PRELIMINARY VERSION

Interpolating Sequences for the Multipliers of the Dirichlet Space

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Abstract. The Dirichlet space $\mathcal{D}$ consists of all analytic functions f defined on the unit disk $\mathbb{D}$ with $\int_{\mathbb{D}} |f'(z)|^2 dA < \infty$. The space of multipliers $\mathcal{M}_{\mathcal{D}}$ consists of analytic functions φ with $\varphi f \in \mathcal{D}$ for all $f \in \mathcal{D}$. A sequence $\{z_n\} \subset \mathbb{D}$ is called an interpolating sequence for $\mathcal{M}_{\mathcal{D}}$ if for each bounded sequence of complex numbers $\{w_n\}$ there exists $\varphi \in \mathcal{M}_{\mathcal{D}}$ with $\varphi(z_n) = w_n$ for all n. Our main result is a geometric characterization of the interpolating sequences for $\mathcal{M}_{\mathcal{D}}$, answering a question of S. Axler.

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§1 Introduction.

The Dirichlet space $\mathcal{D}$ consists of all analytic functions $f(z) = \sum_0^\infty a_n z^n$ defined on the unit disk $\mathbb{D}$ with

$$\int_{\mathbb{D}} |f'(z)|^2 dA/\pi = \sum_{n=1}^{\infty} n|a_n|^2 < \infty,$$

where dA denotes the usual Lebesgue measure on $\mathbb{D}$. Define the norm on $\mathcal{D}$ by

$$||f||_{\mathcal{D}}^2 = \sum_{n=0}^{\infty}(n+1)|a_n|^2 = \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta/2\pi + \int_{\mathbb{D}} |f'(z)|^2 dA/\pi,$$

where $d\theta$ is the usual Lebesgue measure on $[0, 2\pi]$ and $f(e^{i\theta})$ denotes the non-tangential limit of f at $e^{i\theta}$ (a.e. $d\theta$). A function φ is called a *multiplier of* $\mathcal{D}$ if $\varphi f \in \mathcal{D}$ whenever $f \in D$ and we denote the set of multipliers by $\mathcal{M}_{\mathcal{D}}$. Since the constant function 1 is in $\mathcal{D}$, we have $\mathcal{M}_{\mathcal{D}} \subset \mathcal{D}$. We shall see shortly that in fact $\mathcal{M}_{\mathcal{D}} \subset H^\infty$, the Hardy space of bounded analytic functions on $\mathbb{D}$. The algebra $\mathcal{M}_{\mathcal{D}}$ plays a role in the study of the Hilbert space $\mathcal{D}$ which is similar to the role played by H^∞ in the study of the classical Hardy space H^2 of analytic functions with square summable coefficients. Indeed H^∞ is exactly the space of multipliers of H^2: $\varphi f \in H^2$ for all $f \in H^2$ if and only if $\varphi \in H^\infty$.

A sequence $\{z_n\} \subset \mathbb{D}$ is called an *interpolating sequence for* H^∞ if for each bounded sequence of complex numbers $\{w_n\}$ there exists $\varphi \in H^\infty$ such that $\varphi(z_n) = w_n$ for all n. Carleson [C2] gave the following geometric characterization of interpolating sequences for H^∞. Let

$$\rho(z, w) = \left|\frac{z - w}{1 - \overline{z}w}\right|$$

be the pseudo-hyperbolic metric on $\mathbb{D}$. If $I = \{e^{i\theta} : \theta_0 \le \theta \le \theta_0 + a\}$ is an arc on $\partial\mathbb{D}$ whose length $= |I| = a$, let

$$S(I) = \{re^{i\theta} : 1 - a \le r < 1, \quad \theta_0 \le \theta \le \theta_0 + a\}$$

be the approximate square with base I. If $a \ge 1$, let $S(I) = \mathbb{D}$.

Theorem [Carleson]. *A sequence $\{z_n\} \subset \mathbb{D}$ is an interpolating sequence for H^∞ if and only if there is an $\varepsilon > 0$ and $K < \infty$ so that*

(1) $$\rho(z_n, z_m) \ge \varepsilon \text{ for all } n \ne m \text{ and}$$

(2) $$\sum_{z_n \in S(I)} (1 - |z_n|^2) \le K|I|$$

for all arcs $I \subset \partial\mathbb{D}$.

A positive measure μ defined on $\mathbb{D}$ is called a *Carleson measure* for H^2 if there exists a constant $K < \infty$ such that

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq C||f||_{H^2}^2$$

for all $f \in H^2$. Carleson [C2,C3] proved that μ is a Carleson measure if and only if there is a constant $K < \infty$ such that

$$\mu(S(I)) \leq K|I|$$

for all arcs $I \subset \partial\mathbb{D}$. Thus (2) can be rephrased as

$$(2') \qquad \sum (1 - |z_n|^2)\delta_{z_n} \text{ is a Carleson measure,}$$

where δ_z denotes point mass at z. Any Carleson measure can be obtained as a weak-* limit of discrete measures related to interpolating sequences as described in Jones[J2]. Carleson measures and interpolating sequences proved to be of great importance in the proof of the Corona theorem, by Carleson [C3], and in finding L^∞ solutions to $\overline{\partial}$-problems by Jones [J1], for example. Carleson measures were also fundamental in the development of BMO by C. Fefferman [Fe], Jones [J2] and others.

We define interpolating sequences for $\mathcal{M}_\mathcal{D}$ and Carleson measures for $\mathcal{D}$ in an analogous fashion. A sequence $\{z_n\} \subset \mathbb{D}$ is called an *interpolating sequence for* $\mathcal{M}_\mathcal{D}$ if for each bounded sequence of complex numbers $\{w_n\}$ there exists $\varphi \in \mathcal{M}_\mathcal{D}$ such that $\varphi(z_n) = w_n$ for all n. A positive measure μ defined on $\mathbb{D}$ is called a *Carleson measure* for $\mathcal{D}$ if there exists a constant $C < \infty$ such that

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq C||f||_{\mathcal{D}}^2$$

for all $f \in \mathcal{D}$.

Stegenga [St] gave the following geometric characterization of Carleson measures for $\mathcal{M}_\mathcal{D}$. If $E = \cup I_j$ is a finite union of disjoint arcs $\{I_j\} \subset \partial\mathbb{D}$, let $S(E) = \cup S(I_j)$.

Theorem [Stegenga]. *Let μ be a positive measure on $\mathbb{D}$. Then μ is a Carleson measure for $\mathcal{D}$ if and only if there is a constant $C_0 < \infty$ so that whenever E is a finite union of disjoint arcs in $\partial\mathbb{D}$*

$$(3) \qquad \mu(\cup S(E)) \leq C_0 \left(\log \frac{1}{\mathrm{Cap}(E)}\right)^{-1},$$

where $\mathrm{Cap}(E)$ *denotes the usual logarithmic capacity of* E.

If $g(z,\infty)$ is Green's function for $\mathbb{C}^* \setminus E$ with pole at ∞, then $g(z,\infty) = \log|z| + \gamma(E) + o(1)$ near ∞ and $\mathrm{Cap}(E) \equiv e^{-\gamma(E)}$. Stegenga's theorem is a geometric characterization because Fekete and Szegö (see [Ah]) proved that the logarithmic capacity of a set E is the same as the transfinite diameter of E. Some authors call $C_l(E) \equiv \frac{1}{\gamma(E)}$ the capacity of E, in which case the right side of (3) becomes $C_l(E)$, the capacity of $\cup I_j$. The right side of (3) is also comparable to a Bessel capacity [St].

In [Ax], Axler studied interpolating sequences for $\mathcal{M}_{\mathcal{D}}$. He proved that any sequence $\{z_n\} \subset \mathbb{D}$ with $|z_n| \to 1$ contains a subsequence which is interpolating for $\mathcal{M}_{\mathcal{D}}$, though he could not give an explicit example of an interpolating sequence. He asked:

(i) Give a concrete example of an interpolating sequence for $\mathcal{M}_{\mathcal{D}}$.

(ii) Find a growth rate for $\{|z_n|\}$ that implies $\{z_n\}$ is an interpolating sequence for $\mathcal{M}_{\mathcal{D}}$.

(iii) Give a necessary and sufficient condition for $\{z_n\}$ to be an interpolating sequence for $\mathcal{M}_{\mathcal{D}}$.

The following theorem is our main result, answering (iii) above.

Theorem 1. *A sequence* $\{z_n\}$ *is an interpolating sequence for* $\mathcal{M}_{\mathcal{D}}$ *if and only if there is a* $\gamma > 0$ *and* $C_0 < \infty$ *such that*

$$1 - \rho^2(z_n, z_m) \leq (1 - |z_n|^2)^{\gamma} \quad \text{for all } n \neq m \text{ and} \tag{4}$$

$$\sum_{z_n \in S(E)} \left(\log \frac{1}{1-|z_n|^2}\right)^{-1} \leq C_0 \left(\log \frac{1}{\mathrm{Cap}(E)}\right)^{-1} \tag{5}$$

whenever E *is a finite union of disjoint arcs in* $\partial\mathbb{D}$.

By Stegenga's theorem, (5) can be rephrased as

$$\sum \left(\log \frac{1}{1-|z_n|^2}\right)^{-1} \delta_{z_n} \text{ is a Carleson measure for } \mathcal{D}. \tag{5$'$}$$

As consequences of Theorem 1, we also give answers to (i) and (ii) above. In the course of the proof of Theorem 1, we will also prove that (4) and (5) characterize the interpolating sequences for $\mathcal{D}$ (defined below). Our approach is to use the analog in $\mathcal{M}_{\mathcal{D}}$ of Pick's theorem to convert the interpolation problem to an L^2 problem about Riesz sequences. In section 2, we develop the Hilbert space background. In section 3 we give a new proof of Carleson's interpolation theorem for H^∞, based on this approach. In section 4 we prove Agler's theorem that the analog of Pick's theorem for $\mathcal{D}$ holds. In section 5 we prove Theorem 1 and draw some consequences. While this paper was in preparation, we received a manuscript from C. Bishop proving that (4) and (5) characterize interpolating sequences for $\mathcal{D}$ and that (4) and (5) are sufficient for $\{z_n\}$ to be an interpolating sequence for $\mathcal{M}_{\mathcal{D}}$, though he could not prove they are necessary for interpolation in $\mathcal{M}_{\mathcal{D}}$. Bishop also derived the necessary and sufficient condition in Corollary 21 for sequences contained in $\mathbb{R} \cap \mathbb{D}$ to be interpolating, and found an example of the phenomena in Example 24. His techniques are different from ours. We thank B. Solomyak for a useful conversation.

§2 Interpolating Sequences and Bases.

Theorem 1 and Carleson's theorem can be put in a common framework by considering reproducing kernels. If $f(z) = \sum_0^\infty a_n z^n$ and $g(z) = \sum_0^\infty b_n z^n$ then H^2 is a Hilbert space with inner product

$$< f, g >_{H^2} = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})} d\theta / 2\pi = \sum a_n \overline{b_n}.$$

Likewise, $\mathcal{D}$ is a Hilbert space with inner product

$$< f, g >_{\mathcal{D}} = < f, g >_{H^2} + \int_{\mathbb{D}} f'(z)\overline{g'(z)} dA(z)/\pi = \sum (n+1) a_n \overline{b_n}.$$

If $\alpha \in \mathbb{D}$, the function $k_\alpha(z) = (1 - \overline{\alpha} z)^{-1} = \sum \overline{\alpha}^n z^n \in H^2$ satisfies

$$f(\alpha) = < f, k_\alpha >_{H^2}$$

for all $f \in H^2$. Similarly, the function

$$k_\alpha(z) = \frac{1}{\overline{\alpha} z} \log \frac{1}{1 - \overline{\alpha} z} = \sum_{n=0}^{\infty} \frac{1}{n+1} \overline{\alpha}^n z^n \in \mathcal{D}$$

satisfies

$$f(\alpha) = < f, k_\alpha >_{\mathcal{D}}$$

for all $f \in \mathcal{D}$. The functions k_α are called *reproducing kernels* since they "reproduce" the value of functions in the Hilbert space at points of $\mathbb{D}$. The norm of k_α can be easily computed with the useful identity

$$k_\alpha(\alpha) = < k_\alpha, k_\alpha > = ||k_\alpha||^2.$$

The connection between reproducing kernels and interpolation is found by considering the adjoints of multiplication operators: If $\varphi \in \mathcal{M}_{\mathcal{D}}$ let M_φ be the bounded operator on $\mathcal{D}$ given by $M_\varphi f = \varphi f$ for all $f \in \mathcal{D}$. For $f \in \mathcal{D}$ and $\alpha \in \mathbb{D}$

$$(6) \qquad < M_\varphi^* k_\alpha, f > = < k_\alpha, M_\varphi f > = \overline{< \varphi f, k_\alpha >} = \overline{\varphi(\alpha)} < k_\alpha, f > .$$

Thus $M_\varphi^* k_\alpha = \overline{\varphi(\alpha)} k_\alpha$. In other words, each k_α is an eigenvector of M_φ^* with eigenvalue $\overline{\varphi(\alpha)}$. The identity (6), using $f = k_\alpha$ also shows that

$$||\varphi||_{H^\infty} \leq ||M_\varphi^*|| = ||M_\varphi||,$$

and hence $\mathcal{M}_{\mathcal{D}} \subset H^\infty$. Specifying the values of a function φ at points $z_n \in \mathbb{D}$ is then specifying certain eigenvalues of the operator M_φ^*. A similar situation occurs with H^∞ since it is the space of multipliers of H^2. (This view of H^∞ as a space of multipliers is not artificial. In fact, it is the reason H^∞ is important in control theory and other applications. See, for example, [Fr].)

Suppose now that $\mathcal{A}$ is a Hilbert space of analytic functions on a domain $\Omega \subset \mathbb{C}^n$ such that evaluation at $\alpha \in \Omega$ is a bounded linear functional on $\mathcal{A}$:

$$|f(\alpha)| \leq C||f||_{\mathcal{A}}$$

for all $f \in \mathcal{A}$. Then for each $\alpha \in \Omega$, there is a unique function $k_\alpha \in \mathcal{A}$ so that $f(\alpha) =< f, k_\alpha >$, for all $f \in \mathcal{A}$. We will also suppose that for every finite set $\alpha_1, \ldots, \alpha_n \in \Omega$ the corresponding reproducing kernels $\{k_{\alpha_i}\}$ are independent. For example, if for every finite set $\alpha_1, \ldots, \alpha_n \in \Omega$ there is an $f \in \mathcal{A}$ such that $f(\alpha_1) = 1$ and $f(\alpha_j) = 0$, $j = 2, \ldots, n$, then the kernels are independent. A function φ is called a *multiplier of* $\mathcal{A}$ if $\varphi f \in \mathcal{A}$ whenever $f \in \mathcal{A}$ and we denote the set of multipliers by $\mathcal{M}_{\mathcal{A}}$. By the closed graph theorem, if $\varphi \in \mathcal{M}_{\mathcal{A}}$ then the multiplication operator M_φ given by $M_\varphi(f) = \varphi f$ for all $f \in \mathcal{A}$ is a bounded linear operator on $\mathcal{A}$. We define $||\varphi||_{\mathcal{M}_{\mathcal{A}}} = ||M_\varphi||$. By the same calculation as (6),

$$M_\varphi^* k_\alpha = \overline{\varphi(\alpha)} k_\alpha,$$

and thus $\varphi \in H^\infty(\Omega)$ and $||\varphi||_\infty \leq ||\varphi||_{\mathcal{M}_{\mathcal{A}}}$.

Definition: A sequence $\{z_n\} \subset \Omega$ is called an *interpolating sequence for* $\mathcal{M}_{\mathcal{A}}$ if for each bounded sequence of complex numbers $\{w_n\}$ there exists $\varphi \in \mathcal{M}_{\mathcal{A}}$ with $\varphi(z_n) = w_n$ for all n.

By the closed graph theorem, if $\{z_n\}$ is an interpolating sequence, there is a constant $C < \infty$ so that the interpolation can be done with a φ satisfying $||\varphi||_{\mathcal{M}_{\mathcal{A}}} \leq C||\{w_n\}||_{\ell^\infty}$.

