On Fourier Frames

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- One of the fundamental results in Fourier analysis is that $\{e^{i\pi k}\}_{k\in\mathbb{Z}}$ forms an orthogonal basis for $L^2(-\pi,\pi)$.
- An important generalization of an orthogonal basis is a spanning set with possible redundancies in the representation. Such objects are called *frames*:

Definition

A sequence $\{f_n\}_{n\in\mathbb{N}}$ of elements of H a Hilbert spaces is a **discrete frame** for H if:

$$\exists\, A, B > 0 \text{ such that } \forall f \in H, \quad A\|f\|^2 \le \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \le B\|f\|^2.$$

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- The first mention of frames was by Duffin and Schaeffer in the context of non-harmonic Fourier series, where families of complex exponentials satisfying the above frame condition were of interest [2].
- Their objects of study, so-called *Fourier frames*, shall be the topic of these lectures.

Definition

A family of complex exponentials $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$, with $\Lambda = \{\lambda_k\} \subset \mathbb{R}$, is a *Fourier frame* if there exist $0 < A \leq B < \infty$ such that $\forall f \in L^2(-\pi,\pi)$:

$$A\int_{-\pi}^{\pi}|f(x)|^2dx \leq \sum_{k\in\mathbb{Z}}\left|\int_{-\pi}^{\pi}f(x)e^{-i\lambda_k x}dx\right|^2 \leq B\int_{-\pi}^{\pi}|f(x)|^2dx.$$

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- Using this notation, $\Lambda = \mathbb{Z}$ generates a Fourier frame, in fact an orthogonal basis.

- **Central Question**: What $\Lambda \subset \mathbb{R}$ generate a Fourier frame?

- The main contents of Ortega-Cerdà and Seip's paper "On Fourier Frames" is a pair of characterizations of such Λ. Their results will combine a variety of tools from complex analysis, functional analysis and potential theory.

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- A key observation regarding the characterization of Fourier frames is that the frame property may, in the context of complex exponentials, be re-cast.

Definition

The space of entire functions of exponential type at most π whose restriction to $\mathbb{R} \subset \mathbb{C}$ is square-integrable is the *Paley-Wiener space*, denoted PW.

- We introduce a new property of sequences $\Lambda \subset \mathbb{R}$:

Definition

A sequence $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is *sampling* for PW if there exist $0 < A \leq B < \infty$ such that $\forall f \in PW$:

$$A \int_{\mathbb{R}} |f(x)|^2 dx \leq \sum_{k\in\mathbb{Z}} |f(\lambda_k)|^2 \leq B \int_{\mathbb{R}} |f(x)|^2 dx.$$

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- In order to relate Fourier frames to sampling sequences, we recall the Paley-Wiener theorem [7]:

Theorem (Paley-Wiener)

Let $\sigma > 0$ be constant. Then the function $F(x)$ is of the form

$$F(x) = \int_{-\sigma}^{\sigma} f(\xi) e^{i\xi x} dx \text{ for some } f \in L^2(-\sigma, \sigma)$$

if and only if $F(x) \in L^2(\mathbb{R})$ and F can be extended to an entire function of exponential-type at most σ, meaning F extends to an entire function $\tilde{F}$ such that $\exists C > 0$ with the property that $|\tilde{F}(z)| \leq Ce^{\sigma|z|}$ everywhere.

- The Paley-Wiener theorem together with the Plancherel theorem can be used to show that $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is sampling for PW if and only if $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ is a Fourier frame. Hence, we will study sampling sequences in order to understand Fourier frames.

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- Before discussing the main results of the paper of Ortega-Cerdà and Seip, we present more classical results concerning sampling sequences.

Definition

Consider $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ where $\lambda_k \leq \lambda_{k+1}$, $\forall k \in \mathbb{Z}$. Such a sequence is *separated* if $q := \inf_{k\in\mathbb{Z}} (\lambda_{k+1} - \lambda_k) > 0$; q is the *separation constant*. For a separated sequence, define the associated *distribution function* n_Λ as follows:

$$n_\Lambda(0) = 0, \quad \forall a < b, \; n_\Lambda(b) - n_\Lambda(a) = |\Lambda \cap (a, b)|.$$

- In particular, a sequence with an accumulation point is not separated. However, some sequences without accumulation points still fail to be separated, for example if we define $\lambda_k = \sum_{n=1}^{k} \frac{1}{n}$. We shall often assume a sequence to be separated.

