On the multiplier theorem for Fourier transforms of measures with compact support

Paul Malliavin*

Dedicated to Arne Beurling in respectful affection.

$\mathcal{M}$ will denote the algebra of measures defined on the real line and with compact support, $\mathcal{M}_a$ will denote the part of $\mathcal{M}$ consisting of measures having their support contained in an interval of length a. The main problem that we shall consider here can be formulated as follows: "how fast" can the Fourier transform $\hat{\mu}(x)$ of a measure $\mu \in \mathcal{M}_a$ $\mu \not\equiv 0$, tend to zero as x tends to $\pm\infty$ through real values? A well known result is that if we define $I(\hat{\mu})$ by

$$I(\hat{\mu}) = \int_{-\infty}^{+\infty} \log |\hat{\mu}(x)| \frac{dx}{1+x^2},$$

then for all $\mu \in \mathcal{M}$, $\mu \not\equiv 0$ we have

$$I(\hat{\mu}) > -\infty. \tag{1}$$

The condition (1) describes completely how fast the Fourier transform of a measure having its support in the half line can vanish at ∞. (This is a classical result; a proof can be found for instance in Paley—Wiener). However, it is obvious that more than the condition (1) is needed to give a complete description in the case of measures with compact support. We now define the notion of a multiplier. Let $S(x)$ be a given continuous function on the real line. We shall call a measure $\mu \in \mathcal{M}_\varepsilon$ a *multiplier of type* ε *for* S, provided that

$$\hat{\mu}(x)S(x) \quad \text{is bounded for } x \text{ real}.$$

* The results which are presented here come from a joint work with A. Beurling and were exposed in this form at the Stanford conference in 1961. A more concise proof of the Main Theorem was published in Acta Math. 1962 (107), p. 291—309. This paper will be denoted [A. M.]. It is also referred to [A. M.] for a bibliography.

We obtain using (1) an obvious necessary condition for the existence of a multiplier, namely

$$\int \log^{+} |S(x)| \frac{dx}{1+x^2} < \infty$$

or supposing, as we can do without any loss of generality, $|S(x)|>1$,

(2) $\qquad I(S) < \infty,$ (this integral being absolutely convergent).

Conversely we can state

Theorem 1. *Suppose that* $\log |S(x)|$ *is uniformly continuous, and that* (2) *is satisfied then* S *has multipliers of arbitrarly small type.*

Theorem 2. *Suppose that* S *is the restriction to the real axis of an entire function of exponential type and that* (2) *is satisfied. Then* S *has multipliers of arbitrary small type.*

The theorems 1 and 2 can be derived, as it is shown in [A. M.], from a more general statement given later in the Main Theorem 5. The Theorem 3 does not appear in [A. M]. and gives a link between potential theory and representation of even entire functions of exponential type which can be of interest for some other problems. The solution of the multiplier problem is based on the same extremal problem in potential theory as in [A. M.]. We state here this problem in several equivalent forms which show that it is equivalent (cf. Theorem 4) to the multiplier problem. We indicate also in the last remark an iterative scheme which converges to the solution of the extremal problem.

I. Reduction to a potential problem in one dimension

We shall denote by G the function

$$G(u) = \log\left|\frac{1+u}{1-u}\right|, \quad u > 0.$$

G is the restriction to the positive real axis R^* of the Green function for the half plane $\operatorname{Re} z>0$. If φ is a measure with support in R^* we shall denote by U^φ its potential defined by

$$U^\varphi = G * \varphi$$

where $*$ denotes the convolution on the multiplicative group R^*, that is

$$U^\varphi(x) = \int_0^\infty G\left(\frac{x}{t}\right) d\varphi(t).$$

Now we can state the following representation formula.

Theorem 3. *Let f be an even entire function of exponential type a such that*

$$I(f) < \infty, \quad \text{(this integral being absolutely convergent);}$$

then there exists a measure φ such that

$$\log |f(x)| = -xU^{\varphi}(x) \quad x > 0;$$

furthermore φ satisfies

(4) *For all $\varepsilon > 0$ the support of the negative part of $d\varphi + (a+\varepsilon)\frac{dt}{t}$ is compact*

(5) $$\lim_{R\to\infty} \int_0^R d\varphi$$

exists and is finite.