In order to understand interpolating sequences, we ask the following question.
Given $z_1, \ldots, z_n \in \Omega$ and complex numbers $w_1, \ldots, w_n$, when does there exist $\varphi \in \mathcal{M}_{\mathcal{A}}$ such that $||\varphi||_{\mathcal{M}_{\mathcal{A}}} \leq 1$, and $\varphi(z_i) = w_i$, $i = 1, \ldots, n$?

It is easy to give a necessary condition. We say that a finite matrix $P = \{p_{i,j}\}_{i,j=1,\ldots,n}$ is *positive semi-definite* if

$$\sum_{i,j=1,\ldots,n} p_{i,j} a_i \overline{a_j} \geq 0, \qquad \text{for all } a_1, \ldots, a_n \in \mathbb{C}.$$

If P is positive semi-definite we write: $\{p_{i,j}\} \geq 0$. There is an equivalent formulation in terms of determinants. It is easy to see that a positive semi-definite matrix must be self-adjoint: $p_{i,j} = \overline{p_{j,i}}$. A self-adjoint matrix $P = \{p_{i,j}\}_{i,j=1,\ldots,n}$ is positive semi-definite if and only if for all sets $A \subset \{1, \ldots, n\}$ the matrix $P_A = (p_{i,j})_{i,j \in A}$ satisfies

$$\det P_A \geq 0.$$

Let $k_{i,j} =< k_{z_i}, k_{z_j} >$.

Proposition 2. *If $\varphi \in \mathcal{M}_{\mathcal{A}}$ with $||\varphi||_{\mathcal{M}_{\mathcal{A}}} \leq 1$ and $\varphi(z_i) = w_i$ for $i = 1, \ldots, n$ then*

$$\{(1 - \overline{w_i} w_j) k_{i,j}\} \geq 0.$$

Proof.

$$0 \leq || \sum_i a_i k_{z_i} ||^2 - || M_\varphi^* (\sum_i a_i k_{z_i}) ||^2 = \sum_{i,j} (1 - \overline{w_i} w_j) k_{i,j} a_i \overline{a_j}. \tag{7}$$

□

The determinant conditions give concrete and checkable conditions in terms of the data. The proposition can be reformulated to give a condition with fewer determinants to check. A matrix $P = (p_{i,j})_{i,j=1,\ldots,n}$ is called *positive definite* if

$$\sum_{i,j=1,\ldots,n} p_{i,j} a_i \overline{a_j} > 0, \qquad \text{for all } a_1, \ldots, a_n \in \mathbb{C},$$

except when $a_1 = a_2 = \ldots = a_n = 0$. If P is positive definite we write: $\{p_{i,j}\} > 0$. Let P_m be the m-by-m principal submatrix $P_m = (p_{i,j})_{i,j=1,\ldots,m}$. Then an n-by-n self-adjoint matrix P is positive definite if and only if $\det P_m > 0$ for $m = 1, \ldots, n$. If P is positive semi-definite then $\det P_m \geq 0$ for $m = 1, \ldots, n$. The converse, however, is false. See e.g. [BCR, Chapter 3] for these and other elementary facts about positive semi-definite matrices. Proposition 2 can be reworded to the equivalent statement:

If there exists a $\varphi \in \mathcal{M}_{\mathcal{A}}$ with $||\varphi||_{\mathcal{M}_{\mathcal{A}}} < 1$ then $\det \{(1 - \overline{w_i} w_j) k_{i,j}\}_m > 0$, for $m = 1, \ldots, n$.

As a simple application, we give the following Corollary, which is exactly the invariant form of Schwarz's lemma when $\Omega = \mathbb{D}$ and $\mathcal{A} = H^2$.

Corollary 3. *If $\varphi \in \mathcal{M}_{\mathcal{A}}$ with $||\varphi||_{\mathcal{M}_{\mathcal{A}}} \leq 1$ then*

$$\rho^2(\varphi(z_1), \varphi(z_2)) \leq 1 - \frac{|k_{1,2}|^2}{k_{1,1}k_{2,2}},$$

where $\rho(z, w)$ is the pseudo-hyperbolic metric defined in section 1.

Proof. Take the determinant of the 2-by-2 matrix P_2 and use the identity

$$1 - \rho(z, w)^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \overline{z}w|^2},$$

for all $z,\ w \in \mathbb{D}$. □

As a consequence of Proposition 2, if $\{z_n\}$ is an interpolating sequence for $\mathcal{M}_{\mathcal{A}}$ then there is a constant $C < \infty$ so that for all sequences $\{w_n\}$ with $|w_n| \leq 1/C$

$$\{(1 - \overline{w_i}w_j)k_{i,j}\} \geq 0 \tag{8}$$

for all principal m-by-m submatrices. In other words, by the computation (7), if $|b_i| \leq |a_i|$ for all i then

$$||\sum b_i k_{z_i}|| \leq C||\sum a_i k_{z_i}||. \tag{9}$$

In order to understand the inequality (9) we will make a short detour through arbitrary Hilbert spaces. For positive quantities A and B, the notation $A \sim B$ will mean there is a constant $C < \infty$ so that

$$\frac{1}{C} \leq \frac{A}{B} \leq C.$$

If $\{x_n\}$ is a collection of vectors in a Hilbert space H, then $\mathrm{Span}\{x_n\}$ will denote the smallest closed subspace of H containing the collection $\{x_n\}$.

Definition. A sequence of unit vectors $\{u_n\}$ in a Hilbert space H is an *interpolating sequence (IS) for* H if the map

$$x \rightarrow \{< x, u_n >\} \tag{10}$$

maps H onto ℓ^2. In other words, $\mathrm{Span}\{u_n\}$ is mapped one-to-one and onto ℓ^2.

Definition. A sequence of vectors $\{x_n\}$ in a Hilbert space H is called *independent* if for all n, $x_n \notin \mathrm{Span\{x_k : k \neq n\}}$.

Theorem [Köthe-Toeplitz]. *Let $\{u_n\}$ be a sequence of unit vectors contained in a Hilbert space H. Let K be the smallest closed subspace of H containing $\{u_n\}$. Then the following are equivalent.*

(IS) $\{u_n\}$ *is an interpolating sequence for* H

(SS) *For all* $x \in K$, $||x||^2 \sim \sum | < x, u_n > |^2$, *and* $\{u_n\}$ *is independent.*

(RS) $||\sum a_n u_n||^2 \sim \sum |a_n|^2$ *for all* $\{a_n\}$.

(UBS) *There is a* $C < \infty$ *such that* $||\sum b_n u_n|| \leq C||\sum a_n u_n||$ *whenever* $|b_n| \leq |a_n|$ *for all* n.

A sequence satisfying (RS) is called a *Riesz sequence* and a sequence satisfying (UBS) is called an *unconditional basic sequence*. The inequality (9) says that $\{k_{z_n}/||k_{z_n}||\}$ is an unconditional basic sequence. This theorem can be found in [Nk]. We are not sure of the history. The equivalence of (RS) and (UBS) is called the Köthe-Toeplitz theorem in [Nk]. Because it is central to the proof of our main result, we include a self-contained proof.

Proof. First note that (IS) is equivalent to (SS). For if (IS) holds, then $\{u_n\}$ is independent, and by the closed graph and open mapping theorems, (SS) holds. Conversely if (SS) holds, then the map (10) has image I which is a closed subspace of ℓ^2. If $I \neq \ell^2$ then there is a sequence $a = \{a_j\} \in \ell^2$, $a \neq 0$, so that

$$\sum \overline{a_n} < x, u_n >= 0,$$

for all $x \in I$. Fix N and M and set $x = \sum_N^M a_n u_n$. Then

$$||\sum_N^M a_n u_n||^2 = \left|\sum_N^M \overline{a_n} < x, u_n >\right|$$

$$\leq \left(\sum_N^M |a_n|^2\right)^{\frac{1}{2}} \left(\sum_N^M | < x, u_n > |^2\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_N^M |a_n|^2\right)^{\frac{1}{2}} C||x||,$$

since (SS) holds. Thus

$$||\sum_N^M a_n u_n|| \leq C \left(\sum_N^M |a_n|^2\right)^{\frac{1}{2}}$$

and hence $\sum_1^N a_n u_n$ is a Cauchy sequence. Set $x0 = \sum_1^\infty a_n u_n$. Then $||x||^2 = \sum_1^\infty a_n < x, u_n >= 0$. Since $\{u_n\}$ ARe independent, this implies $a_n = 0$ for all n, contradicting $a \neq 0$. Thus (IS) holds.

Next note that (RS) is equivalent to (SS). For if (RS) holds, then clearly $\{u_n\}$ is independent and

$$\begin{aligned} ||x|| &= \sup\{| < x, \sum a_n u_n > | : || \sum a_n u_n|| \le 1\} \\ &\sim \sup\{| \sum \overline{a_n} < x, u_n > | : \sum |a_n|^2 \le 1\} \\ &= (\sum | < x, u_n > |^2)^{\frac{1}{2}}, \end{aligned}$$

proving (SS). If (SS) holds, then for $x \in K$

$$\begin{aligned} || \sum a_n u_n|| &= \sup\{| < x, \sum a_n u_n > | : x \in K,\ ||x|| \le 1\} \\ &\sim \sup\{| \sum \overline{a_n} < x, u_n > | : \sum | < x, u_n > |^2 \le 1\} \\ &= (\sum |a_n|^2)^{\frac{1}{2}}, \end{aligned}$$

and hence (RS) holds. Here we used that $x \to \{< x, u_n >\}$ maps K onto ℓ^2.

Now suppose that (RS) holds. If $|b_n| \le |a_n|$ for all n,

$$|| \sum b_n u_n||^2 \le C_1 \sum |b_n|^2 \le C_1 \sum |a_n|^2 \le C_2 || \sum a_n u_n||^2,$$

proving (UBS).

Finally suppose (UBS) holds. To show (RS), it suffices to consider finite sequences $\{u_n\}$ and prove (RS) with constants which are independent of the number of elements in $\{u_n\}$.

Lemma [Orlicz]. *If $x_n \in H$, then there exists complex numbers ε_n with $|\varepsilon_n| = 1$ and*

$$\sum ||x_n||^2 \le || \sum \varepsilon_n x_n||^2.$$

Proof. Use induction and

$$\Big\|x + \frac{< x, y >}{| < x, y > |} y\Big\|^2 \ge ||x||^2 + ||y||^2.$$

□

By Orlicz's lemma and (UBS)

$$\sum |a_n|^2 = \sum ||a_n u_n||^2 \le || \sum \varepsilon_n a_n u_n||^2 \le C|| \sum a_n u_n||^2,$$

which is the lower estimate of the norm in (RS). Note that by (UBS)

$$|a_n| \le C|| \sum a_n u_n||,$$

so there exists $v_n \in K$ with $||v_n|| \leq C$ and $< \sum a_m u_m, v_n >= a_n$ for all $x = \sum a_m u_m \in K$. To prove the upper estimate of the norm in (RS) we will first show that $\{v_n\}$ satisfies (UBS). Suppose $b_n = \lambda_n a_n$ with $|\lambda_n| \leq 1$ for all n. Then

$$\begin{aligned} |< \sum b_n v_n, \sum c_n u_n >| &= \sum b_n \overline{c_n} = \sum a_n \overline{\overline{\lambda_n} c_n} \\ &= |< \sum a_n v_n, \sum \overline{\lambda_n} c_n u_n >| \\ &\leq || \sum a_n v_n|| \, || \sum \overline{\lambda_n} c_n u_n|| \\ &\leq C|| \sum a_n v_n|| \, || \sum c_n u_n||. \end{aligned}$$

By duality, $|| \sum b_n v_n|| \leq C|| \sum a_n v_n||$. Again by Orlicz's lemma and (UBS) for $\{v_n\}$

$$\sum |b_n|^2 \leq \sum |b_n|^2 ||v_n||^2 \leq || \sum \varepsilon_n b_n v_n||^2 \leq C^2 || \sum b_n v_n||^2$$

and hence

$$(11) \quad |< \sum a_n u_n, \sum b_m v_m >|^2 = | \sum a_n \overline{b_n}|^2 \leq \sum |a_n|^2 \sum |b_n|^2 \leq \sum |a_n|^2 C^2 || \sum b_n v_n||^2.$$

Each $x \in K$ is of the form $\sum b_m v_m$, since 0 is the only $x \in K$ orthogonal to all such sums. Thus by duality and (11)

$$|| \sum a_n u_n||^2 \leq C^2 \sum |a_n|^2.$$

proving (RS) holds and completing the proof of the theorem. □

Note that $(5')$ is equivalent to the lower estimate of the norm in (SS):

$$\sum |< f, \frac{k_n}{||k_n||} >|^2 = \sum \frac{|f(z_n)|^2}{||k_n||^2} \leq C||f||^2_{\mathcal{D}},$$

for all $f \in \mathcal{D}$.

§3 Pick's Theorem and H^∞ Interpolation

It is a fundamental result of Pick [P] that the converse to Proposition 2 holds for H^∞. As mentioned above, when $\mathcal{A} = H^2$, the space of multipliers is H^∞ and $k_{i,j} =< k_{z_i}, k_{z_j} >= (1 - \overline{z_i} z_j)^{-1}$. In this case, it is easy to show

$$||\varphi||_{\mathcal{M}_{H^2}} = ||\varphi||_\infty.$$

Theorem [Pick]. *If $z_1, \ldots, z_n \in \mathbb{D}$ and $w_1, \ldots, w_n \in \mathbb{D}$ then there is a $\varphi \in H^\infty$ with $||\varphi||_\infty \le 1$ and $\varphi(z_i) = w_i$, $i = 1, \ldots, n$ if and only if the n-by-n matrix*

$$\left\{ \frac{1 - \overline{w_i} w_j}{1 - \overline{z_i} z_j} \right\} \ge 0.$$

A classical technique in harmonic analysis, in vague terms, is to convert an "L^∞" problem to an "L^1" problem via duality, and then convert to an "L^2" problem, which can then be solved. In many situations, though, duality arguments do not work. The philosophy of this paper is that the Pick theorem plays the role of converting to an "L^2" problem, namely proving a collection of reproducing kernels forms a Riesz sequence. To further motivate the method, in this section we give a self-contained proof of Carleson's interpolation theorem along these lines. Amar[Am] proved Carleson's theorem from Pick's theorem, using Carleson's geometric characterization of Carleson measures. The proof below does not depend on this latter result of Carleson.

Let $z_1, \ldots, z_n$ be distinct points in $\mathbb{D}$, and let

$$B(z) = \prod_{i=1}^{n} \frac{z - z_i}{1 - \overline{z_i} z} \qquad B_j(z) = \prod_{i \ne j} \frac{z - z_i}{1 - \overline{z_i} z},$$

$$k_j(z) = \frac{1}{1 - \overline{z_j} z}$$

$$C_1 = \sup_j \sum_i \frac{(1 - |z_i|^2)(1 - |z_j|^2)}{|1 - \overline{z_j} z_i|^2} \quad \text{and} \quad \delta = \inf_i |B_i(z_i)|$$

The next Lemma is due to Shapiro and Shields [SS1].