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- A relatively straightforward inequality related to sampling for PW is:

$$n_\Lambda(b) - n_\Lambda(a) \geq (1+\epsilon)(b-a) - C, \ \forall a < b \Longrightarrow \Lambda \text{ is sampling.}$$

Here C, ϵ are of course independent of a, b. The following more sophisticated result gives a necessary condition for sampling, one which involves a logarithmic growth condition on the distribution function.

Theorem (Landau)

If Λ is a separated sampling sequence for PW, then there exist constants A, B, independent of a, b, such that for for all $a < b$:

$$n_\Lambda(b) - n_\Lambda(a) \geq b - a - A\log^+(b-a) - B.$$

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- Of great use in understanding sampling sequences is the notion of *lower Beurling density*:

Definition

For a separated sequence $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ with associated distribution n_Λ, the *lower Beurling uniform density* is

$$D^-(\Lambda) := \lim_{R\to\infty} \frac{\min_{x\in\mathbb{R}}(n_\Lambda(x+R) - n_\Lambda(x))}{R}.$$

- Beurling lower density is one way to measure density of a sequence of reals, relative to the integers.

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- Beurling lower density is one way to measure density of a sequence of reals, relative to the integers.

- Moreover, Beurling lower density gives an almost complete characterization of sampling sequences [3]:

- $D^{-}(\Lambda) > 1 \implies \Lambda$ is sampling for PW.
- $D^{-}(\Lambda) < 1 \implies \Lambda$ is not sampling for PW.

- So, the critical case is when $D^{-}(\Lambda) = 1$. This corresponds to a set Λ that is "of the same size as the integers," in the sense of lower Beurling density.

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- One of the novelties of the results of Ortega-Cerdà and Seip is their applicability even in the critical case of $D^{-}(\Lambda) = 1$.

- We shall now present these two results. We shall present the minimal background required for the results to be intelligible, then proceed to discuss them at length in the second and third lectures.

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- The de Branges theory of entire functions will be crucial. A space of particular import is:

Definition

The *Hermite-Biehler* space, $\overline{HB}$, is the space of entire functions f without roots in the upper half plane $\mathbb{H}$ and such that $|f(z)| \geq |f(\bar{z})|$ whenever $\Im(z) > 0$.

- A relevant construction involving $\overline{HB}$ allows us to construct Hilbert spaces from elements of $\overline{HB}$. More explicitly, given $E \in \overline{HB}$, we associate a Hilbert space of entire functions:

$$H(E) := \left\{ f \text{ entire} \,\middle|\, \frac{f(z)}{E(z)}, \frac{f^*(z)}{E(z)} \in H^2(\mathbb{H}) \right\}, \ \|f\|^2_{H(E)} := \int_{-\infty}^{\infty} \frac{|f(t)|^2}{|E(t)|^2} dt.$$

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Theorem (Main Result 1)

$\Lambda \subset \mathbb{R}$ is sampling for PW if and only if there exist $E, F \in \overline{HB}$ such that $H(E) = PW$ and Λ is the zero sequence of $EF + E^* F^*$.

- We shall develop the theory necessary to make sense of this condition in the coming days.

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- The second major result concerns generating functions for sampling sequences. More precisely, we shall consider $\psi \in \mathcal{C}^1(\mathbb{R})$ non-decreasing with the properties that:
  1. $\psi(\infty) - \psi(-\infty) = \infty$.
  2. $\psi'(x) = o(1)$ as $|x| \to \infty$ (Recall: $\psi^{'}(x) = o(1)$ if $\forall \epsilon > 0, \exists x_0$ such that $|\psi^{'}(x)| < \epsilon, \ \forall x > x_0$).

- We are interested in the sequence generated by ψ in the following manner. Consider $\Lambda(\psi) = \{\lambda_k\}_{k\in\mathbb{Z}}$ given by $\lambda_k = k - \psi(\lambda_k)$. Alternatively, setting $\psi(0) = 0$, this means that $n_\Lambda(t) = [t + \psi(t)]$.

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    2. $\psi'(x) = o(1)$ as $|x| \to \infty$ (Recall: $\psi^{'}(x) = o(1)$ if $\forall \epsilon > 0, \exists x_0$ such that $|\psi^{'}(x)| < \epsilon,\ \forall x > x_0$).

- We are interested in the sequence generated by ψ in the following manner. Consider $\Lambda(\psi) = \{\lambda_k\}_{k \in \mathbb{Z}}$ given by $\lambda_k = k - \psi(\lambda_k)$. Alternatively, setting $\psi(0) = 0$, this means that $n_\Lambda(t) = [t + \psi(t)]$.