Conversely, let φ_1 be a measure satisfying (4) *and* (5); *then for every $\eta > 0$, it is possible to find a positive measure ϱ such that*

(6) $$h(z) = \int_0^\infty \log |1 - z^2 t^{-2}|\, d\varrho(t)$$

will satisfy

$$h(z) < \pi(a+\eta)|z| + O(1)$$

(7) $$h(x) = -xU^{\varphi_1}(x) + O(\log x), \quad \textit{for } x \textit{ real large enough.}$$

Remark. It is possible to go from the function h given by a positive continuous measure ϱ to an entire function of exponential type. We shall call a function h of the kind given by (6) and satisfying $\log |S(x)| + h(x) < 0$ a *multiplier in the wide sense.* (cf. [A.M.] p. 295—297).

Proof of Theorem 3. Let us first suppose that all the zeros of f are real; let $n(t)$=numbers of zeros of $f \in (0, t)$. Then

$$\log |f(x)| = \int_0^\infty \log |1 - x^2 t^{-2}|\, dn(t)$$

or by making an integration by part

$$\frac{\log |f(x)|}{x} = \text{P.V.} \int_0^\infty \frac{2x}{x^2 - t^2} \frac{n(t)}{t}\, dt$$

or by defining the measure φ by

$$\int_0^R d\varphi = \frac{n(R)}{R},$$

we get by another integration by part

$$\log |f(x)| = -x \int_0^\infty G\left(\frac{x}{t}\right) d\varphi(t) = -xU^{\varphi}(x).$$

Now the fact that $I(f)<\infty$ implies (cf., Boas, *Entire Functions*) that $\lim \frac{n(R)}{R}$ exists and is equal to the type of f. We have

$$d\varphi(t)=\frac{dn(t)}{t}-\frac{n(t)}{t}\frac{dt}{t}>-(a+\varepsilon)\frac{dt}{t}$$

if t is sufficiently large, and this proves (4).

Let us now consider the case where the zeros of f are not real. We shall reduce it in an obvious way to the case where f has only real zeros if we prove the following lemma.

Lemma 1. *Let f be an even entire function of exponential type such that*

$$I(f)<\infty;$$

then there exists a measure $d\mu$, positive, with support on R^*, such that

$$\log|f(x)|=\int_0^\infty \log|1-x^2t^{-2}|\,d\mu(t), \quad \textit{for} \quad x\in R^*.$$

Furthermore, denoting by $N(r)$ the numbers of zeros of f in $|z|<R$ we have

$$N(R)=2\int_0^R d\mu(t)+o(R). \tag{8}$$

Proof. Let

$$W_\theta(x)=\log|1-x^2e^{-2i\theta}|,$$

Let Λ the sequence of zeros of f contained in the angle

$$\frac{-\pi}{2}<\arg z\leqq\frac{\pi}{2}$$

and let

θ_λ=argument of λ,

$\delta_{|\lambda|}$=Dirac Mass put at the point $|\lambda|$.

With these notations the Weierstrass factorization of f can be written, $*$ denoting always the convolution on the multiplicative group of positive reals, in the following form:

$$\log|f(x)|=\sum_{\lambda\in\Lambda}(W_{\theta_\lambda}*\delta_{|\lambda|})(x).$$

Now we shall use a factorization form of the balayage

$$W_\theta=W_0*K_\theta$$

where

$$K_\theta(x) = \frac{2}{\pi}\frac{x(x^2+1)\,|\sin\theta|}{x^4-2x^2\cos 2\theta+1};$$

factorization which can be proved by looking at the Mellin transform of both members. Then we will get

$$\log|f(x)| = W_0 * \sum_{\lambda\in\Lambda} K_{\theta_\lambda} * \delta_{|\lambda|}$$

the interversion of integration used to obtain this formula being justified by the fact that $I(f)<\infty$ implies

(9) $$\sum |\theta_\lambda|\,|\lambda|^{-1}<\infty$$

which with the fact

$$K_\theta(x) = O(\theta x) \quad 0<x<\frac{1}{2}$$

implies the absolute convergence of

$$\sum K_{\theta_\lambda} * \delta_{|\lambda|}.$$

Let $d\mu$ be the measure equal to the sum of this series. $d\mu$ is positive. Finally (8) follows from (9), as an elementary computation shows.