Lemma 4. *Let $K = \sum c_j B_j^2 k_j^2 / ||k_j||^3$. Then*

$$< K, \frac{k_i}{||k_i||} >= c_i B_i(z_i)^2 \quad \textit{and } ||K||^2 \le 2C_1 \sum |c_j|^2.$$

Proof.

$$\begin{aligned} < B_j^2 k_j^2, B_i^2, k_i^2 > &=< \frac{B^2}{(z - z_j)^2}, \frac{B^2}{(z - z_i)^2} > \\ &= \int_0^{2\pi} \frac{1}{(e^{i\theta} - z_j)^2} \frac{e^{i\theta}}{(1 - \overline{z_i} e^{i\theta})^2} \frac{i e^{i\theta} d\theta}{2\pi i} \\ &= \frac{d}{dz} \frac{z}{(1 - \overline{z_i} z)^2} \bigg|_{z = z_j} \\ &= \frac{1 + \overline{z_i} z_j}{(1 - \overline{z_i} z_j)^3}. \end{aligned}$$

Since

$$(12) \qquad \frac{(1-|z_i|^2)(1-|z_j|^2)}{|1-\overline{z_i}z_j|^2} = 1 - \left|\frac{z_i - z_j}{1-\overline{z_i}z_j}\right|^2 \le 1,$$

we have

$$|< B_j^2 k_j, B_i^2 k_i >| \le \frac{2(1-|z_j|^2)^{-\frac{1}{2}}(1-|z_i|^2)^{-\frac{1}{2}}}{|1-\overline{z_i}z_j|^2}.$$

By Schur's lemma

$$\begin{aligned} ||K||^2 &= \sum_{i,j} \frac{c_j}{||k_j||^3}\frac{\overline{c_i}}{||k_i||^3} < B_j^2 k_j^2, B_i^2, k_i^2 > \\ &\le 2\sum_{i,j} |c_j|\ |c_i| \frac{(1-|z_j|^2)(1-|z_i|^2)}{|1-\overline{z_i}z_j|^2} \le 2C_1 \sum_j |c_j|^2. \end{aligned}$$

(Schur's lemma in this context is just two applications of Cauchy-Schwarz, followed by an interchange in the order of a double sum.) Clearly $< K, \frac{k_i}{||k_i||} >= K(z_i)/||k_i|| = c_i B_i^2(z_i)$. □

The next Lemma says that $\{k_i/||k_i||\}$ is a (RS).

Lemma 5.

$$\frac{\delta^4}{2C_1}\sum |a_i|^2 \le \left\|\sum a_i \frac{k_i}{||k_i||}\right\|^2 \le \frac{2C_1}{\delta^2}\sum |a_i|^2.$$

Proof.

$$\left|\sum \overline{a_i} c_i B_i^2(z_i)\right|^2 = |< K, \sum a_i \frac{k_i}{||k_i||} >|^2 \le 2C_1\left(\sum |c_i|^2\right)\left\|\sum a_i \frac{k_i}{||k_i||}\right\|^2.$$

If $c_i = a_i|B_i^2(z_i)|/B_i^2(z_i)$ then

$$\delta^4\left(\sum |a_i|^2\right)^2 \le 2C_1\left(\sum |a_i|^2\right)\left\|\sum a_i \frac{k_i}{||k_i||}\right\|^2,$$

which proves the left inequality. To prove the right inequality, note that

$$< B_i k_i, B_j k_j >=< \frac{B}{z-z_i}, \frac{B}{z-z_j} >=< \frac{1}{1-z_i\overline{z}}, \frac{1}{1-z_j\overline{z}} >= \overline{< k_i, k_j >}.$$

Thus

$$\left\|\sum a_j \frac{k_j}{||k_i||}\right\|^2 = \left\|\sum \overline{a_j} B_j \frac{k_j}{||k_j||}\right\|^2.$$

Note that $h = \sum \overline{a_j} B_j k_j/||k_j|| \in \mathcal{M} \equiv$ Span $\{k_j : j = 1, \ldots, n\}$ by partial fractions, and

$$< h, \frac{k_i}{||k_i||} >= \overline{a_i} B_i(z_i).$$

Let $c_i = \overline{a_i}/B_i(z_i)$ and form K as above. Then $h - K$ is orthogonal to $\mathcal{M}$ and hence

$$\left\|\sum a_j \frac{k_j}{||k_j||}\right\|^2 = ||h||^2 \leq ||K||^2 \leq \frac{2C_1}{\delta^2}\sum |a_j|^2.$$

$\square$

We can now easily prove Carleson's theorem. If $|w_i| \leq \delta^3/(2C_1)$ then using both inequalities in Lemma 5

$$0 \leq ||\sum a_i k_i||^2 - ||\sum a_i \overline{w_i} k_i||^2 = \sum_{i,j}(1 - \overline{w_i} w_j) < k_i, k_j > a_i \overline{a_j}.$$

By Pick's theorem there is a function $\varphi \in H^\infty$ with $||\varphi||_\infty \leq 1$ and $\varphi(z_i) = w_i$, for $i = 1, \dots, n$.

Suppose now that $\{z_i\}_{i=1}^\infty \subset \mathbb{D}$ and

$$\inf_{i \neq j}\left|\frac{z_i - z_j}{1 - \overline{z_i} z_j}\right| = \varepsilon > 0 \quad \text{and} \tag{13}$$

$$C_1 = \sup_j \sum_i \frac{(1 - |z_i|^2)(1 - |z_j|^2)}{|1 - \overline{z_j} z_i|^2} < \infty. \tag{14}$$

It is a standard elementary calculus argument using logarithms (see [G]) that

$$\inf_i |B_i(z_i)| \equiv \delta > 0.$$

Thus there is a constant $K(\varepsilon, C_1)$ depending only on ε and C_1 so that if

$$|w_i| \leq K(\varepsilon, C_1)$$

then there exist a $\varphi \in H^\infty$ with $||\varphi||_\infty \leq 1$ and $\varphi(z_i) = w_i$, for $i = 1, 2, \dots$. The passage from finite sequences to a countable sequence is just a normal families argument. It is elementary to prove that (14) and (2) are equivalent (see [G]), thereby proving Carleson's theorem. $\square$

The key idea of the above proof was to show (Lemma 5) that $k_i/||k_i||$ forms a Riesz sequence and then use Pick's theorem.

§4 Pick Interpolation in $\mathcal{M}_\mathcal{A}$

We now return to our assumptions that $\mathcal{A}$ is a Hilbert space on a domain $\Omega \subset \mathbb{C}^n$ and that $\{k_\alpha : \alpha \in \Omega\}$ is an independent collection of reproducing kernels for $\mathcal{A}$.

Definition. We say that $\mathcal{A}$ has the *Pick property* if whenever $\{z_i\}_1^n \subset \Omega$ and $\{w_i\}_1^n \in \mathbb{C}$ satisfy

$$\{(1-\overline{w_i}w_j)k_{i,j}\} \geq 0,$$

then there exists $\varphi \in \mathcal{M}_{\mathcal{A}}$ with $||\varphi||_{\mathcal{M}_{\mathcal{A}}} \leq 1$ and $\varphi(z_i) = w_i$, for $i = 1, \ldots, n$.

We remark that if the Pick property holds, then it also holds if we replace finite sequences by countable sequences. That is, if the Pick property holds, and if $\{z_i\}_{i=1}^{\infty} \subset \Omega$ and if $\{w_i\}_{i=1}^{\infty} \subset \mathbb{C}$ such that for each n, the n-by-n matrix

$$\{(1-\overline{w_i}w_j)k_{i,j}\} \geq 0,$$

then there exists $\varphi \in \mathcal{M}_{\mathcal{A}}$ with $||\varphi||_{\mathcal{M}_{\mathcal{A}}} \leq 1$ and $\varphi(z_i) = w_i$ for all i. Indeed, for each n there is a $\varphi_n \in \mathcal{M}_{\mathcal{A}}$ with $||\varphi_n||_{\mathcal{M}_{\mathcal{A}}} \leq 1$ and $\varphi(z_i) = w_i$ for $i = 1, \ldots, n$. Since the unit ball of the bounded operators on $\mathcal{A}$ is compact in the weak-operator topology and since $\mathcal{A}$ is separable, we can select a subseqence φ_{n_j} and an operator T such that

$$\lim_j < M_{\varphi_{n_j}} f, g > = < Tf, g >$$

for all $f, g \in \mathcal{A}$. Applying the above with $g = k_\alpha$ for $\alpha \in \Omega$ shows that $T = M_\varphi$ for some $\varphi \in \mathcal{M}_{\mathcal{A}}$ with $||\varphi||_{\mathcal{M}_{\mathcal{A}}} \leq 1$ and $\varphi(z_i) = w_i$ for all i. Of course we can extend this to uncountable sequences, by selecting a countable subsequence with a cluster point in Ω and using Proposition 2.

The Pick property can be used to describe zero sets. Suppose $\{z_j\}_{j=1}^{\infty} \subset \Omega$. Let $z_0 \in \Omega$, with $z_0 \neq z_j$ for $j = 1, \ldots$, and let

$$c_{\mathcal{A}} = \inf\{||f||_{\mathcal{A}} : f(z_0) = 1 \text{ and } f(z_j) = 0,\ j = 1, \ldots\}, \tag{15}$$

and let

$$c_{\mathcal{M}_{\mathcal{A}}} = \inf\{||\varphi||_{\mathcal{M}_{\mathcal{A}}} : \varphi(z_0) = 1 \text{ and } \varphi(z_j) = 0,\ j = 1, \ldots\}. \tag{16}$$

Proposition 6. *Suppose $\mathcal{A}$ has the Pick property and $\{z_j\} \subset \Omega$. Then there is a non-zero $f \in \mathcal{A}$ with $f(z_j) = 0$ for all j if and only if there is a non-zero $\varphi \in \mathcal{M}_{\mathcal{A}}$ with $\varphi(z_j) = 0$ for all j. Moreover,*

$$c_{\mathcal{M}_{\mathcal{A}}} = c_{\mathcal{A}} ||k_{z_0}||_{\mathcal{A}}.$$

There is a unique f_0 and a unique φ_0 which are extremal for (15) and (16) respectively and they satisfy

$$f_0 = \varphi_0 \frac{k_{z_0}}{||k_{z_0}||^2}.$$

Proof. Let K be the smallest closed subspace of $\mathcal{A}$ containing $\{z_j\}_{j=0}^{\infty}$. Define $\Lambda \in K^*$ by

$$\Lambda(\sum a_j k_{z_j}) = a_0.$$

Then there exists a $f \in \mathcal{A}$ with $||f||_{\mathcal{A}} = ||\Lambda||$ and $\Lambda(g) =< g, f >$ for all $g \in K$. Clearly $f(z_0) = 1$ and $f(z_j) = 0$ for $j \geq 1$, and it is the unique element of K with these values. Since projection into K decreases norm, $||\Lambda|| = c_{\mathcal{A}}$.

Likewise, define an operator S on K by

$$S(\sum a_j k_{z_j}) = \frac{a_0}{||\Lambda||} \frac{k_{z_0}}{||k_{z_0}||}.$$

Then $||S|| = 1$. By the Pick property, this occurs if and only if there is $\varphi \in \mathcal{M}_{\mathcal{A}}$ with $||\varphi||_{\mathcal{M}_{\mathcal{A}}} \leq 1$ and $\varphi(z_j) = 0$ for $j \geq 1$ and $\varphi(z_0) = 1/(||\Lambda||||k_{z_0}||)$. Thus $c_{\mathcal{M}_{\mathcal{A}}} = ||\Lambda||||k_{z_0}|| = c_{\mathcal{A}}||k_{z_0}||_{\mathcal{A}}$. If φ is an extremal function for (16) then

$$f = \varphi \frac{k_{z_0}}{||k_{z_0}||^2}$$

satisfies

$$c_{\mathcal{A}} \leq ||f||_{\mathcal{A}} \leq ||\varphi||_{\mathcal{M}_{\mathcal{A}}}/||k_{z_0}|| = c_{\mathcal{A}}$$

and $f(z_0) = 1$ and $f(z_j) = 0$ for $j \geq 1$. By the uniqueness of the extremal function for (15), φ is also unique. $\square$

The zero sets of $\mathcal{A}$ and $\mathcal{M}_{\mathcal{A}}$ are not always the same. For example, Horowitz [Ho] proved that the zero sets of the Bergman space A^2 are different from the zero sets of $\mathcal{M}_{A^2} = H^{\infty}$. Note that Proposition 6 does not guarantee that the extremal functions vanish exactly on the given collection $\{z_j\}$, but in some cases we can prove this is the case (see Corollary 13). We also note that Proposition 23 contains an estimate of $c_{\mathcal{M}_{\mathcal{A}}}$ in some important special cases.

We now apply the results of section 2.

Corollary 7. *Let $\{z_n\} \subset \Omega$ and let $u_n = k_{z_n}/||k_{z_n}||$. If $\mathcal{A}$ has the Pick property, then the following statements are equivalent:*

(a) *$\{z_n\}$ is an interpolating sequence for $\mathcal{M}_{\mathcal{A}}$.*

(b) *$\{u_n\}$ is an interpolating sequence for $\mathcal{A}$.*

(c) *$\{u_n\}$ is an unconditional basic sequence in $\mathcal{A}$.*

(d) *$\{u_n\}$ is a Riesz sequence in $\mathcal{A}$.*

In many texts, $\{z_n\}$ is called an interpolating sequence for $\mathcal{A}$ when (b) holds.

Proof. By the Köthe-Toeplitz theorem, (b), (c), and (d) are equivalent. Proposition 2 and the inequality (9) show that if z_n is an interpolating sequence for $\mathcal{M}_{\mathcal{A}}$ then (c) holds. Finally suppose that (c) holds. To prove (a) it suffices to prove each finite subsequence of $\{z_n\}$ is interpolating with uniformly bounded norms, by weak convergence of operators of the form M_φ. Property (a) now follows from the Pick property and the equivalence of (8) and (9).

We are interested in conditions on $\{k_\alpha\}$ under which the converse to Proposition 2 is valid.

Theorem 8. *Suppose that for every $m \geq 1$ and every $\alpha_1, \ldots, \alpha_{m+1} \in \Omega$, the m-by-m matrix*

$$\left\{1 - \frac{< k_{\alpha_i}, k_{\alpha_{m+1}} > \overline{< k_{\alpha_j}, k_{\alpha_{m+1}} >}}{< k_{\alpha_i}, k_{\alpha_j} >< k_{\alpha_{m+1}}, k_{\alpha_{m+1}} >}\right\} \geq 0. \tag{17}$$

Then given $z_1, \ldots, z_n \in \Omega$ and $w_1, \ldots, w_n \in \mathbb{C}$ there is a $\varphi \in \mathcal{M}_{\mathcal{A}}$ with $||\varphi||_{\mathcal{M}_{\mathcal{A}}} \leq 1$ and $\varphi(z_i) = w_i$ for $i = 1, \ldots, n$ if and only if the n-by-n matrix

$$\left\{(1 - \overline{w_i} w_j) < k_{z_i}, k_{z_j} >\right\} \geq 0. \tag{18}$$

Theorem 8 says that if (17) holds then $\mathcal{A}$ has the Pick property.

Nevanlinna's approach to the finite interpolation problem in H^∞ was to reduce the problem of interpolation at n points to an equivalent problem at $n-1$ points. See [M] for a proof of Pick's theorem along those lines. In the present case, we do not have a good description of the set of functions in $\mathcal{M}_{\mathcal{A}}$ vanishing at a point of Ω; thus this approach won't work. Instead, notice that the computation (7) says that the operator T defined only on the linear span of $\{k_{z_i}, i = 1, \ldots, n\}$ by $T(k_{z_i}) = \overline{w_i} k_{z_i}$ has norm 1 if and only if (18) holds. Akin to the usual proof of the Hahn-Banach theorem, using condition (17) we will show that if z_{n+1} is any other point in Ω then we can extend condition (18) to an (n+1)-by-(n+1) matrix of the same form, by judiciously choosing w_{n+1}. By induction we can extend T to any subspace spanned by a finite collection $\{k_{z_1}, \ldots, k_{z_m}\}$, with $m \geq n$. Passing to a limit, this will give the operator M_φ^* and hence φ.

Choose $z_{n+1} \in \Omega$ and write $k_i = k_{z_i}$ for $i = 1, \ldots, n+1$. Let M_p be the smallest closed subspace containing $k_1, \ldots, k_p$, and let $\widetilde{M}$ be the subspace of M_{n+1} orthogonal to k_{n+1}. Then the projection of k_j onto $\widetilde{M}$ is

$$\widetilde{k}_j = P_{\widetilde{M}} k_j = k_j - \frac{< k_j, k_{n+1} >}{< k_{n+1}, k_{n+1} >} k_{n+1}.$$

Define T on M_n and $\widetilde{T}$ on $\widetilde{M}$ by

$$T k_j = \overline{w_j} k_j \quad \text{and} \quad \widetilde{T}\widetilde{k}_j = \overline{w_j}\widetilde{k}_j,$$

for $j = 1, \ldots, n$. For $w \in \mathbb{C}$, define an extension of T to M_{n+1} by

$$T_w k_j = \begin{cases} \overline{w_j} k_j & \text{for } j = 1, \ldots, n \\ \overline{w} k_{n+1} & \text{for } j = n+1. \end{cases}$$

Thus $\widetilde{T} = P_{\widetilde{M}} T_w|_{\widetilde{M}}$.