- **Major Question**: For what ψ is $\Lambda(\psi)$ a sampling sequence for PW?

- A characterization of such ψ is given in terms of the extent to which the potential of ψ can be approximated by elements of $\overline{HB}$.

Definition

For a ψ with the above properties, the corresponding *potential* is given by:

$$U_\psi(z) := \int_{-\infty}^{\infty} \left[\log\left|1 - \frac{z}{t}\right| + \Re\left(\frac{z}{t}\right)\right] d\psi(t),$$

taken in the principle value sense.

- A crucial property of U_ψ is that it is sub-harmonic.

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- Our characterization will involve a notion of *logarithmically regular partition*.

Definition

Let ψ be as above, and WLOG assume that $\psi(x) \equiv 0$ for $x \leq 0$. Let $\{t_n\}_{n=0}^{\infty}$ be such that $t_0 = 0$ and $\psi(t_n) = n$, $\forall n \geq 1$. Set $d_n := t_n - t_{n-1}$. We say ψ induces a *logarithmically regular partition* if $d_n \simeq d_{n+1}$ and

$$\sup_{x>0} \sum_{\frac{x}{2}<t_n<2x} \frac{d_n^2}{(x-t_n)^2 + d_n^2} = \sup_{x>0} \sum_{\frac{x}{2}<t_n<2x} \frac{(t_n - t_{n-1})^2}{(x-t_n)^2 + (t_n - t_{n-1})^2} < \infty.$$

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Theorem (Main Result 2)

Let ψ be as above, and WLOG assume that $\psi(x) \equiv 0$ for $x \leq 0$. Then:

1. *If $\psi'(x) = \frac{1}{O(x)}$ when $x \to \infty$ and ψ induces a logarithmically regular partition, then $\Lambda(\psi)$ is sampling for PW.*
2. *If $\psi'(x) = o(\frac{1}{x})$ when $x \to \infty$, then $\Lambda(\psi)$ is not sampling for PW.*

- Note that $\psi^{'}(x) = \frac{1}{O(x)}$ if $\exists M > 0, x_0$ such that $\left|\frac{1}{Mx}\right| \leq |\psi^{'}(x)|,\ \forall x > x_0$.
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- Having presented the two fundamental results of Ortega-Cerdà and Seip's "On Fourier Frames" in relative isolation, we introduce the machinery deployed to prove them.

- Main result 1 will be proven using a series of lemmas and theorems from complex and functional analysis, while main result 2 will be somewhat more self-contained. The proof of the latter will however draw heavily from ideas of Lyubarskii and Malinnikova [4] and their work on approximating subharmonic functions.

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- We begin by introducing de Branges' theory of Hilbert spaces of entire functions.

Definition

A *de Branges space* is a Hilbert space H of entire functions with the following three properties:

1. If $f \in H$, ζ non-real such that $f(\zeta) = 0$, then $g \in H$, where $g(z) := \frac{f(x)(z-\bar{\zeta})}{z-\zeta}$. Moreover, $\|f\|_H = \|g\|_H$.
2. For every ζ non-real, the linear functional on H given by $\zeta \mapsto f(\zeta)$ is continuous.
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The *Hermite-Biehler* space, $\overline{HB}$, is the space of entire functions f without roots in the upper half plane $\mathbb{H}$ and such that $|f(z)| \geq |f(\bar{z})| = |f^*(z)|$ whenever $\Im(z) > 0$.

- Note that by the maximum modulus principle, the above condition may be replaced by $|f(z)| > |f(\bar{z})|$ when $\Im(z) > 0$.
- It is not difficult to show that $\overline{HB}$ is a de Branges space. What is of far greater interest is the following characterization of de Branges spaces.

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- Recall our construction for $H(E)$: given $E \in \overline{HB}$, we associate a Hilbert space of entire functions:

$$H(E) := \left\{ f \text{ entire} \,\middle|\, \frac{f(z)}{E(z)}, \frac{f^*(z)}{E(z)} \in H^2(\mathbb{H}) \right\}, \quad \|f\|^2_{H(E)} := \int_{-\infty}^{\infty} \frac{|f(t)|^2}{|E(t)|^2} dt.$$

Theorem (Characterization of de Branges spaces)

A Hilbert space of entire functions satisfying the three criterion of a de Branges space is equal isometrically to $H(E)$, some $E \in \overline{HB}$.