Now with Lemma 1 the first part of the Theorem 3 is proved in the case of complex zeros as in the case of real zeros.

Proof of the constructive part of Theorem 3: We now have to show how, given a potential U^{φ_1}, we can construct a function h, of the form given in (6), such that (7) holds. Denote by

$$\varphi_1(R) = \int_0^R d\varphi_1, \quad \beta = \lim_{R\to\infty}\varphi(R)$$

and let ω be the measure defined by its differential $d\omega$:

$$d\omega = d\big(x[\varphi_1(x)-\beta+a+\eta]\big).$$

We then have

$$d\left(\frac{\omega(x)}{x}\right) = d\varphi_1(x);$$

hence

$$\int_0^\infty \log|1-x^2t^{-2}|\,d\omega(t) = -xU^{\varphi_1}(x).$$

Furthermore, for x sufficiently large, $x>M$, ω is a positive measure. Let ϱ = positive part of ω. Then

$$\int_0^\infty \log|1-x^2t^{-2}|[d\varrho(t)-d\omega(t)] = \int_0^M \log|1-x^2t^{-2}|[d\varrho(t)-d\omega(t)],$$

and this last integral is $O(\log x)$ for x large, which proves (7) as

$$\int \log\left|1-\frac{z^2}{t^2}\right| d\varrho(t) \leqq \int \log\left|1+\frac{r^2}{t^2}\right| d\varrho(t) \quad \text{where} \quad r=|z| \quad \text{and}$$

$$\frac{1}{\pi r}\int \log\left|1+\frac{r^2}{t^2}\right| d\varrho(t) = \int Q\left(\frac{r}{t}\right)\frac{\varrho(t)}{t}\frac{dt}{t} \quad \text{where} \quad Q(v)=\frac{2v}{\pi(1+v^2)}.$$

When $r\to\infty$ the last integral is majorized by the lim sup of $t^{-1}\varrho(t)$.

II. Solution of an extremal problem

We shall reduce the multiplier problem to a problem on the potentials U^φ. Let

$$\sigma(x)=\frac{\log|S(x)|}{|x|}$$

(We can suppose without loss of generality that $\sigma>0$, that $\sigma(x)=O(1)$ near zero, and that $\sigma(x)$ is even — if not, we shall introduce $S_1(x)=S(x)S(-x)$). Now a being a positive number given, let us consider the convex set $\mathscr{A}=\mathscr{A}(\sigma, a)$ of measures φ defined by

$$\mathscr{A}(\sigma, a)=\left\{\varphi \,|\, U^\varphi \geqq \sigma \quad \text{and} \quad d\varphi \geqq -a\frac{dt}{t}\right\}. \tag{10}$$

Now according to Theorem 3 we have: *S has multipliers in the wide sense of type arbitrarily small if and only if for every $b>0$ there exists $\varphi\in\mathscr{A}(\sigma, b)$ such that*

$$\lim_{R\to\infty}\int_0^R d\varphi \tag{11}$$

exists and is finite.

We can delete the factor $c\log x$ appearing in (7) multiplying by the entire function $\dfrac{\sin \varepsilon z}{P(z)}$ where P is a polynomial of degree $>c$.

The next step consists in replacing the condition (11) by a condition on U^φ. As $\sigma>0$ we have, for all $\varphi\in\mathscr{A}(\sigma, a)$,

$$U^\varphi>0.$$

Let

$$I(\varphi)=\int_0^\infty U^\varphi(x)\frac{dx}{x}. \tag{12}$$

Then $I(\varphi)$ is a number finite or infinite well defined. We have

Lemma 2. *The condition* (11) *holds if and only if*

$$I(\varphi) < \infty.$$

Proof. We have

$$\int_0^R U^\varphi(x)\frac{dx}{x} = \int_0^\infty G\left(\frac{R}{t}\right)\varphi(t)\frac{dt}{t}.$$

If $\varphi(t)$ tends to a finite limit when $t\to\infty$ it will be the same for the first integral as $R\to\infty$.