Lemma 9. *The following are equivalent:*

(i) $||T|| \leq 1$ *and* $||\widetilde{T}|| \leq 1$

(ii) There exists $w \in \mathbb{C}$ *such that* $||T_w|| \leq 1$.

Proof. The ideas in this proof came from the preprint of Agler [Ag]. If (ii) holds, clearly $||T|| \leq ||T_w|| \leq 1$. Since $\widetilde{T} = P_{\widetilde{M}} T_w|_{\widetilde{M}}$

$$||\widetilde{T}|| = ||P_{\widetilde{M}} T_w|_{\widetilde{M}}|| \leq ||T_w|_{\widetilde{M}}|| \leq ||T_w|| \leq 1,$$

proving (i). Now suppose (i) holds. Choose w_0 so that

$$||T_{w_0}|| = \inf_w ||T_w||$$

and suppose $||T_{w_0}|| > 1$. Choose $\Lambda \in M_{n+1}^*$ so that $\Lambda(k_{n+1}) = 1$ and $\Lambda(k_j) = 0$ for $j = 1, \ldots, n$. Then

$$T_w f = T_{w_0} f + (w - w_0)\Lambda(f) k_{n+1}. \tag{19}$$

Since $T_w^* T_w$ is self-adjoint, it has largest eigenvalue $||T_w^* T_w|| = ||T_w||^2$. Choose an eigenvector f_w of $T_w^* T_w$ so that $||f_w|| = 1$ and $< f_w, f_{w_0} > \geq 0$. Note that if $T_{w_0}^* T_{w_0} f_i = ||T_{w_0}||^2 f_i$ for $i = 1, 2$, with f_1, f_2 linearly independent, then we can choose $c_1, c_2 \in \mathbb{C}$ so that $f = c_1 f_1 + c_2 f_2 \in M_n$ and $||f|| = 1$. Then

$$||T||^2 = ||T^* T|| \geq ||T^* T f|| = ||T_{w_0}||^2 > 1,$$

contradicting (i). Thus the eigenspace for $T_{w_0}^* T_{w_0}$ is one dimensional. If $w_n \to w_0$ and $f_{w_n} \to f_0$ then by (19), f_0 must be an eigenvector for $T_{w_0}^* T_{w_0}$ with eigenvalue $||T_{w_0}||^2$, and hence $f_0 = c f_{w_0}$ for some $c \in \mathbb{C}$ with $|c| = 1$. Since $< f_w, f_{w_0} > \geq 0$, we have c=1. This shows that $\{f_w\}$ has only one cluster point in the finite dimensional space M_{n+1} as $w \to w_0$, and hence

$$\lim_{w \to w_0} f_w = f_{w_0}.$$

Since $||T_{w_0}||$ is minimal,

$$
\begin{aligned}
&0 \leq ||T_w||^2 - ||T_{w_0}||^2 = ||T_w f_w||^2 - ||T_{w_0}||^2 \\
&\quad = ||T_{w_0} f_w + (w-w_0)\Lambda(f_w) k_{n+1}||^2 - ||T_{w_0}||^2 \\
&\quad \leq ||T_{w_0}||^2 + 2\mathrm{Re}(w-w_0)\Lambda(f_w) < k_{n+1}, T_{w_0} f_w > + |w-w_0|^2 |\Lambda(f_w)|^2 ||k_{n+1}||^2 - ||T_{w_0}||^2 \\
&\quad = 2\mathrm{Re}(w-w_0)\Lambda(f_w) < k_{n+1}, T_{w_0} f_w > + O(|w-w_0|^2).
\end{aligned} \tag{20}
$$

Write $w - w_0 = re^{i\theta}$, with $r > 0$. Fix θ, divide the last line of (20) by $r = |w - w_0|$ and let $r \to 0$. We obtain

$$0 \leq \mathrm{Re}\ e^{i\theta}\Lambda(f_{w_0}) < k_{n+1}, T_{w_0} f_{w_0} >$$

for all θ and hence

$$\Lambda(f_{w_0}) < k_{n+1}, T_{w_0} f_{w_0} >= 0.$$

Now if $\Lambda(f_{w_0}) = 0$, then $f_{w_0} \in M_n$ and hence

$$1 < ||T_{w_0}||^2 = ||T_{w_0}^* T_{w_0} f_{w_0}|| = ||T^* T f_{w_0}|| \leq ||T||^2,$$

contradicting (i). Thus $< k_{n+1}, T_{w_0} f_{w_0} >= 0$, and

$$
\begin{aligned}
||T_{w_0}||^2 < k_{n+1}, f_{w_0} > =< k_{n+1}, T_{w_0}^* T_{w_0} f_{w_0} > =< T_{w_0} k_{n+1}, T_{w_0} f_{w_0} > \\
= \overline{w_0} < k_{n+1}, T_{w_0} f_{w_0} >= 0.
\end{aligned}
$$

In other words, $f_{w_0} \in \widetilde{M}$ and $T_{w_0} f_{w_0} \in \widetilde{M}$, and so $T_{w_0} f_{w_0} = \widetilde{T} f_{w_0}$. We conclude that

$$||\widetilde{T}||^2 \geq ||\widetilde{T} f_{w_0}||^2 = ||T_{w_0} f_{w_0}||^2 = ||T_{w_0}||^2 > 1,$$

contradicting the second assumption in (i), and completing the proof of the lemma. □

Proof of Theorem 8. The necessity of (18) is given by Proposition 2. Suppose now that (17) and (18) hold. Choose $z_{n+1} \in \Omega$. Note that by the computation (7), $||T|| \leq 1$ is equivalent to (18). Also

$$< \widetilde{k}_i, \widetilde{k}_j >= k_{i,j} - \frac{k_{i,n+1}\overline{k_{j,n+1}}}{k_{n+1,n+1}},$$

so by the same computation (7), $||\widetilde{T}|| \leq 1$ is equivalent to

$$\left\{ (1 - \overline{w_i} w_j) \left(k_{i,j} - \frac{k_{i,n+1}\overline{k_{j,n+1}}}{k_{n+1,n+1}} \right) \right\} \geq 0. \tag{21}$$

Let $\alpha_j = z_j,\ \ j = 1, \ldots, n+1$ in (17). Then the matrix in (21) is the entry-by-entry product of the positive semi-definite matrices (17) and (18) and hence must be positive semi-definite by Schur's Theorem [BCR, p.69]. By Lemma 9, there is a w_{n+1} so that the (n+1)-by-(n+1) matrix

$$\left\{(1 - \overline{w_i} w_j) < k_{z_i}, k_{z_j} >\right\} \geq 0.$$

By induction, given any $z_{n+1}, \ldots, z_m \in \Omega$ we can find $w_{n+1}, \ldots, w_m$ so that the m-by-m matrix

$$\left\{(1 - \overline{w_i} w_j) < k_{z_i}, k_{z_j} >\right\} \geq 0.$$

By choosing a countable dense subset $\{z_m\}$ of Ω, this allows us to find an operator T on the dense subset of $\mathcal{A}$, consisting of finite linear combinations of $\{k_{z_n}\}$, with $||T|| \leq 1$ and $T(k_{z_m}) = \overline{w_m} k_{z_m}$ for $m = 1, 2, \ldots$. By continuity, T extends to all of $\mathcal{A}$ and $T(k_z) = \overline{w_z} k_z$ for each $z \in \Omega$. If we define $\varphi(z) = w_z$, then

$$T^* f(z) = \varphi(z) f(z)$$

for each $f \in \mathcal{A}$ and hence $\varphi \in \mathcal{M}_{\mathcal{A}}$, with $||\varphi||_{\mathcal{M}_{\mathcal{A}}} \leq 1$. $\square$

We do not know if (17) is necessary for this process to work. However it is possible to show that the one-step extension from two points to three points is possible for all 2-by-2 matrices satisfying (18) if and only if (17) holds with $m = 2$. We leave this latter fact as an exercise for the reader.

The results above hold in greater generality than stated. Suppose that $\{k_\alpha\}_{\alpha \in A}$ is a collection of vectors in a separable Hilbert space H such that any finite subcollection is an independent set. Let $\mathcal{M}$ be the space of bounded operators S on H with the property that each k_α is an eigenvector for S:

$$S k_\alpha = \lambda_\alpha k_\alpha$$

for some $\lambda_\alpha \in \mathbb{C}$. We can regard each k_α as a reproducing kernel by defining

$$\hat{x}(\alpha) = < x, k_\alpha >$$

for all $\alpha \in A$. Then the adjoint of such an operator S corresponds to a multiplication operator:

$$M_\varphi \hat{x} = \varphi \hat{x} = \widehat{S^* x}$$

where φ is defined by $\varphi(\alpha) = \overline{\lambda_\alpha}$.

It is natural to ask which eigenvalues can occur. Given $\alpha_1, \ldots, \alpha_n \in A$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, when does there exist $S \in \mathcal{M}$ with $||S|| \leq 1$ and $S k_{\alpha_i} = \lambda_i k_{\alpha_i}$ for $i = 1, \ldots, n$? In other words, given φ defined by $\varphi(\alpha_i) = \overline{\lambda_i}$, $i = 1, \ldots, n$, when can we find an extension of φ to all of A so that the multiplication operator M_φ has norm at most 1? Without loss of generality, we may suppose that finite linear combination of $\{k_\alpha\}$ are dense in H. The results of this section up to this point, then remain valid in this greater context. Proposition 2 gives a necessary condition. If (17) holds then Theorem 8 gives a necessary and sufficient condition. There are many important contexts in which reproducing kernel spaces arise besides analytic function spaces. See for example the seminal paper [Ar] and the introduction [Hi].

We will now investigate conditions under which (17) holds. Suppose $c_n > 0$ for $n = 0, 1, \ldots$. Define an inner product on analytic functions on the disk by

$$< f, g >= \sum_{n=0}^{\infty} \frac{1}{c_n} b_n \overline{d_n}, \tag{22}$$

when $f(z) = \sum b_n z^n$ and $g(z) = \sum d_n z^n$. Define

$$k(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Then $k(\overline{\alpha} z)$ is the reproducing kernel at $\alpha \in \mathbb{D}$ for the Hilbert space D_k of analytic functions with $< f, f >\equiv ||f||^2 < \infty$. For example, consider the function $k^\beta(z) = (1 - z)^{-\beta}$, with $0 < \beta \leq 1$. In this case, we will denote D_k by D^β. When $\mathcal{A} = H^2$, as mentioned above, $k(z) = (1 - z)^{-1}$. In this case, it is easy to verify the identity

$$\left(1 - \frac{k_{i,0}\overline{k_{j,0}}}{k_{i,j} k_{0,0}}\right) = \overline{C_0(z_i)} C_0(z_j),$$

where $C_0(z) = (z - z_0)/(1 - \overline{z_0} z)$ and hence (17) holds. Thus Theorem 8 gives another, albeit more difficult, proof of Pick's theorem.

Lemma 10. *Suppose k is analytic in $\mathbb{D}$ with $k(r) > 0$, when $r > 0$, and $k(0) = 1$. Suppose further that*

$$1 - \frac{1}{k(z)} = \sum_{n=1}^{\infty} a_n z^n$$

converges in $\mathbb{D}$ and satisfies $a_n \geq 0$ for all n. Let $k_i(z) = k(\overline{z_i} z)$ and $k_{i,j} = k_i(z_j)$, for $j = 0, \ldots, m$. Then the m-by-m matrix

$$\left\{1 - \frac{k_{i,0}\overline{k_{j,0}}}{k_{i,j} k_{0,0}}\right\} \geq 0.$$

The conditions on k can be stated solely in terms of the coefficients a_n as: $a_n \geq 0$ and $\sum a_n \leq 1$, and thus can be used to define a k satisfying the hypotheses. Usually, though, we are given a kernel

k and it is easier to check $k(r) > 0$ than to check the equivalent condition $\sum a_n \le 1$. Indeed, the coefficients of k are all positive for the weighted inner products (22), and hence $k(r) > 0$ for $r > 0$.

Proof.

$$\frac{1}{k_{i,0}\overline{\overline{k_{j,0}}}} - \frac{1}{k_{i,j}k_{0,0}} = \frac{1}{k_{0,0}}\left(\frac{1}{k_{i,0}} + \frac{1}{\overline{\overline{k_{j,0}}}} - \frac{1}{k_{0,0}} - \frac{1}{k_{i,j}}\right) + \left(\frac{1}{k_{i,0}} - \frac{1}{k_{0,0}}\right)\left(\frac{1}{\overline{\overline{k_{j,0}}}} - \frac{1}{k_{0,0}}\right)$$

$$= \frac{1}{k_{0,0}}\sum_{n=1}^{\infty} a_n(\overline{z_i^n} z_j^n + \overline{z_0^n} z_0^n - \overline{z_i^n} z_0^n - \overline{z_0^n} z_j^n) + \left(\frac{1}{k_{i,0}} - \frac{1}{k_{0,0}}\right)\left(\frac{1}{\overline{\overline{k_{j,0}}}} - \frac{1}{k_{0,0}}\right)$$

$$= \frac{1}{k_{0,0}}\sum_{n=1}^{\infty} a_n\overline{(z_i^n - z_0^n)}(z_j^n - z_0^n) + \left(\frac{1}{k_{i,0}} - \frac{1}{k_{0,0}}\right)\overline{\left(\frac{1}{k_{j,0}} - \frac{1}{k_{0,0}}\right)}.$$

Multiplying by $b_i\overline{b_j}k_{i,0}\overline{k_{j,0}}$ and summing over i and j, we obtain

$$\sum_{i,j}\left(1 - \frac{k_{i,0}\overline{k_{j,0}}}{k_{i,j}k_{0,0}}\right)b_i\overline{b_j} = \frac{1}{k_{0,0}}\sum_{n=1}^{\infty} a_n\left|\sum_i b_i(\overline{z}_i^n - \overline{z}_0^n)k_{i,0}\right|^2 + \left|\sum_i b_i k_{i,0}\left(\frac{1}{k_{i,0}} - \frac{1}{k_{0,0}}\right)\right|^2.$$

Since $a_n \ge 0$ and $k_{0,0} \ge 0$, this proves the Lemma. □

Corollary 11. *Suppose $\mathcal{A}$ is a Hilbert space of analytic functions on $\mathbb{D}$ with reproducing kernels $k(\overline{\alpha}z)$, $\alpha \in \mathbb{D}$, such that any finite subcollection is independent. Suppose further that $k(r) > 0$, when $r > 0$ and*

$$1 - \frac{1}{k(z)} = \sum_{n=1}^{\infty} a_n z^n$$

is analytic in $\mathbb{D}$ and satisfies $a_n \ge 0$ for all n. Then $\mathcal{A}$ has the Pick property.

For $0 < \beta < 1$ the reproducing kernels $k^\beta(z) = (1-z)^{-\beta}$ clearly satisfy $k(0) = 1$ and $k(r) > 0$ when $r > 0$. The coefficients of $1 - 1/k^\beta$ are

$$a_n = \frac{\beta(1-\beta)\cdots(n-1-\beta)}{1\cdot 2\cdots n} > 0.$$

By Corollary 11, D^β has the Pick property for $0 < \beta \le 1$. When $\beta = 1$, k^β is the reproducing kernel for the classical Hardy space H^2, and we obtain another proof of Pick's theorem. Note also that

$$\lim_{\beta\to 0}\frac{k^\beta(z) - 1}{z\beta} = \frac{1}{z}\log\frac{1}{1-z} \equiv k_{\mathcal{D}}(z). \tag{23}$$

The latter kernel gives the reproducing kernel for the Dirichlet space, $\mathcal{D}$, the main topic of this paper. However, we do not see a direct way to use (23) to prove the Pick property for $\mathcal{D}$.