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- The second condition of a de Branges space H, namely that for every non-real ζ, the linear functional on H given by $f \mapsto f(\zeta)$ is continuous, has the consequence that each such ζ yields a reproducing kernel $K_E(\zeta, z)$:

Theorem (Reproducing Kernel for $H(E)$)

Let $E \in \overline{HB}$. For each fixed $\zeta \in \mathbb{C}$, the function

$$K_E(\zeta, z) := \frac{i}{2} \frac{E(z)\overline{E(\zeta)} - E^*(z)\overline{E^*(\zeta)}}{\pi(z - \bar{\zeta})}$$

as a function of z is in $H(E)$. Moreover, K_E is a reproducing kernel for $H(E)$:

$$\forall f \in H(E),\ \langle f, K_E(\zeta, \cdot)\rangle_E = \int_{-\infty}^{\infty} \frac{f(t)\overline{K_E(\zeta, t)}}{|E(t)|^2} dt = f(\zeta).$$

- Useful in analyzing $E \in \overline{HB}$ will be a notion of *phase function*.

Proposition

For $x \in \mathbb{R}$, we may write $E(x) = |E(x)|e^{-i\phi(x)}$, where $\phi(x) \in \mathcal{C}(\mathbb{R})$ is such that $E(x)e^{i\phi(x)} \in \mathbb{R}$, for all $x \in \mathbb{R}$.

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Theorem (Generalized Plancherel)

Let $H(E)$ be a de Branges space, ϕ the phase function associated to E. Suppose $\alpha \in \mathbb{R}$ and let $\Gamma := \{\gamma_k\}_{k\in\mathbb{Z}}$ be the sequence of real numbers such that $\phi(\gamma_k) = \alpha + k\pi, k \in \mathbb{Z}$. Then if $e^{i\alpha}E - e^{-i\alpha}E^* \notin H(E)$, the family of normalized reproducing kernels

$$\left\{ \frac{K_E(\gamma_k, z)}{\|K_E(\gamma_k, \cdot)\|_E} \right\}_{k\in\mathbb{Z}}$$

is an orthonormal basis for $H(E)$. In particular:

$$\|f\|_E^2 = \sum_k \frac{\pi |f(\gamma_k)|^2}{\phi'(\gamma_k)|E(\gamma_k)|^2}, \ \forall f \in H(E).$$

- Note that $e^{i\alpha}E - e^{-i\alpha}E^* \in H(E)$ for at most one $\alpha \in [0, \pi)$, so the conditions of the theorem are easily met.

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- One more crucial theorem of the de Branges theory will be needed to prove main result 1. It relates the classical Poisson transform to the norm on $H(E)$.

Theorem

Let μ be a measure on $\mathbb{R}$, and $E \in \overline{HB}$. Then:

$$\int_{\mathbb{R}} \frac{|f(t)|^2}{|E(t)|^2} d\mu(t) = \int_{\mathbb{R}} \frac{|f(t)|^2}{|E(t)|^2} dt$$

if and only if there exists a bounded holomorphic function A on $\mathbb{H}$ such that $\|A\|_\infty := \sup_{z\in\mathbb{H}} |A(z)| \leq 1$ and:

$$\frac{y}{\pi} \int_{\mathbb{R}} \frac{d\mu(t)}{(t-x)^2 + y^2} = \Re\left(\frac{E + E^* A}{E - E^* A}\right).$$

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- We are now ready to sketch a proof of main result 1.

- It has been shown by Seip [6] that any Λ, a sampling sequence for PW, contains $\Lambda' \subseteq \Lambda$ that is sampling and *separated*. Thus, we may WLOG restrict ourselves to separated Λ.

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$\Lambda \subset \mathbb{R}$ is a separated sampling sequence for PW if and only if there exist $E, F \in \overline{HB}$ such that $H(E) = PW$ and Λ is the zero sequence of $EF + E^*F^*$.

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- We first prove the $\Rightarrow$ implication, so assume $\Lambda \subset \mathbb{R}$ is a separated sampling sequence for PW.

- PW with norm $f \mapsto \sqrt{\sum_k |f(\lambda_k)|^2}$ is a de Branges space. So by our characterization of de Branges spaces, there exists $E \in \overline{HB}$ such that $H(E) = PW$ and $\sum_k |f(\lambda_k)|^2 = \int_{\mathbb{R}} \frac{|f(t)|^2}{|E(t)|^2} dt$.

