|\varphi_2 * \psi_y(t)| &\leqslant \sum_{R : R \in \mathcal{D},\, |R| \geqslant 2^{-N}} \int_0^1 \int_0^1 |(\varphi^\alpha, h_R)|\, |h_{I_{\alpha_1}} * \psi_{y_1}(t_1)|\, |h_{J_{\alpha_2}} * \psi_{y_2}(t_2)|\, d\alpha \\
&\leqslant \sum_{R : R \in \mathcal{D},\, |R| \geqslant 2^{-N}} C_{\mathrm{d}} |R|^{1/2} 6 y_1 |I|^{-1/2} \int_0^1 |J|^{-1/2}\, d\alpha_2 \\
&= \sum_{R : R \in \mathcal{D},\, |R| \geqslant 2^{-N}} C_{\mathrm{d}} 6 y_1 \\
&\leqslant C C_{\mathrm{d}} (N+1) 2^{N+2} y_1.
\end{aligned}$$

Thus

$$
\begin{aligned}
&\int_{(t_1,t_2)\in\Omega}\int_{0<y_1<2^{-K}}\int_{\delta<y_2<1} |\varphi_2 * \psi_y(t)|^2 \,\frac{dt_1\,dt_2\,dy_1\,dy_2}{y_1 y_2}\\
&\quad\leqslant \int_{(t_1,t_2)\in\Omega}\int_{0<y_1<2^{-K}}\int_{\delta<y_2<1} \left[CC_{\mathrm{d}}(N+1)2^{N+2}y_1\right]^2 \,\frac{dt_1\,dt_2\,dy_1\,dy_2}{y_1 y_2}\\
&\quad= \left[CC_{\mathrm{d}}(N+1)2^{N+2}\right]^2 |\Omega| \int_0^{2^{-K}} y_1^2\,\frac{dy_1}{y_1}\int_\delta^1 \frac{dy_2}{y_2}\\
&\quad= \left[CC_{\mathrm{d}}(N+1)2^{N+2}\right]^2 |\Omega|\,\frac{2^{-2K}}{2}\,\log\frac{1}{\delta}\\
&\quad\leqslant \varepsilon|\Omega|,
\end{aligned}
$$

if $K = K(\varepsilon, N, \delta)$ is chosen sufficiently large.

We have shown that the translation-average φ is in $\mathrm{VMO}(\mathbb{T}\otimes\mathbb{T})$, as required. □

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