The following lemma of Kaluza is useful for verifying many spaces, such as the Dirichlet space, have the Pick property. See [Ha, Ch IV, Theorem 22].

Lemma [Kaluza]. *Suppose $c_n > 0$, with $c_0 = 1$ and*

$$\frac{c_n}{c_{n-1}} \leq \frac{c_{n+1}}{c_n} \leq 1.$$

Let $k(z) = \sum_{n=0}^{\infty} c_n z^n$. Then

$$1 - \frac{1}{k(z)} = \sum_{n=1}^{\infty} a_n z^n$$

with $a_n \geq 0$.

Since the proof is so simple, we record it here for the convenience of the reader.

Proof. That $c_{n+1}/c_n \leq 1$ just says that k is analytic on $\mathbb{D}$. Write $1/k(z) = \sum b_n z^n$. Then

$$c_0 b_n + c_1 b_{n-1} + \ldots + c_n b_0 = 0$$

and

$$c_0 b_{n+1} + c_1 b_n + \ldots + c_{n+1} b_0 = 0.$$

Multiplying the first line by $c_{n+1}/(c_n c_0)$ and subtracting it from $1/c_0$ times the second,

$$b_{n+1} = \sum_{k=1}^{n} b_k \frac{c_{n-k}}{c_0} \left(\frac{c_{n+1}}{c_n} - \frac{c_{n-k+1}}{c_{n-k}} \right).$$

Since $b_1 = -c_1 < 0$, by induction, $b_{n+1} < 0$ and hence $a_n > 0$. $\square$

For example if the coefficients $1/c_n$ in the norm in D_k are concave:

$$\frac{1/c_{n-1} + 1/c_{n+1}}{2} \leq \frac{1}{c_n},$$

then by the inequality between the arithmetic and geometric means $a^{\frac{1}{2}} b^{\frac{1}{2}} \leq (a+b)/2$, the coefficients c_n satisfy the hypothesis of Kaluza's lemma.

Theorem [Agler]. *The Dirichlet space, $\mathcal{D}$, has the Pick property.*

Proof. As shown in section 1, $c_n = 1/(n+1)$. By Kaluza's lemma and Corollary 11, $\mathcal{D}$ has the Pick property. $\square$

More generally, since $c_n > 0$ for all n implies $k(r) > 0$ when $r > 0$, we have:

Corollary 12. *Suppose $c_n > 0$ and $c_n^2 \leq c_{n-1}c_{n+1}$ and suppose D_k is the Hilbert space of analytic functions on $\mathbb{D}$ with inner product given by (22). Then the D_k has the Pick property.*

The space D^β can alternately be described as the functions $\sum b_n z^n$ such that

$$\sum_{n=0}^{\infty}(1+n)^{1-\beta}|b_n|^2 < \infty,$$

since c_n is asymptotic to $\Gamma(\beta)^{-1}n^{\beta-1}$. Note also that if we define the norm on D_β using the above, we obtain reproducing kernels from

$$k(z) = \sum_{n=0}^{\infty} \frac{z^n}{(1+n)^{1-\beta}}.$$

In this case it is easy to verify that the hypothesis of Kaluza's lemma hold. When $\beta = 0$ or 1, in other words in the cases $\mathcal{A} = H^2$ or $\mathcal{A} = \mathcal{D}$, the norm is exactly the same as the one given by the coefficients of k^β and $k_{\mathcal{D}}$ above. When $0 < \beta < 1$, the above norms are different from the norms given by the coefficients of k^β, but D^β still satisfies the Pick property by Corollary 12. Another commonly used norm on D^β is

$$\left\|\sum_0^\infty b_n z^n\right\| = \sum_0^\infty (1+n^2)^{\beta/2}|b_n|^2.$$

The hypotheses of Kaluza's lemma fail in this case, but only when $n = 1$. If we modify the norm slightly to be

$$\left\|\sum_0^\infty b_n z^n\right\| = \left(\frac{4}{5}\right)^{\beta/2}|b_0|^2 + \sum_1^\infty (1+n^2)^{\beta/2}|b_n|^2,$$

then Kaluza's lemma applies and D^β has the Pick property. Yet another norm on D_β is given by

$$\left\|\sum b_n z^n\right\|^2 = \sum_{n=0}^{\infty}(1+n^\beta)|b_n|^2.$$

By taking second derivatives in n, $1+n^\beta$ is concave if $0 \leq \beta \leq 1$, and so $\mathcal{A}$ again has the Pick property under this norm. We remark that these spaces have been studied in a number of contexts. Carleson gave a sufficient condition for a sequence to be a zero set of a function in D^β in his thesis [C1], and Shapiro and Shields [SS2] improved upon Carleson's result. Stegenga [St] characterized the Carleson measures for these spaces.

We can use the norms

$$||f||^2_{\mathcal{D}_\beta} = \sum_{n=0}^{\infty}(n+1)^{1-\beta}|a_n|^2$$

for $f = \sum a_n z^n$, to give a more precise version of Proposition 6, when $\mathcal{A} = \mathcal{D}_\beta$. We say that $Z \subset \mathbb{D}$ is a zero set of a Banach space X of functions defined on $\mathbb{D}$ if there is a $f \in X$ with $Z = \{z \in \mathbb{D} : f(z) = 0\}$.

Corollary 13. *A set $Z \subset \mathbb{D}$ is a zero set of $\mathcal{D}_\beta$ if and only if it is a zero set of $\mathcal{M}_{\mathcal{D}_\beta}$, $0 \le \beta \le 1$. If $z_0 \notin Z$ let $f \in \mathcal{D}_\beta$ be the function of least norm vanishing on Z and equal to 1 at z_0 and let $\varphi \in \mathcal{M}_{\mathcal{D}}$ be the multiplier of least norm vanishing on Z and equal to 1 at z_0. Then f and φ are unique, have zero set Z and satisfy*

$$f = \varphi \frac{k_{z_0}}{||k_{z_0}||^2}.$$

Richter and Sundberg [RS] proved a more general result than this theorem when $\mathcal{D}_\beta = \mathcal{D}$, the Dirichlet space. They proved that the extremal function for any invariant subspace is a multiplier.

Proof. First consider the case when $\beta = 0$. Write

$$\mathcal{D}(f) = \iint_{\mathbb{D}} |f'|^2 dA$$

If $a \in \mathbb{D} \setminus Z$ and $f \in \mathcal{D}$, with $f(a) = 0$, let

$$\tau(z) = \frac{z-a}{1-\overline{a}z} \quad \text{and} \quad g(z) = \frac{f(z)}{\tau(z)}.$$

Then $||g||_{H^2} = ||f||_{H^2}$ and

$$\mathcal{D}(g) = \mathcal{D}\left(\frac{f}{\tau}\right) = \mathcal{D}\left(\frac{f \circ \tau^{-1}}{z}\right) < \mathcal{D}\left(f \circ \tau^{-1}\right) = \mathcal{D}(f).$$

Thus

$$||\tau(z_0)g||_{\mathcal{D}} < ||g||_{\mathcal{D}} < ||f||_{\mathcal{D}},$$

and so an extremal function for (15) with $\mathcal{A} = \mathcal{D}$ vanishes only on the set Z.

Now consider the operator

$$T(f) = \frac{f}{\tau}$$

defined on the functions in $\mathcal{D}_\beta$ which vanish at a, for $0 \le \beta \le 1$. The operator T is bounded by 1 when $\beta = 0$ and when $\beta = 1$, and by an interpolation result of Stein and Weiss (see e.g. [BL]), T is also bounded by 1 on each $\mathcal{D}_\beta$, for $0 < \beta < 1$. Then

$$||\tau(z_0)\frac{f}{\tau}|| < ||f||,$$

so if $a \notin Z$, f cannot be extremal. By the comments above, D_β has the Pick property under this norm. The Corollary now follows from Proposition 6. $\square$

As a consequence, the Pick condition gives a (albeit complicated) characterization of the zero sets of $\mathcal{M}_{\mathcal{D}_\beta}$.

We can give a Pick Theorem on the unit ball in $\mathbb{C}^2$, for example, by letting

$$k_\alpha(z) = \frac{1}{1 - < z, \alpha >} = \sum_{m,n} \binom{n+m}{m} \overline{\alpha_1}^n \overline{\alpha_2}^n z_1^n z_2^m,$$

where $< z, \alpha >= z_1\overline{\alpha}_1 + z_2\overline{\alpha}_2$ is the usual inner product. In this case the natural Hilbert space $\mathcal{H}$ for which these are reproducing kernels are those analytic functions $f(z) = \sum a_{n,m} z_1^n z_2^m$ on $\mathbb{B}^2$ for which

$$||f||^2 = \sum \frac{|a_{n,m}|^2}{\binom{n+m}{m}} < \infty.$$

The analog of the equality given just before Lemma 10 holds in this case. See Theorem 2.2.2 (iii) in [R]. Thus (17) holds and the space has the Pick property. In this case $\mathcal{M}_{\mathcal{H}} \neq H^\infty$ since

$$||(z_1z_2)^n||_{H^\infty}^2 \leq \frac{C}{n^{\frac{1}{2}}} ||(z_1z_2)^n||_{\mathcal{H}}^2 \leq \frac{C}{n^{\frac{1}{2}}} ||(z_1z_2)^n||_{\mathcal{M}_{\mathcal{H}}}^2.$$

The natural H^2 space of analytic functions with square integrable boundary values has a similar norm with $\binom{n+m}{m}$ replaced by $\binom{n+m}{m}(n+m+1)$. See Proposition 1.4.9 in [R]. We can thus view the Hilbert space whose reproducing kernels are k_α as

$$\mathcal{H} = \{f \text{ analytic} : (I + z\partial_z + w\partial_w)^{\frac{1}{2}} f \in H^2\}.$$

A similar result holds for the unit ball in $\mathbb{C}^k$ where the usual H^2 norm is altered multiplying the coefficients by $(|c| + k - 1)!/(|c|!(k-1)!)$.

The Bergman space A^2 consists of analytic function on $\mathbb{D}$ for which

$$||f||_{A^2}^2 = \int_{\mathbb{D}} |f|^2 dA/\pi < \infty.$$

The reproducing kernels are

$$k_\alpha(z) = \frac{1}{(1 - \overline{\alpha}z)^2}.$$

The space of multipliers is just H^∞, with the usual sup norm, and it is easy to check that the Pick property fails. For example, if $z_1 = w_1 = 0$ and $z_2 = \frac{1}{2}$, then the Pick matrix with the above kernel is positive semi-definite if and only if $|w_2| \leq \sqrt{\frac{7}{16}}$, but Schwarz's lemma says that H^∞ functions have the stronger requirement that $|w_2| \leq \frac{1}{2}$. In this case, Siep [Si] has characterized the interpolating sequences for A^2 and they are different from the interpolating sequences in $\mathcal{M}_{A^2} = H^\infty$. Thus statements (a) and (b) in Corollary 7 are not equivalent in this case.

If $\mathcal{A}$ is the Hardy space of analytic functions on the ball with square intergrable boundary values, the (Cauchy) kernels defined on the unit ball in $\mathbb{C}^2$ by

$$k_\alpha(z) = (1- < z, \overline{\alpha} >)^{-2},$$

are the reproducing kernels, and the multipliers are the bounded analytic functions. The same example as used for the Bergman space shows that the Pick property fails. Similar examples can be constructed in the unit ball in $\mathbb{C}^n$. The failure of the Pick Theorem for the usual H^2 kernel on the ball and the polydisk in $\mathbb{C}^n$ was proved by Amar [Am].

§5 Interpolating Sequences for $\mathcal{M}_\mathcal{D}$.

In this section we will prove Theorem 1, the geometric characterization of interpolating sequences for $\mathcal{M}_\mathcal{D}$. Suppose now that (4) and (5) hold. We will work first with *harmonic Dirichlet space*. If u is harmonic on $\mathbb{D}$, let

$$|\nabla u|^2 = |\nabla \text{Re } u|^2 + |\nabla \text{Im } u|^2.$$

If

$$u(re^{i\theta}) = \sum_{-\infty}^{\infty} a_n r^{|n|} e^{in\theta}$$

let

$$||u||_\mathcal{D}^2 = \sum_{n=-\infty}^{\infty} (|n|+1)|a_n|^2$$

and define the *harmonic Dirichlet space* , $\mathcal{D}_h$, to be the harmonic functions u defined on $\mathbb{D}$ for which $||u||_\mathcal{D} < \infty$. This norm can be rewritten as

$$||u||_\mathcal{D}^2 = \int_0^{2\pi} |u(e^{i\theta})|^2 d\theta/2\pi + \frac{1}{2} \iint\limits_{\mathbb{D}} |\nabla u|^2 dA/\pi,$$

where $u(e^{i\theta})$ denotes the non-tangential limit of u at $e^{i\theta}$, (a.e. $d\theta$). Then $\mathcal{D}_h$ is a Hilbert space on $\mathbb{D}$ with reproducing kernel

$$K_\alpha(z) = 2\text{Re } k_\alpha(z) - 1 = 2\text{Re } \frac{1}{\overline{\alpha}z} \log \frac{1}{1-\overline{\alpha z}} - 1,$$

where $k_\alpha(z)$ is the reproducing kernel for $\mathcal{D}$ as given in section 1. Note that $k_\alpha = P_\mathcal{D} K_\alpha$ where $P_\mathcal{D}$ is the orthogonal projection onto $\mathcal{D}$. Write $K_n = K_{z_n}$.

Theorem 14. *If (4) and (5) hold, then $K_n/||K_n||_{\mathcal{D}}$ is an unconditional basic sequence.*

Proof. We will show:

Claim. *There is a constant $C < \infty$ so that if $u \in \mathcal{D}_h$ is real-valued and if $\{t_n\} \subset \mathbb{R}$ with $|t_n| \leq 1$ for all n, then there is a real-valued $v \in \mathcal{D}_h$ with $v(z_n) = t_n u(z_n)$ for all n, and $||v||_{\mathcal{D}} \leq C$.*

If the claim is true, then clearly it is true for complex-valued u, v, and t_n, with a larger C. If we define the operator T on $\mathcal{D}_h$ by $T(u) = v$, then $||T^*|| = ||T|| \leq C$. Since $T^*(K_n) = \overline{t_n} K_n$, this proves

$$\left\|\sum \overline{t_n} b_n \frac{K_n}{||K_n||_{\mathcal{D}}}\right\|_{\mathcal{D}} = \left\|T^*\left(\sum b_n \frac{K_n}{||K_n||_{\mathcal{D}}}\right)\right\|_{\mathcal{D}} \leq C \left\|\sum b_n \frac{K_n}{||K_n||_{\mathcal{D}}}\right\|_{\mathcal{D}},$$

which proves Theorem 14.

There are an unfortunate number of arbitrary constants to be chosen in the proof, and it might be best to list them here. Without loss of generality, we may suppose $0 < \gamma < 1$ in (4). Choose $\alpha, \beta, \varepsilon, \eta, \lambda, p$, and q such that

$$0 < \beta < \alpha < 1$$

$$1 < \eta < \frac{2\beta - 1}{1 - \gamma}$$

$$\varepsilon > 0$$

$$\eta(\beta - \varepsilon) > p$$

$$1 < p < \frac{\alpha}{2\alpha - 1}$$

$$\frac{1}{p} + \frac{1}{q} = 1$$

$$0 < \lambda < \frac{\beta}{2\alpha}.$$

It is possible to find such numbers by taking β very close to 1. The letter C_0 will be reserved for the inequality in Stegenga's Theorem. The letter C will be used for various constants, depending at most on $\alpha, \beta, \varepsilon, \eta, \lambda, p, q$ and C_0. Finally note that we can throw away any finite number of $z_n's$ in the proof. One way we'll exploit this is by tacitly assuming $|z_n|$ is "sufficiently close to 1" for all n.