- Setting $\mu = \sum_k |E(\lambda_k)|^2 \delta_k$ and applying our theorem on the Poisson transform, we get a bounded holomorphic function A in $\mathbb{H}$ such that $\|A\|_\infty \leq 1$, and $a \in \mathbb{R}$ such that:

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- We now analyze this equation. Notice the LHS is meromorphic, but the RHS is holmorphic in $\mathbb{H}$. Set

$$M(z) := -i \sum_k |E(\lambda_k)|^2 \left(\frac{1}{z - \lambda_k} + \frac{1}{\lambda_k} \right) + ia$$

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- It is readily verified that

$$A = \frac{M-1}{M+1} \frac{E}{E^*}.$$

- Notice $M - 1$ has poles exactly at the points λ_k. Moreover, our main equality tells us $M - 1$ vanishes whenever E^* does.

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$$A = \frac{M-1}{M+1} \frac{E}{E^*}.$$

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$$M - 1 = -\frac{E^* F^*}{G}, \; F \text{ entire}, \; G(z) := \prod_k \left(1 - \frac{z}{\lambda_k}\right) e^{\frac{z}{\lambda_k}}.$$

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- We now analyze this equation. Notice the LHS is meromorphic, but the RHS is holmorphic in $\mathbb{H}$. Set

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- This implies $\dfrac{F^*}{F} = -A$ in $\mathbb{H}$ and F has no zeroes in $\mathbb{H}$. Since $\|A\|_\infty \leq 1$, we conclude $F \in \overline{HB}$.

- We now claim that $G = \dfrac{EF + E^*F^*}{2}$, which will imply Λ is the zero sequence of $EF + E^*F^*$, since Λ is by definition the zero sequence for G.

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- We now turn to the converse. Assume we have $E, F \in \overline{HB}$ such that $PW = H(E)$ and Λ is the zero sequence of $EF + E^* F^*$. We shall prove Λ is sampling for PW.

- Notice $H(E) = PW$ implies E has no real zeroes. WLOG, F also has no real zeroes.

- For $\alpha \in (0, \pi]$, we define $\Lambda_\alpha = \{\lambda_{\alpha,k}\}_{k \in \mathbb{Z}}$ by $\phi_{EF}(\lambda_{\alpha,k}) = \alpha + k\pi$.

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- Citing the Plancherel-Pólya inequality, we conclude that there exists C, independent of α, such that:

$$\sum_k |f(\lambda_{\alpha,k})|^2 \leq C\|f\|^2_{PW}.$$

- Now, applying our generalized Plancherel theorem, we have that for all but at most one $\alpha \in (0, \pi]$:

$$\forall g \in H(EF),\ \int_{\mathbb{R}} \frac{|g(t)|^2}{|E(t)F(t)|^2} dt = \sum_k \frac{|g(\lambda_{\alpha,k})|^2}{|E(\lambda_{\alpha,k})F(\lambda_{\alpha,k})|^2 \phi'_{EF}(\lambda_{\alpha,k})}.$$

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$$1 = \sup_{f \in PW, \|f\|_{PW}^2 \leq 1} |f(x)|^2 \simeq \sup_{f \in H(E), \|f\|_{H(E)}^2 \leq 1} |f(x)|^2 = K_E(x,x) = \frac{1}{\pi} \phi_E^{'}(x) |E(x)|^2$$

- Since $\phi_{EF}^{'} = \phi_E^{'} + \phi_F^{'} \geq \phi_E^{'}$, the above inequality along with the previous string of equalities yields:

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- If $\alpha = \frac{\pi}{2}$, then let $\{\alpha_n\} \to \frac{\pi}{2}$. Notice that $\sum_k |f(\lambda_{\alpha_n,k})|^2 \leq C\|f\|_{PW}^2$.

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- We now present an interpretation of the function F in main result 1.

Definition

$\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is a *complete interpolating sequence* if the interpolation problem $f(\lambda_k) = a_k, k \in \mathbb{Z}$, has a unique solution $f \in PW$ for all l^2 data $\{a_k\}_{k\in\mathbb{Z}}$.

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- If Λ is a complete interpolating sequence, then our characterization of de Branges spaces and our generalized Plancherel theorem imply $\exists E \in \overline{HB}$ such that $H(E) = PW$ and Λ constitutes the zero sequence of $E + E^*$.

- In this sense, we may understand F as accounting for the redundancy in Λ.

- In particular, if $D^-(\Lambda) > 1$, Seip has shown $\Lambda = \Lambda^{'} \cup (\Lambda \setminus \Lambda^{'})$, where $\Lambda^{'}$ is a complete interpolating sequence. In this case, the hypotheses of main result 1 are met if we choose E to correspond as above with $\Lambda^{'}$ and set F to be:

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Let H be a de Branges space. $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is a *complete interpolating sequence for a de Branges space* if the interpolation problem $f(\lambda_k) = a_k, k \in \mathbb{Z}$ has a unique solution $f \in H$ for all $\{a_k\}_{k\in\mathbb{Z}}$ such that

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- This relates to a general result of Seip [6], which states that we cannot in general take a sampling sequence Λ and acquire a complete interpolating sequence as a subsequence, that is to say by making Λ *thinner*. Instead, we can make the space *larger* so that Λ becomes a complete interpolating sequence for the larger space.

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- An interesting corollary of main result 1 relates separated sampling sequences and complete interpolating sequences:

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If Λ is a separated sampling sequence for PW, there exists a complete interpolating sequence $\Gamma = \{\gamma\}_{k \in \mathbb{Z}}$ such that for every $k \in \mathbb{Z}$, there is at least one $\lambda \in \Lambda$ such that $\gamma_k \leq \lambda \leq \gamma_{k+1}$.

Proof.

We have mentioned that Λ sampling implies Λ consists of those points $\{\lambda\}$ such that $\phi_{EF}(\lambda) = \frac{\pi}{2} + k\pi$, for some $k \in \mathbb{Z}$. On the other hand, ϕ_E is increasing and increases more slowly than ϕ_{EF}. The set $\Gamma_\alpha = \{\gamma \mid \phi_E(\gamma) = \alpha + k\pi\}$ thus has the desired property. Moreover, since $H(E) = PW$, generalized Plancherel implies Γ_α is a complete interpolating sequence except for at most one $\alpha \in [0, \pi)$. Pick a good one. □

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- We now move to the case when $D^{-}(\Lambda) = 1$ but there does not exist $\Lambda^{'} \subset \Lambda$ such that $\Lambda^{'}$ is a complete interpolating sequence.
- Let $\psi \in \mathcal{C}^1(\mathbb{R})$ be non-decreasing such that $\psi(\infty) - \psi(-\infty) = \infty$ and $\psi^{'}(x) = o(1)$ as $|x| \to \infty$.
- To ψ we associate a sequence $\Lambda(\psi) = \{\lambda_k\}_{k\in\mathbb{Z}}$ given by $\lambda_k = k - \psi(\lambda_k)$.
- Alternatively, setting $\psi(0) = 0$, $n_{\Lambda(\psi)}(t) = [t + \psi(t)]$.

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- All such $\Lambda(\psi)$ are *sets of uniqueness*, i.e. every trigonometric series vanishing off of $\Lambda(\psi)$ is identically zero.

- However, Seip [6] has shown that no $\Lambda(\psi)$ can contain a complete interpolating sequence as a subset.

- In order to understand $\Lambda(\psi)$ in the context of sampling, we introduce the following potential function:

$$U_\psi(z) := \int_{\mathbb{R}} \left(\log \left| 1 - \frac{z}{t} \right| + \Re \left(\frac{z}{t} \right) \right) d\psi(t).$$

- Note that $\psi^{'} \geq 0$ implies U_ψ is sub-harmonic.

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- We recall the second main result of the paper of Oretega-Cerdà and Seip:

Definition

Let ψ be as above, and WLOG assume that $\psi(x) \equiv 0$ for $x \leq 0$. Let $\{t_n\}_{n=0}^{\infty}$ be such that $t_0 = 0$ and $\psi(t_n) = n$, $\forall n \geq 1$. Set $d_n := t_n - t_{n-1}$. We say ψ induces a *logarithmically regular partition* if $d_n \simeq d_{n+1}$ and

$$\sup_{x>0} \sum_{\frac{x}{2}<t_n<2x} \frac{d_n^2}{(x-t_n)^2 + d_n^2} < \infty.$$

Theorem (Main Result 2)

Let ψ be as above, and WLOG assume that $\psi(x) \equiv 0$ for $x \leq 0$. Then:

1. *If $\psi'(x) = \frac{1}{O(x)}$ when $x \to \infty$ and ψ induces a logarithmically regular partition, then $\Lambda(\psi)$ is sampling for PW.*
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- In order to prove main result 2, we first establish the relationship between sampling for PW and the extent to which U_ψ can be approximated by the logarithm of the modulus of an entire function. This will follow in part from main result 1.