We now assume $|z_1| \leq |z_2| \leq \ldots$ and let $w_n = z_n/|z_n|$, and define the sets

$$U_n = \{z \in \overline{\mathbb{D}} : |z - w_n| \leq (1 - |z_n|^2)^\alpha\},$$

and

$$V_n = \{z \in \overline{\mathbb{D}} : |z - w_n| \le (1 - |z_n|^2)^\beta\}.$$

Since $\beta < \alpha < 1$ we have $z_n \in U_n \subset V_n$.

Suppose $n > m$ and $V_m \cap V_n \neq \emptyset$, so that

$$|w_m - w_n| \le (1 - |z_m|^2)^\beta + (1 - |z_n|^2)^\beta.$$

Since $|z_n| \ge |z_m|$ we have by (4) and (12)

$$\frac{(1 - |z_m|^2)(1 - |z_n|^2)}{|1 - \overline{z_m} z_n|^2} \le (1 - |z_n|^2)^\gamma,$$

so that

$$\begin{aligned}(1 - |z_m|^2)(1 - |z_n|^2)^{1-\gamma} &\le |1 - \overline{z_m} z_n|^2 \\ &= |z_m||z_n||w_m - w_n|^2 + (1 - |z_m||z_n|)^2 \\ &\le 4(1 - |z_m|^2)^{2\beta} + (1 - |z_m|^2)^2 \\ &\le 5(1 - |z_m|^2)^{2\beta}\end{aligned},$$

and hence

$$1 - |z_n|^2 \le 5^{\frac{1}{1-\gamma}} (1 - |z_m|^2)^{\frac{2\beta - 1}{1-\gamma}} \le (1 - |z_m|^2)^\eta.$$

In the last inequality, we used that $|z_m|$ is "sufficiently close to 1". In particular, if $z_m \in V_n$, then

$$1 - |z_m| \le |z_m - w_n| \le (1 - |z_n|^2)^\beta \le (1 - |z_m|^2)^{\eta\beta} \le 2^{\eta\beta}(1 - |z_m|)^{\eta\beta},$$

which is a contradiction, since $\eta\beta > 1$ and $|z_m|$ is sufficiently close to 1. We've shown that

(24) $$\text{If } n > m \text{ and } V_n \cap V_m \neq \emptyset \quad \text{then} \quad (1 - |z_n|^2) \le (1 - |z_m|^2)^\eta$$

and

(25) $$\text{If } n > m \quad \text{then} \quad z_m \notin V_n.$$

Define functions

$$\varphi_n(z) = \begin{cases} 0 & z \in \overline{\mathbb{D}} \setminus V_n \\ 1 & z \in U_n \\ \dfrac{\log \dfrac{(1 - |z_n|^2)^\beta}{|z - w_n|}}{(\alpha - \beta) \log \dfrac{1}{1 - |z_n|^2}} & z \in V_n \setminus U_n. \end{cases}$$

Lemma 15. *Given $t_j \in \mathbb{R}$, with $|t_j| \leq 1$, there exists $a_j \in \mathbb{R}$ such that $|a_j| \leq 3$ and*

$$\left\|\sum_{j=0}^{\infty} a_j \varphi_j\right\|_{\infty} \leq 2 \quad \text{and} \quad \sum_{j=0}^{\infty} a_j \varphi_j(z_n) = t_n \quad \text{for all } n.$$

Proof of Lemma 15. By (25), $\varphi_j(z_n) = 0$ if $j > n$ so we can inductively choose $a_j \in \mathbb{R}$ such that

$$t_n = \sum_{j=1}^{n} a_j \varphi_j(z_n) = \sum_{j=1}^{\infty} a_j \varphi_j(z_n)$$

for all n.

Assume $|| \sum_{j=1}^{n-1} a_j \varphi_j ||_\infty \leq 2$. Since $|t_n| \leq 1$ and $\varphi_n(z_n) = 1$, we have $|a_n| \leq 3$. If $z \in V_n$, denote by $n_1, \ldots, n_k$ those indices smaller than n for which $z \in V_{n_j}$. By (24) $(1 - |z_n|^2) \leq (1 - |z_{n_j}|^2)^\eta$ so that

$$\begin{aligned}
\left|\nabla \sum_{j=1}^{n-1} a_j \varphi_j(z)\right| &= \left|\sum a_{n_j} \nabla \varphi_{n_j}(z)\right| \\
&\leq 3 \sum \frac{1}{(\alpha - \beta) \log \frac{1}{1-|z_{n_j}|^2}} \frac{1}{(1 - |z_{n_j}|^2)^\alpha} \\
&\leq \frac{3}{\alpha - \beta} \; \frac{1}{(1 - |z_n|^2)^{\alpha/\eta}} \sum \frac{1}{\log \frac{1}{1-|z_{n_j}|^2}} \\
&\leq \frac{3C_0}{\alpha - \beta} \frac{1}{(1 - |z_n|^2)^{\alpha/\eta}},
\end{aligned} \tag{26}$$

by (5). Hence for $z \in V_n$,

$$\left|\sum_{j=1}^{n-1} a_j \varphi_j(z) - \sum_{j=1}^{n-1} a_j \varphi_j(z_n)\right| \leq \frac{6C_0}{\alpha - \beta} (1 - |z_n|^2)^{\beta - \frac{\alpha}{\eta}} < 1. \tag{27}$$

In the last inequality, we used that $\beta\eta > 1 > \alpha$ and that z_n is sufficiently close to 1. Thus if $a_n \leq 0$

$$\sum_{j=1}^{n} a_j \varphi_j(z) \leq \sum_{j=1}^{n-1} a_j \varphi_j(z) \leq 2$$

and by (27), for $z \in V_n$,

$$\begin{aligned}
\sum_{j=1}^{n} a_j \varphi_j(z) &\geq a_n + \sum_{j=1}^{n-1} a_j \varphi_j(z) \\
&= t_n + \sum_{j=1}^{n-1} a_j \varphi_j(z) - \sum_{j=1}^{n-1} a_j \varphi_j(z_n) \\
&\geq -1 - 1 = -2.
\end{aligned}$$

So we've shown $|\sum_{j=1}^{n} a_j\varphi_j(z)| \le 2$, if $z \in V_n$ and $a_n \le 0$. A similar argument works if $a_n \ge 0$. If $z \in \overline{\mathbb{D}} \setminus V_n$ then

$$\left|\sum_{j=1}^{n} a_j\varphi_j(z)\right| = \left|\sum_{j=1}^{n-1} a_j\varphi_j(z)\right| \le 2,$$

proving Lemma 15. □

Now let $\varphi = \sum_{j=0}^{\infty} a_j\varphi_j$.

Lemma 16. *$|\nabla\varphi|^2 dA$ is a Carleson measure for the Dirichlet space $\mathcal{D}$.*

Proof. We will verify Stegenga's criterion (3). Suppose $E \subset \partial\mathbb{D}$ is a union of finitely many disjoint arcs $I_1, \dots, I_N$. Denote by $\widetilde{I_j}$ the arc with the same center as I_j and $|\widetilde{I_j}| = |I_j|^\lambda$. We may assume that $\sum |I_j|$ is small, since φ is supported on $\cup V_n$ and all z_n are sufficiently close to $\partial\mathbb{D}$. Define

$$\mathbb{N}_1 = \{j : V_j \subset \cup_{k=1}^{N} S(\widetilde{I}_k)\}$$

and

$$\mathbb{N}_2 = \mathbb{N} \setminus \mathbb{N}_1.$$

Then

$$\begin{aligned} \iint\limits_{\cup S(I_k)} |\nabla\varphi|^2 dA &\le 2\iint\limits_{\mathbb{D}} |\sum_{j\in\mathbb{N}_1} a_j\nabla\varphi_j|^2 dA + 2\iint\limits_{\mathbb{D}} |\sum_{j\in\mathbb{N}_2} a_j\nabla\varphi_j|^2 dA \\ &\equiv \quad Int_1 \quad + \quad Int_2. \end{aligned} \tag{28}$$

We first estimate Int_1: Let $n \in \mathbb{N}_1$ and let

$$z \in W_n \equiv V_n \setminus \cup_{\{m\in\mathbb{N}_1 : m>n\}} V_m.$$

The argument in (26) yields

$$\left|\sum_{\{j\in\mathbb{N}_1 : j<n\}} a_j\nabla\varphi_j(z)\right| \le \frac{3C_0}{\alpha-\beta}\frac{1}{(1-|z_n|^2)^{\alpha/\eta}},$$

and hence

$$\begin{aligned} \iint\limits_{W_n}\left|\sum_{j\in\mathbb{N}_1} a_j\nabla\varphi_j(z)\right|^2 dA(z) &\le 2\iint\limits_{V_n}\left[\frac{3C_0}{\alpha-\beta}\frac{1}{(1-|z_n|^2)^{\alpha/\eta}}\right]^2 dA(z) + 2\iint\limits_{\mathbb{D}} |3\nabla\varphi_n(z)|^2\, dA(z) \\ &\le \frac{18C_0^2}{(\alpha-\beta)^2}\frac{\pi}{2}(1-|z_n|^2)^{2(\beta-\alpha/\eta)} + \frac{18\pi}{\alpha-\beta}\frac{1}{\log\frac{1}{1-|z_n|^2}} \\ &\le \frac{36\pi}{\alpha-\beta}\frac{1}{\log\frac{1}{1-|z_n|^2}} \end{aligned}$$

since $\beta > \alpha/\eta$, and $|z_n|$ is sufficiently close to 1. Iterating this estimate we obtain

$$\iint_{V_n} \Big| \sum_{j\in\mathbb{N}_1} a_j \nabla\varphi_j(z) \Big|^2 dA(z) \le \frac{36\pi}{\alpha-\beta} \sum_{\substack{\mathbb{N}_1 \ni m > n \\ V_m \cap V_n \neq \emptyset}} \frac{1}{\log \frac{1}{1-|z_n|^2}}.$$

Let J_n be the arc on $\partial\mathbb{D}$ with the same center as V_n, yet 3 times as long as $V_n \cap \partial\mathbb{D}$. Then the above is

$$\begin{aligned} &\le \frac{36\pi}{\alpha-\beta} \sum_{\substack{m\in\mathbb{N}_1 \\ z_m \in S(J_n)}} \frac{1}{\log \frac{1}{1-|z_n|^2}} \\ &\le \frac{36\pi}{\alpha-\beta} C_0 \left(\log \frac{1}{\mathrm{Cap}(J_n)} \right)^{-1} \\ &\le C \left(\log \frac{1}{1-|z_n|^2} \right)^{-1}. \end{aligned}$$

Hence

$$\begin{aligned} Int_1 &\le 2 \sum_{n\in\mathbb{N}_1} \iint_{V_n} \Big| \sum_{j\in\mathbb{N}_1} a_j \nabla\varphi_j \Big|^2 dA \\ &\le C \sum_{n\in\mathbb{N}_1} \left(\log \frac{1}{1-|z_n|^2} \right)^{-1} \\ &\le C \sum_{z_n \in S(\cup \widetilde{I_k})} \left(\log \frac{1}{1-|z_n|^2} \right)^{-1} \\ &\le CC_0 \left(\log \frac{1}{\mathrm{Cap}(\cup \widetilde{I_k})} \right)^{-1}. \end{aligned}$$

We'll complete the estimate for Int_1 by showing

$$\gamma(\cup \widetilde{I_k}) = \log \frac{1}{\mathrm{Cap}(\cup \widetilde{I_k})} \ge C \log \frac{1}{\mathrm{Cap}(\cup I_k)} = C\gamma(\cup I_k). \tag{29}$$

The following Lemma is well known, but we could not find a reference.

Lemma 17. *Suppose $E \subset \partial\mathbb{D}$ and $\gamma(E) = \lim_{z\to\infty} g(z,\infty) - \log|z|$ is the Robin constant for E. Then*

$$\frac{\log 2}{\log 2 + \gamma(E)} \le \omega(0, E, 3\mathbb{D} \setminus E) \le \frac{\log 4}{\log 4 + \gamma(E)},$$

where $\omega(z, E, 3\mathbb{D} \setminus E)$ is the harmonic measure of E in $\{z : |z| < 3\} \setminus E$.

Proof of Lemma 17. By consideration of boundary values,

$$g(z,\infty) - g(\frac{1}{\bar{z}}, \infty) = \log|z|.$$

Letting $z \to \infty$ we obtain $g(0,\infty) = \gamma(E)$. Now

$$g(z,\infty) - \gamma(E) = \int_E \log|z-\zeta| d\mu(\zeta),$$

where μ is a probability measure, and hence if $|z| = 3$

$$\log 2 \leq g(z,\infty) - \gamma(E) \leq \log 4.$$

By consideration of boundary values

$$\frac{\log 2 - g(z,\infty) + \gamma(E)}{\log 2 + \gamma(E)} \leq \omega(z, E, 3\mathbb{D} \setminus E) \leq \frac{\log 4 - g(z,\infty) + \gamma(E)}{\log 4 + \gamma(E)}.$$

Lemma 17 now follows by letting $z = 0$, since $\gamma(E) = g(0,\infty)$. $\square$

We now continue our estimation of Int_1. Using an explicit conformal map of $\mathbb{C}^* \setminus I_k$ onto $\mathbb{D}$ and making a easy estimate, shows that

$$\omega(z, I_k, 3\mathbb{D} \setminus I_k) \geq C > 0$$

for $z \in \widetilde{I}_k$. Thus if $z \in \widetilde{I}_k$ then

$$\omega(z, \cup I_j, 3\mathbb{D} \setminus \cup I_j) \geq \omega(z, I_k, 3\mathbb{D} \setminus I_k) \geq C.$$

and hence

$$\omega(z, \cup I_j, 3\mathbb{D} \setminus \cup I_j) \geq C\omega(z, \cup \widetilde{I}_j, 3\mathbb{D} \setminus \cup \widetilde{I}_j),$$

for $z \in 3\mathbb{D} \setminus \cup \widetilde{I}_j$. Letting $z = 0$ and applying Lemma 17, we obtain (29), since $\sum |I_j|$ is small. Thus

$$Int_1 \leq C \left(\log \frac{1}{\text{Cap}(\cup I_k)} \right)^{-1}.$$

We now estimate Int_2 in (28). Suppose $V_j \cap S(I_k) \neq \emptyset$, but $V_j \not\subset S(\widetilde{I}_k)$. Then since $\widetilde{I}_k$ and I_k have the same center,

$$2(1-|z_j|^2)^\beta = |V_j \cap \mathbb{R}| \geq \frac{1}{2}(|\widetilde{I}_k| - |I_k|) \geq \frac{1}{3}|I_k|^\lambda.$$

Hence

$$||\nabla \varphi_j||_\infty \leq \frac{1}{(\alpha-\beta)(1-|z_j|^2)^\alpha \log \frac{1}{1-|z_j|^2}} \leq \frac{C}{|I_k|^{\alpha\lambda/\beta}} \frac{1}{\log \frac{1}{1-|z_j|^2}}.$$

This shows that for $z \in S(I_k)$,

$$\Big|\sum_{j\in\mathbb{N}_2} a_j \nabla\varphi_j(z)\Big|^2 \le \frac{9C}{|I_k|^{2\alpha\lambda/\beta}} \left(\sum_{j\in\mathbb{N}_2} \frac{1}{\log\frac{1}{1-|z_j|^2}}\right)^2 \le CC_0^2 \frac{1}{|I_k|^{2\alpha\lambda/\beta}}.$$

since $|a_j| \le 3$ and since (5) holds. Hence

$$\int_{S(I_k)} \Big|\sum_{j\in\mathbb{N}_2} a_j \nabla\varphi_j\Big|^2 dA \le C|I_k|^{2-2\alpha\lambda/\beta} \le C|I_k|,$$

since $\lambda < \beta/(2\alpha)$. Thus

$$Int_2 \le C|\cup I_k| < Ce^{-\gamma(\cup I_k)} \le \frac{C}{\gamma(\cup I_k)} = C\left(\log\frac{1}{\mathrm{Cap}(\cup I_k)}\right)^{-1}.$$

Adding Int_1 and Int_2, we obtain

$$\iint_{\cup S(I_k)} |\nabla\varphi|^2 dA \le C\left(\log\frac{1}{\mathrm{Cap}(\cup I_k)}\right)^{-1}.$$

By Stegenga's Theorem, $|\nabla\varphi|^2 dA$ is a Carleson measure for $\mathcal{D}$, with constant depending only on $C_0, \alpha, \beta, \gamma, \varepsilon, \lambda, \eta$, completing the proof of Lemma 16. $\square$

Now suppose $u \in \mathcal{D}_h$ is real-valued. Then $u = \mathrm{Re} f$, for some $f \in \mathcal{D}$ and $|\nabla u| = |f'|$. By Stegenga's Theorem, Lemma 13 and Lemma 14,

$$\iint_{\mathbb{D}} |\nabla(\varphi u)|^2 dA \le C^2 \iint_{\mathbb{D}} |\nabla u|^2 dA.$$

Let u^* be the boundary function of u and let $P[\varphi^* u^*]$ be the Poisson integral of the boundary function $\varphi^* u^*$. By the Dirichlet principle,

$$\iint_{\mathbb{D}} |\nabla P[\varphi^* u^*]|^2 dA \le C^2 \iint |\nabla(\varphi u)|^2 dA.$$

Since $||\varphi||_\infty \le 2$, we see that

$$||P[\varphi^* u^*]||_{\mathcal{D}} \le C||u||_{\mathcal{D}}.$$

The estimate in the next Lemma will allow us to "correct" $P[\varphi^* u^*]$ so as to obtain $v \in \mathcal{D}_h$ with $v(z_n) = t_n u(z_n)$.