Corollary

$\Lambda(\psi)$ is sampling for PW if there exists $f \in \overline{HB}$ such that $\phi_f'(x) = o(1)$ when $|x| \to \infty$ and such that:

$$|U_\psi(z) - \log|f(z)|\,| \lesssim 1, \ \Im(z) \geq 0.$$

- Notice if we could find $e \in \overline{HB}$ such that $\phi_e(x) = \pi x + \pi\psi(x) - \phi_f(x)$, we would be done. This is because $\Lambda(\psi)$ would be the zero sequence of $ef + e^* f^*$ and $|e(z)| = e^{\pi\Im(z)}$ for $\Im(z) \geq 0 \implies H(e) = PW$. We could thus apply main result 1 and conclude $\Lambda(\psi)$ is sampling.

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- Sadly, such an e is elusive. Instead, we shall prove the result using the following perturbation principle, proved originally by Duffin and Schaeffer [2]:

Proposition (Perturbation Principle)

If $\Gamma = \{\gamma_k\}$ is sampling, then $\Gamma' = \{\gamma_k + \delta_k\}$ is sampling whenever each $\gamma_k + \delta_k$ is distinct and $\delta_k \to 0$ as $|k| \to \infty$.

- So, it suffices to find $E \in \overline{HB}$ such that $\phi_E(x) - \pi x - \pi\psi(x) + \phi_f(x) = o(1)$ as $|x| \to \infty$ and $|E(x)| \simeq e^{\pi \Im(z)}$ for $\Im(z) \geq 0$.

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- WLOG, $\omega(x) := x + \psi(x) + \frac{1}{\pi}\phi_f(x)$ satisfies $1 + \psi^{'}(x) + \frac{1}{\pi}\phi_f^{'} = \omega^{'} \simeq 1$.

- Partition $\mathbb{R}$ into a sequence of disjoint intervals $I_k = [x_k, x_{k+1}], k \in \mathbb{Z}, x_0 = 0$, such that:

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- Ortega-Cerdá and Seip have previously shown that:

$$|A(z)|e^{-U_\omega(z)} \simeq \min(1, \mathrm{dist}(z, \Gamma)).$$

- Now choose two monomials P, Q of the same degree with only real zeroes such that the function B defined by:

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- By a theorem of Meiman, either $A - iB \in \overline{HB}$ or $A + iB \in \overline{HB}$; we set $E = A - iB$ and WLOG (since P may be replaced by $-P$) we may assume $E \in \overline{HB}$.

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Let ψ be as above, and WLOG assume that $\psi(x) \equiv 0$ for $x \leq 0$. Then:

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- We shall prove the first claim of this theorem via our corollary, and the second claim directly. The key for the second claim is to show $\psi^{'}(x) = \frac{1}{O(x)}$ is critical.
- We begin with the first claim, namely that $\psi^{'}(x) = \frac{1}{O(x)}$ as $x \to \infty$ and ψ inducing a logarithmically regular partition together imply $\Lambda(\psi)$ is sampling for PW.
- By our corollary, it suffices to produce an appropriate $f \in \overline{HB}$. We define the zeroes of f as follows: let $\{t_n\}$ be the logarithmically regular partition, and define $r_n \in (t_{n-1}, t_n)$ by:

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- Now, choose f so that:

$$\log|f(z)| = \sum_{n=1}^{\infty} \int_{r_{n-1}}^{r_n} \left(\log\left|1 - \frac{z}{z_n}\right| + \Re\left(\frac{z}{t}\right) \right) d\psi(t).$$

- We see $f \in \overline{HB}$. Set $V := U_\psi - \log|f|$; this satisfies:

$$V(z) = \sum_{n=1}^{\infty} \int_{r_{n-1}}^{r_n} \left(\log\left|1 - \frac{z}{t}\right| - \log\left|1 - \frac{z}{z_n}\right| \right) d\psi(t) =: \sum_{n=1}^{\infty} j_n(z).$$

- It suffices to prove this series converges uniformly on compact subsets of $\mathbb{C}$, and that $V(z) = O(1)$ for $\Im(z) \geq 0$, since we then apply our corollary and conclude $\Lambda(\psi)$ is sampling for PW.

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$$\begin{aligned} j_n(z) &= \int_{r_{n-1}}^{r_n} \left(\log \left| 1 - \frac{z}{t} \right| - \log \left| 1 - \frac{z}{z_n} \right| \right) d\psi(t) \\ &= \int_{r_{n-1}}^{r_n} \left(\log \left| 1 - \frac{t}{z} \right| - \log \left| 1 - \frac{z_n}{z} \right| \right) d\psi(t) \\ &= \int_{r_{n-1}}^{r_n} \log \left(\left| \frac{z-t}{z - r_n e^{\frac{-icd_n}{r_n}}} \right| \right) d\psi(t). \end{aligned}$$

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- It remains to show $V(z) \simeq 1$. To do so, we shall split the sum defining V into two pieces, and analyze each separately.
- More precisely, define the set of *essential indices* by:

$$N(z) := \{n^-(z), n^-(z)+1, ..., n^+(z)-1, n^+(z)\} \setminus \{n(z)-1, n(z), n(z)+1\}$$

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$$V(z) \simeq V_1(z) + V_2(z) = \sum_{n \in N(z)} \int_{r_{n-1}}^{r_n} \left(\log\left|1 - \frac{z}{t}\right| - \log\left|1 - \frac{z}{r_n}\right| \right) d\psi(t) + \sum_{n \in N(z)} \left(\log|z - r_n| - \log|z - z_n| \right),$$

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- We first estimate V_1 by introducing the function $L(w) := \log(1 - ze^{-\omega})$, which has the property that for each $n \in N(z)$, L is analytic in a domain containing $\{\omega \mid e^{\omega} \in [t_{n-1}, t_n]\}$. Note that z is fixed here.
- Therefore, we may write a series expansion:

$$L(\omega) - L(\omega_n) = (\omega - \omega_n)L^{'}(\omega_n) + \int_{\omega_n}^{\omega} L^{''}(\sigma)(\omega - \sigma)d\sigma$$
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- Moving on to V_2, we write

$$\log|z-r_n| - \log|z-z_n| = \Re\left(\int_{z_n}^{r_n} \frac{d\zeta}{\zeta - z}\right).$$

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- We now move on to the second result of this theorem: if $\psi'(x) = o(\frac{1}{x})$ when $x \to \infty$, then $\Lambda(\psi)$ is not sampling for PW.

- We will show this by exhibiting a sequence $\{f_n\} \subset PW$ such that for $\Lambda(\psi) = \{\lambda_k\}$:

$$\lim_{n\to\infty} \frac{\sum_k |f_n(\lambda_k)|^2}{\|f_n\|_{PW}^2} \to 0.$$

- Let $\{t_n\}$ be such that $\psi(t_n) = n$ and suppose n is sufficiently large for the following construction to work. We require $\xi_n \in (t_n, \frac{t_{n+1}}{2})$ to be such that $\psi(\xi_n) = n + \frac{1}{2} + \epsilon_n$, where ϵ_n will be chosen below.

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- Define a bounded, continuous ϕ_n with the following properties:
  1. $\phi_n(t) = -t, |t| < \frac{1}{2}$.
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  4. Linear everywhere else.

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- We define a subharmonic function

$$U_n(z) := \lim_{R \to \infty} \int_{-R}^{R} \left(\log \left| 1 - \frac{z}{t} \right| \right) (1 + \phi_n'(t)) dt.$$

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- Setting $\Omega_n := [t_n, t_{n+1}] \cup [2\xi_n, t_{n+1}]$, we see:

$$\int_{\Omega_n} e^{2U_n(x)} dx \to \infty \text{ but } \int_{\mathbb{R}\setminus\Omega_n} e^{2U_n(x)} dx < C < \infty$$

- Thus, $e^{2U_n} \in L^2(\mathbb{R})$ but $\|e^{U_n}\|_2 \to \infty$.

- Similarly, if we have a sequence of reals $\Gamma = \{\gamma_k\}_{k=1}^{\infty}$ satisfiying $\gamma_{k+1} - \gamma_k \simeq 1$, we have:

$$\sum_{\gamma \in \mathbb{R} \setminus \Omega_n} e^{2U_n(\gamma_k)} < C.$$

- Following our construction in the corollary, we define f_n as follows:

$$f_n(z) := \lim_{R \to \infty} \prod_{|\gamma_k| < R} \left(1 - \frac{z}{\gamma_k}\right)$$

where $\Gamma = \{\gamma_k\}$ is a separated real sequence such that

$$|f_n(z)| e^{-U_n(z)} \simeq \min(1, \operatorname{dist}(z, \Gamma)).$$

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- We see f_n is of exponential type π. Moreover, our earlier estimate yields $f_n \in PW$ but $\|f_n\|_{PW} \to \infty$.
- Finally, notice that $\text{dist}(\lambda_k, \Gamma) \to 0$ uniformly when $n \to \infty$, and that $\gamma_k = k \; \forall k < 0$. Along with our earlier estimate, we see that:

$$\frac{\sum_k |f_n(\lambda_k)|^2}{\|f_n\|_{PW}^2} \to 0.$$

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