Lemma 18.

$$\sum_n |P[\varphi^* u^*](z_n) - t_n u(z_n)| \le C||u||_{\mathcal{D}}.$$

Proof. Fix n and let

$$W_1 = \partial\mathbb{D} \setminus U_n,$$

$$W_2 = \partial\mathbb{D} \cap U_n \cap (\cup_{j>n} V_j), \quad \text{and}$$

$$W_3 = \partial\mathbb{D} \cap U_n \setminus \cup_{j>n} V_j.$$

Let P_{z_n} be the Poisson kernel for z_n. Then

$$\begin{aligned} |P[\varphi^* u^*](z_n) - t_n u(z_n)| &\le \int_{W_1} + \int_{W_2} + \int_{W_3} |\varphi(e^{i\theta})u(e^{i\theta}) - t_n u(e^{i\theta})| P_{z_n}(e^{i\theta}) \frac{d\theta}{2\pi} \\ &= \ Int_1 + \ Int_2 + \ Int_3. \end{aligned}$$

Note that if I is an arc on $\partial\mathbb{D}$ then

$$\int_{S(I)} |\nabla u|^2 (1 - |z|^2) dA \le 2|I| \ ||u||^2_{\mathcal{D}},$$

so $u \in BMO$ and for all $q < \infty$ $\ ||u^*||_q \le C_q ||u||_{BMO} \le \widetilde{C_q} ||u||_{\mathcal{D}}$, by Fefferman's characterization of BMO (see [G]).

Estimate for Int_1 (on $W_1 = \partial\mathbb{D} \setminus U_n$):

Writing $e^{is} = w_n = z_n/|z_n|$ we see that

$$\begin{aligned} P_{z_n}(e^{i\theta}) &= \frac{1 - |z_n|^2}{1 - 2|z_n|\cos(\theta - s) + |z_n|^2} \\ &= \frac{1 - |z_n|^2}{(1 - |z_n|^2)^2 + 2|z_n|(1 - \cos(\theta - s))} \\ &\le \frac{1 - |z_n|^2}{2|z_n|(1 - \cos(\theta - s))}, \end{aligned}$$

and hence

$$\int_{W_1} P_{z_n}(e^{i\theta})^p \frac{d\theta}{2\pi} \le C(1 - |z_n|^2)^p \int_{(1-|z_n|^2)^\alpha}^{\pi} \frac{dt}{t^{2p}} \le C(1 - |z_n|^2)^{p - \alpha(2p-1)}.$$

Note that $\alpha(2 - 1/p) < 1$. Thus

$$\begin{aligned} Int_1 &\le C\left(1 - |z_n|^2\right)^{1 - \alpha(2 - 1/p)} ||u^*||_q \\ &\le C||u||_{\mathcal{D}} \left(1 - |z_n|^2)\right)^{1 - \alpha(2-1/p)} \\ &\le C||u||_{\mathcal{D}} \left(\log \frac{1}{1 - |z_n|^2}\right)^{-1}. \end{aligned}$$

Estimate for Int_2 (on $W_2 = \partial\mathbb{D} \cap U_n \cap (\cup_{j>n} V_j)$):

By (24), if $j > n$ and $V_j \cap V_n \neq \emptyset$, then $1 - |z_j|^2 \leq (1 - |z_n|^2)^\eta$, so

$$\begin{aligned}
\left| \bigcup_{\substack{j>n \\ V_j \cap U_n \neq \emptyset}} (V_j \cap \partial\mathbb{D}) \right| &\leq \sum_{\substack{j>n \\ V_j \cap U_n \neq \emptyset}} 2(1 - |z_j|^2)^\beta \\
&= \sum_{\substack{j>n \\ V_j \cap U_n \neq \emptyset}} 2(1 - |z_j|^2)^\beta \left(\log \frac{1}{1 - |z_j|^2} \right) \frac{1}{\log \frac{1}{1-|z_j|^2}} \\
&\leq \sum_{\substack{j>n \\ V_j \cap U_n \neq \emptyset}} 2(1 - |z_j|^2)^{\beta - \varepsilon} \frac{1}{\log \frac{1}{1-|z_j|^2}} \\
&\leq C_0 (1 - |z_n|^2)^{\eta(\beta - \varepsilon)}.
\end{aligned}$$

Hence

$$\int_{W_2} P_{z_n}(e^{i\theta})^p \frac{d\theta}{2\pi} \leq C ||P_{z_n}||_\infty^p C_0 (1 - |z_n|^2)^{\eta(\beta-\varepsilon)} \leq C(1 - |z_n|^2)^{\eta(\beta-\varepsilon)-p},$$

and so

$$Int_2 \leq C(1 - |z_n|^2)^{\eta(\beta-\varepsilon)/p-1} ||u^*||_q \leq C||u||_{\mathcal{D}} \left(\log \frac{1}{1 - |z_n|^2} \right)^{-1}.$$

Estimate for Int_3 (on $W_3 = \partial\mathbb{D} \cap [U_n \setminus (\cup_{j>n} V_j)]$):

Since $\varphi(z_n) = t_n$, for $z \in \partial\mathbb{D} \cap [U_n \setminus (\cup_{j>n} V_j)]$ we have by (26) again

$$\begin{aligned}
|t_n - \varphi(z)| &= |\sum_{j=1}^{n} a_j \varphi_j(z_n) - \sum_{j=1}^{n} a_j \varphi_j(z)| \\
&= |\sum_{j=1}^{n-1} a_j \varphi_j(z_n) - \sum_{j=1}^{n-1} a_j \varphi_j(z)| \\
&\leq \frac{3C_0}{\alpha - \beta} \frac{1}{(1 - |z_n|^2)^{\alpha/\eta}} 2(1 - |z_n|^2)^\alpha \\
&\leq C(1 - |z_n|^2)^{\alpha(1-1/\eta)}.
\end{aligned}$$

Hence

$$Int_3 \leq C(1 - |z_n|^2)^{\alpha(1-1/\eta)} P[|u^*|](z_n) \leq C P[|u^*|](z_n) \left(\log \frac{1}{1 - |z_n|^2} \right)^{-1}.$$

Combining the estimates for $Int_1,\ Int_2,\ Int_3$ and summing over n yields

$$\begin{aligned}
\sum_n |P[\varphi^* u^*](z_n) - t_n u(z_n)| &\leq C||u||_{\mathcal{D}} \sum \left(\log \frac{1}{1 - |z_n|^2} \right)^{-1} + \sum_n P[|u^*|](z_n) \left(\log \frac{1}{1 - |z_n|^2} \right)^{-1} \\
&\leq C||u||_{\mathcal{D}} + C \left[\sum_n P[|u^*|](z_n)^2 \left(\log \frac{1}{1 - |z_n|^2} \right)^{-1} \right]^{\frac{1}{2}} \left[\sum \left(\log \frac{1}{1 - |z_n|^2} \right)^{-1} \right]^{\frac{1}{2}} \\
&\leq C||u||_{\mathcal{D}} + C||P[|u^*|]||_{\mathcal{D}} \\
&\leq C||u||_{\mathcal{D}}.
\end{aligned}$$

The last two inequalities above follow from (5′) and the Dirichlet principle. This completes the proof of Lemma 18. $\square$

We now finish the proof of the claim stated at the start of the proof of Theorem 14. Shapiro and Shields [SS2] proved that if

$$\sum \left(\log \frac{1}{1-|z_n|^2}\right)^{-1} \le M,$$

then there exists $g \in \mathcal{D}$ with $g(z_n) = 0$, $g'(0) = 1$ and $\iint_{\mathbb{D}} |g'|^2 dA \le C(M)$ where $C(M)$ is a constant depending on M. By (4), $1 - \rho^2(z_m, z_n) \le (1 - |z_n|^2)^\gamma$ and thus

$$\sum_{m \ne n} \left(\log \frac{1}{1-\rho^2(z_m, z_n)}\right)^{-1} \le \frac{C_0}{\gamma}.$$

Fix n. Then this estimate together with the result of Shapiro and Shields shows there exists $g_n \in \mathbb{D}$ such that

$$g_n\left(\frac{z_m - z_n}{1 - \overline{z_n} z_m}\right) = 0 \quad \text{if } m \ne n$$

$$g_n'(0) = 1 \quad \text{and}$$

$$\iint_{\mathbb{D}} |g_n'|^2 dA < C.$$

We can also arrange that $g_n(-z_n) = 0$. Define

$$f_n(z) = \frac{g_n\left(\frac{z - z_n}{1 - \overline{z_n} z}\right)}{\frac{z - z_n}{1 - \overline{z_n} z}}.$$

Then $f_n(z_m) = \delta_{m,n}$ and $\iint_{\mathbb{D}} |f_n'|^2 dA < C$. Since $f_n(0) = -g_n(-z_n)/z_n = 0$, we have $||f_n||_{\mathcal{D}} < 2C$, as can be seen from the series expansion. Defining

$$v = P[\varphi^* u^*] + \sum \left(t_n u(z_n) - P[\varphi^* u^*](z_n)\right) \mathrm{Re} f_n,$$

we see that $v \in \mathcal{D}_h$ is real-valued, with $v(z_n) = t_n u(z_n)$ for all n and $||v||_{\mathcal{D}} \le C||u||_{\mathcal{D}}$ by Lemma 18, proving the claim and finishing the proof of Theorem 14. $\square$

We now prove our main result.

Proof of Theorem 1. Suppose (4) and (5) hold. As above, let $k_n = k_{z_n}$ and $K_n = K_{z_n}$. By Theorem 14, $\{K_n/||K_n||_{\mathcal{D}}\}$ is a (UBS), so by the Köthe-Toeplitz Theorem, it is a (RS). We will use this fact to show that $k_n/||k_n||$ is a Riesz sequence (RS).

One inequality is now easy. Let $P_{\mathcal{D}}$ be the orthogonal projection onto $\mathcal{D}$. If $\alpha_n \in \mathbb{C}$, then

$$\begin{aligned}\left\|\sum \alpha_n \frac{k_n}{||k_n||}\right\| &= \left\|P_{\mathcal{D}} \sum \alpha_n \frac{K_n}{||k_n||}\right\| \\ &\le \left\|\sum \alpha_n \frac{K_n}{||k_n||}\right\| \\ &= \left\|\sum \alpha_n \frac{||K_n||}{||k_n||} \frac{K_n}{||K_n||}\right\| \\ &\le C\left(\sum |\alpha_n|^2\right)^{\frac{1}{2}},\end{aligned}$$

since $\{K_n/||K_n||\}$ is a (RS) and $||K_n|| \sim ||k_n||$.

For the opposite inequality, write $\alpha_n = a_n + ib_n$, with $a_n, b_n \in \mathbb{R}$. Replacing α_n by $\lambda\alpha_n$ with $|\lambda| = 1$, we may suppose $\mathrm{Im}\sum \alpha_n/||k_n|| = 0$. If f is analytic and $\mathrm{Im} f(0) = 0$, then $||f||_{\mathcal{D}} = ||\mathrm{Re} f||_{\mathcal{D}}$. Thus

$$\begin{aligned}\left\|\sum \alpha_n \frac{k_n}{||k_n||}\right\|^2 &= \left\|\sum \left(a_n \frac{\mathrm{Re} k_n}{||k_n||} - b_n \frac{\mathrm{Im} k_n}{||k_n||}\right)\right\|^2 \\ &= \left\|\sum a_n \frac{\mathrm{Re} k_n}{||k_n||}\right\|^2 + \left\|\sum b_n \frac{\mathrm{Im} k_n}{||k_n||}\right\|^2 - 2\sum a_m b_n \frac{< \mathrm{Re} k_m, \mathrm{Im} k_n >}{||k_m||\,||k_n||}.\end{aligned}$$

Now

$$\begin{aligned}\left\|\sum a_n \frac{\mathrm{Re} k_n}{||k_n||}\right\|^2 &= \frac{1}{2}\left\|\sum a_n \frac{K_n + 1}{||k_n||}\right\|^2 \\ &= \frac{1}{2}\left\|\sum a_n \frac{K_n}{||k_n||} + \left(\sum a_n \frac{K_n}{||k_n||}\right)(0)\right\|^2 \\ &\ge \frac{1}{2}\left\|\sum a_n \frac{K_n}{||k_n||}\right\|^2 \\ &\ge \frac{1}{C}\sum |a_n|^2,\end{aligned} \tag{30}$$

since $||u+u(0)||^2 \ge ||u||^2$, as can be seen from the norm expressed in terms of the Fourier coefficients. Since $\sum b_n/||k_n|| = \mathrm{Im}\sum \alpha_n/||k_n|| = 0$,

$$\left\|\sum b_n \frac{\mathrm{Im} k_n}{||k_n||}\right\|^2 = \left\|\sum b_n \frac{\mathrm{Re} k_n}{||k_n||}\right\|^2 \ge \frac{1}{C}\sum |b_n|^2,$$

by (30).

For the cross terms, note that

$$\mathrm{Im}k_n(z_m) =< \mathrm{Im}k_n, K_m >= 2 < \mathrm{Im}k_n, \mathrm{Re}k_m >,$$

since $\mathrm{Im}k_n(0) = 0$. When $\zeta = e^{i\theta}$, the elementary estimate $|\frac{1}{2}\mathrm{Im}\left(\frac{1}{\zeta}\log(1-\zeta)\right)| \leq 5$ holds, and hence the same inequality holds for all $\zeta \in \mathbb{D}$. Thus

$$| < \mathrm{Im}k_n, \mathrm{Re}k_m > | \leq \frac{1}{2}|\mathrm{Im}k_n(z_m)| \leq 5,$$

and so

$$\left|\sum a_m b_n \frac{< \mathrm{Re}k_m, \mathrm{Im}k_n >}{||k_m||\,||k_n||}\right| \leq 5\left(\sum |a_m|\frac{1}{||k_m||}\right)\left(\sum |b_n|\frac{1}{||k_n||}\right)$$

$$\leq 5\left(\sum |a_n|^2\right)^{\frac{1}{2}}\left(\sum |b_n|^2\right)^{\frac{1}{2}}\sum\left(\log\frac{1}{1-|z_n|^2}\right)^{-1}.$$

By throwing away finitely many z_n we may assume that

$$\sum\left(\log\frac{1}{1-|z_n|^2}\right)^{-1} < \frac{1}{10C},$$

where C is the constant appearing in (30). Thus

$$\left\|\sum \alpha_n \frac{k_n}{||k_n||}\right\|^2 \geq \frac{1}{C}\sum\left(|a_n|^2 + |b_n|^2\right) - 10\left(\sum |a_n|^2\right)^{\frac{1}{2}}\left(\sum |b_n|^2\right)^{\frac{1}{2}}\sum\left(\log\frac{1}{1-|z_n|^2}\right)^{-1}$$

$$\geq \frac{1}{C}\left[\sum\left(|a_n|^2 + |b_n|^2\right) - \left(\sum |a_n|^2\right)^{\frac{1}{2}}\left(\sum |b_n|^2\right)^{\frac{1}{2}}\right]$$

$$\geq \frac{1}{2C}\sum\left(|a_n|^2 + |b_n|^2\right)$$

$$= \frac{1}{2C}\sum |\alpha_n|^2.$$

Thus $k_n/||k_n||$ is a Riesz sequence. By Agler's Theorem, $\mathcal{D}$ has the Pick property. By Corollary 7, $\{z_n\}$ is an interpolating sequence for $\mathcal{M}_{\mathcal{D}}$.

Now suppose $\{z_n\}$ is an interpolating sequence for $\mathcal{M}_{\mathcal{D}}$. As noted previously, by Stegenga's theorem, (5) is equivalent to

$$\sum \frac{\delta_{z_n}}{||k_n||^2}$$

is a Carleson measure. In other words,

$$\sum |< f, \frac{k_n}{||k_n||} >|^2 = \sum \frac{|f(z_n)|^2}{||k_n||^2} \le C||f||_{\mathcal{D}}$$

for all $f \in \mathcal{D}$. This is exactly the lower estimate of the norm in (SS). Thus by the Corollary 7, if $\{z_n\}$ is an interpolating sequence for $\mathcal{M}_{\mathcal{D}}$ then (5) holds. To interpret (4), we prove the following elementary lemma.

Lemma 19. *There is a constant $\gamma > 0$ so that*

$$1 - \rho^2(z_n, z_m) \le (1 - |z_n|^2)^\gamma \text{ for all } n \ne m \tag{31}$$

if and only if there is a $\sigma < 1$ so that

$$\left|< \frac{k_n}{||k_n||}, \frac{k_m}{||k_m||} >\right| \le \sigma \text{ for all } n \ne m. \tag{32}$$

Proof of Lemma 19. If (32) holds, then since $2ab \le a^2 + b^2$, we have

$$\log \frac{1}{|1 - \overline{z_n} z_m|^2} \le 2 \left(\sigma^2 \log \frac{1}{1 - |z_n|^2}\right)^{\frac{1}{2}} \left(\log \frac{1}{1 - |z_m|^2}\right)^{\frac{1}{2}} \le \sigma^2 \log \frac{1}{1 - |z_n|^2} + \log \frac{1}{1 - |z_m|^2}.$$

Exponentiating, we get

$$\frac{1}{|1 - \overline{z_n} z_m|^2} \le \frac{1}{(1 - |z_n|^2)^{\sigma^2}} \frac{1}{1 - |z_m|^2}.$$

Together with (12) this gives (31) with $\gamma = 1 - \sigma^2$. Conversely suppose (31) holds and without loss of generality, suppose that

$$\log \frac{1}{1 - |z_m|^2} \le \log \frac{1}{1 - |z_n|^2}.$$

By (12) and (31)

$$\log \frac{1}{|1 - \overline{z_n} z_m|^2} \le (1 - \gamma) \log \frac{1}{1 - |z_n|^2} + \log \frac{1}{1 - |z_m|^2}. \tag{33}$$

Note that if $0 \le a \le b \le (1 + \gamma)a$, then $(1 - \gamma)b + a \le (2 - \gamma^2)a^{\frac{1}{2}} b^{\frac{1}{2}}$. Thus if

$$\log \frac{1}{1 - |z_n|^2} \le (1 + \gamma) \log \frac{1}{1 - |z_m|^2},$$

then by (33)

$$\log \frac{1}{|1 - \overline{z_n} z_m|^2} \le 2(1 - \frac{\gamma^2}{2}) \left(\log \frac{1}{1 - |z_n|^2}\right)^{\frac{1}{2}} \left(\log \frac{1}{1 - |z_m|^2}\right)^{\frac{1}{2}}.$$

If

$$\log \frac{1}{1-|z_n|^2} \geq (1+\gamma) \log \frac{1}{1-|z_m|^2},$$

then

$$\begin{aligned}
\log \frac{1}{|1-\overline{z_n} z_m|} &\leq \log \frac{1}{1-|z_n||z_m|} \leq \log \frac{1}{1-|z_m|} \\
&\leq \left(\frac{1}{1+\gamma}\right)^{\frac{1}{2}} \left(\log \frac{1}{1-|z_n|^2}\right)^{\frac{1}{2}} \left(\log \frac{1}{1-|z_m|^2}\right)^{\frac{1}{2}}.
\end{aligned}$$

Since $|\arg(1-\overline{z_n} z_m)| \leq \pi/2$, if $1-|z_n|^2$ and $1-|z_m|^2$ are sufficiently small then

$$\left|\log \frac{1}{1-\overline{z_n} z_m}\right| \leq \sigma \left(\log \frac{1}{1-|z_n|^2}\right)^{\frac{1}{2}} \left(\log \frac{1}{1-|z_m|^2}\right)^{\frac{1}{2}},$$

for some $\sigma < 1$. This gives (32) for n and m sufficiently large. Since the inequality (32) is obviously true for any fixed z_m, (32) holds for all n and m with $n \neq m$ by increasing σ slightly. $\square$

If $\{z_n\}$ is an interpolating sequence for $\mathcal{M}_{\mathcal{D}}$ then for each pair z_n, z_m with $n \neq m$ we can find an $f \in \mathcal{M}_{\mathcal{D}}$ so that $f(z_n) = 1$, $f(z_m) = 0$, and $||f||_{\mathcal{M}_{\mathcal{D}}} \leq C$. If $\varphi = f/C$ then and Corollary 3,

$$1/C \leq 1 - \frac{|k_{n,m}|^2}{k_{n,n} k_{m,m}}$$

which is exactly (32) with $\sigma = (1-1/C)^{\frac{1}{2}}$. Thus by Lemma 19, if $\{z_n\}$ is an interpolating sequence (4) must hold. This completes the proof of Theorem 1. $\square$

The next Corollary follows immediately from Corollary 7 and Theorem 1.

Corollary 20. *A sequence $\{z_n\}$ is interpolating for $\mathcal{D}$ if and only if both (4) and (5) hold.*

If $z_n \in (0,1) \to 1$ then a standard estimate using (12) shows that there is a constant $\sigma < 1$ so that

$$(1-z_{n+1})^\sigma \leq (1-z_n)$$

for all n if and only if both (4) and (5) hold. This yields the following Corollary.

Corollary 21. *If $z_n \in (0,1)$ then z_n is an interpolating sequence for $\mathcal{M}_{\mathcal{D}}$ if and only if there is a constant $\sigma < 1$ so that*

$$(1-z_{n+1})^\sigma \leq (1-z_n)$$

for all n.

We can now answer Axler's question (i) by noting that for $\sigma < 1$,

$$z_n = 1 - \sigma^{\sigma^{-n}},$$

satisfies the hypotheses of Corollary 21 and hence is interpolating.

Since the inequalities in (4) and (5) are harder to satisfy if we replace z_n by $|z_n|$ for all n, we can answer Axler's question (ii).

Corollary 22. *If $\{z_n\} \subset \mathbb{D}$ and if there is a constant $\sigma < 1$ so that*

$$(1 - |z_{n+1}|)^\sigma \leq (1 - |z_n|)$$

for all n, then z_n is interpolating for $\mathcal{M}_\mathcal{D}$.

By Corollary 21, the growth rate in Corollary 22 is best possible.

There is a sequence $\{z_n\}$ which is not interpolating for $\mathcal{M}_\mathcal{D}$, yet there are functions $\varphi_n \in \mathcal{M}_\mathcal{D}$ with $\varphi_n(z_m) = \delta_{n,m}$ the Dirac delta function, and $||\varphi_n||_{\mathcal{M}_\mathcal{D}} \leq C$. To see this, we first give a sufficient condition for the existence of such functions.

Proposition 23. *Suppose $\mathcal{A}$ is a Hilbert space of analytic functions on $\mathbb{D}$ with reproducing kernels $k(\overline{\alpha}z)$ which satisfy the hypotheses of Corollary 11. Let $k_{i,j} =< k_{z_i}, k_{z_j} >$, $i, j = 0, 1, 2, \ldots,$ and suppose*

$$C = \prod_{i=1}^{\infty} \left(1 - \frac{|k_{i,0}|^2}{k_{i,i}k_{0,0}}\right)^{-1} < \infty.$$

Then there is a $\varphi \in \mathcal{M}_\mathcal{A}$ with $||\varphi||_{\mathcal{M}_\mathcal{A}} \leq C$, $\varphi(z_0) = 1$, and $\varphi(z_j) = 0$, for $j = 1, 2, \ldots$.

Proof. By Corollary 11, $\mathcal{A}$ has the Pick property. Fix m and let

$$M_m = \{k_{i,j}\}_{i,j=1,\ldots,m}$$

and

$$M_m^0 = \{k_{i,j}\}_{i,j=0,\ldots,m}$$

If $w_1 = w_2 = \ldots = w_m = 0$ then the Pick matrix

$$P_m = \{(1 - \overline{w_i}w_j)k_{i,j}\}_{i,j=0,\ldots,m}$$

satisfies

$$\det P_m = \det M_m^0 - |w_0|^2 k_{0,0} \det M_m.$$

Thus there is a $\varphi_m \in \mathcal{M}_{\mathcal{A}}$ with $||\varphi_m||_{\mathcal{M}_{\mathcal{A}}} < 1$, $\varphi_m(z_0) = w_0$, and $\varphi_m(z_j) = 0$, $j = 1, \ldots, m$ if and only if

$$|w_0|^2 < \frac{\det M_p^0}{k_{0,0} \det M_p} \tag{34}$$

for $p = 1, \ldots, m$. As in Shapiro and Shields [SS2], subtracting $k_{i,0}/k_{0,0}$ times the first row of M_m^0 from the i^{th} row does not affect the determinant. Thus

$$\det M_m^0 = k_{0,0} \det \left(\left\{ k_{i,j} - \frac{k_{i,0}\overline{k_{j,0}}}{k_{0,0}} \right\} \right).$$

Since

$$k_{i,j} - \frac{k_{i,0}\overline{k_{j,0}}}{k_{0,0}} = k_{i,j} \left[1 - \frac{k_{i,0}\overline{k_{j,0}}}{k_{i,j}k_{0,0}} \right],$$

by Lemma 10 and a result of Oppenheim[O],

$$\det M_m^0 \geq k_{0,0} \det M_m \prod_{i=1}^{m} \left(1 - \frac{|k_{i,0}|^2}{k_{i,i}k_{0,0}} \right).$$

Thus if

$$|w_0|^2 \leq \prod_{i=1}^{\infty} \left(1 - \frac{|k_{i,0}|^2}{k_{i,i}k_{0,0}} \right)$$

then there is a $\varphi \in \mathcal{M}_{\mathcal{A}}$ with $||\varphi||_{\mathcal{M}_{\mathcal{A}}} \leq 1$, $\varphi(z_0) = w_0$ and $\varphi(z_j) = 0$, for $j = 1, 2, \ldots$. This easily implies the Proposition. □

We remark that the determinant conditions (34) characterize the zero sets of $\mathcal{M}_{\mathcal{D}^\beta}$ by Corollary 13, though it is difficult to extract more geometric information than the sufficent condition of the Proposition. In the case of $\mathcal{A} = H^2$, the conditions (1) and (2) can be stated in terms of the product C, by interchanging the role of z_0 and each z_n (see (14)).

Example 24. *There is a sequence $\{z_n\} \subset \mathbb{D}$, $C < \infty$, and functions $\varphi_n \in \mathcal{M}_{\mathcal{D}}$ with $||\varphi_n|| \leq C$ for all n, and $\varphi_n(z_m) = \delta_{n,m}$, yet $\{z_n\}$ is not an interpolating sequence for $\mathcal{M}_{\mathcal{D}}$.*

Here $\delta_{n,m}$ denotes the Dirac delta function which equals 1 if $m = n$ and equals 0 if $m \neq n$.

Proof. As in [St] we construct a Cantor-like set. For $m \geq 1$, E_m consists of 2^m intervals of length e^{-2^m}. Let $E_0 = [0, e^{-1}]$ and let E_1 consists of two intervals $[0, e^{-2}] \cup [1 - e^{-2}, 1]$. In other words, we removed an open interval from the middle of E_0 so that the remaining intervals have length e^{-2}. Form E_m from E_{m-1} by removing an open interval from the middle of each interval in E_{m-1} . The removed intervals are chosen so that each interval in E_m has length e^{-2^m}. Fix m, let $r = 1 - e^{-2^m}$ and let $z_n = re^{i\theta_n}$ where θ_n is the center of the n-th interval in E_m, for $n = 1, \ldots, 2^m$.

Then by fairly straightforward estimates we have

$$1 - \rho^2(z_n, z_k) \leq 5(1 - |z_n|^2)$$

and

$$\sum_{z_n \in S(I)} \left(\log \frac{1}{1 - |z_n|} \right)^{-1} \leq \left(\log \frac{1}{|I|} \right)^{-1}$$

for all arcs $I \subset \partial\mathbb{D}$. We also have that

$$\sum_{z_n} \left(\log \frac{1}{1 - |z_n|} \right)^{-1} = 1.$$

However, (see [St]), $\mathrm{Cap}(E_m) \to 0$ and hence

$$\frac{1}{\log \frac{1}{\mathrm{Cap}(E_m)}} \to 0.$$

In other words, the sequences satisfy (4) uniformly and satisfy (5) uniformly *for intervals*, but do not satisfy (5) uniformly for all finite unions of intervals. By rescaling the initial interval E_0, and taking a union over a subsequence m_j, we obtain a sequence $\{z_n\}$ which does not satisfy (5), and hence is not interpolating, yet

$$\sum_{z_n \in S(I)} \left(\log \frac{1}{1 - |z_n|} \right)^{-1} \leq 2 \left(\log \frac{1}{|I|} \right)^{-1} \tag{35}$$

To prove the example, it suffices by Proposition 23 to show

$$\sup_m \sum_i \frac{|\log \frac{1}{1 - \overline{z_i} z_m}|^2}{\log \frac{1}{1 - |z_i|^2}} \leq C \log \frac{1}{1 - |z_m|^2}. \tag{36}$$

Let I_m be the arc on $\partial\mathbb{D}$ with center $z_m/|z_m|$ and length $2(1 - |z_m|)$. Then for $z_i \in S(I_m)$ we have

$$|\log \frac{1}{1 - \overline{z_i} z_m}| \leq C \log \frac{1}{1 - |z_m|^2},$$

for some universal constant $C < \infty$, and hence by (35)

$$\sum_{z_i \in S(I_m)} \frac{|\log \frac{1}{1 - \overline{z_i} z_m}|^2}{\log \frac{1}{1 - |z_i|^2} \log \frac{1}{1 - |z_m|^2}} \leq C \log \frac{1}{1 - |z_m|^2} \sum_{z_i \in S(I_m)} \frac{1}{\log \frac{1}{1 - |z_i|^2}} \leq 2C.$$

Let I_m^k be the arc with center $z_m/|z_m|$ and length $2(1-|z_m|)^{1/2^k}$, for $k=1,2,\ldots$. Then for $z_i \in S(I_m^k) \setminus S(I_m^{k-1})$ we have

$$\left|\log \frac{1}{1-\overline{z_i} z_m}\right| \leq \frac{C}{2^k} \log \frac{1}{1-|z_m|^2},$$

and hence by (35)

$$\sum_{k=1}^{\infty} \sum_{z_i \in S(I_m^k) \setminus S(I_m^{k-1})} \frac{|\log \frac{1}{1-\overline{z_i} z_m}|^2}{\log \frac{1}{1-|z_i|^2} \log \frac{1}{1-|z_m|^2}} \leq \sum_k \frac{C}{2^k} = 2C$$

proving (36) and completing the proof of the example. □

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