To prove the converse we shall proceed as follows. Let α denote a given number $\alpha>1$, and denote by h_α a four times differentiable function such that

$$h_\alpha(t) = 1, \quad \text{for} \quad t > \alpha$$

$$h_\alpha(t) = 0 \quad \text{for} \quad t < 1.$$

Then let k_α be the bounded function defined by the convolution equation

$$U^{\omega_\alpha} = h_\alpha \quad \text{where} \quad d\omega_\alpha = k_\alpha(t)\frac{dt}{t}.$$

We have on the Mellin transforms $\hat{k_\alpha}(s)=(\frac{\pi}{2}s)\operatorname{cotg}(\frac{\pi}{2}s)\hat{h_\alpha}(s)$, $-1<\operatorname{Re} s<0$. Then

$$k_\alpha(t)\to\frac{2}{\pi} \quad \text{when} \quad t\to\infty.$$

Hence

$$\lim_{x\to\infty}\int h_\alpha\left(\frac{x}{t}\right)d\varphi(t) = \frac{2}{\pi}\int_0^\infty U^\varphi(t)\frac{dt}{t}.$$

This is true for all $\alpha>1$. Furthermore,

$$d\varphi \geqq -b\frac{dt}{t}.$$

These two facts imply that (11) is satisfied and the lemma is proved.

Now we can state our extremal problem:

Minimize the integral $I(\varphi)$ when $\varphi\in\mathscr{A}(\sigma, b)$.

Let us denote by σ_b the function defined by

(13) $$\sigma_b(x) = \inf U^\varphi(x) \quad \text{for all} \quad \varphi\in\mathscr{A}(\sigma, b).$$

Now using the theorem of the infimum envelope of a family of potentials, we see that

(14) $$\sigma_b = U^{\theta_b}$$

where θ_b is a measure satisfying

$$d\theta_b \geqq -b\frac{dt}{t}.$$

Hence $\theta_b \in \mathscr{A}(\sigma, b)$ and as we obviously have

$$l(\theta_b) \leqq l(\varphi) \quad \text{for all} \quad \varphi \in \mathscr{A}(\sigma, b)$$

θ_b will give us the solution of our extremal problem. We then have

Theorem 4. *S has multipliers in the wide sense of arbitrarily small type if and only if for all* $b>0$

$$\int_1^{+\infty} \sigma_b(x)\frac{dx}{x} < \infty \tag{15}$$

(where σ_b *is defined by (10) and (13)).*

Now the problem is to evaluate (15). We shall do that using the properties of the extremal measure θ_b. Let us denote by $\| \; \|_2$ the energy norm on the measures defined by

$$\|\varphi\|_2 = \left(\int U^\varphi \, d\varphi\right)^{\frac{1}{2}}$$

If σ can be written as a potential

$$\sigma = U^\varrho,$$

we shall denote by $\mathscr{D}(\sigma)$ the Dirichlet integral of σ defined by

$$\mathscr{D}(\sigma) = \|\varrho\|_2^2. \tag{16}$$

This integral coincides with the integral defined in [A. M.], formula 3.2. We shall denote by $(u|v)_{\mathscr{D}}$ the scalar product associated to the Dirichlet integral.

Let us define

$$\mathscr{D}_1(\sigma) = \int_0^\infty |\sigma(x)|^2 \frac{dx}{x} + \int_{-\infty}^{+\infty} \frac{du}{u^2} \int_0^\infty |\sigma(xe^u) - \sigma(x)|^2 \frac{dx}{x}. \tag{17}$$

Then looking at the Mellin transform of G it is easy to show that $\mathscr{D}(\sigma)$ and $\mathscr{D}_1(\sigma)$ define equivalent norms. (cf. A.M. p. 303)

We can now state our main theorem.

Main Theorem 5. *Suppose that* σ *has a finite Dirichlet integral and that* (2) *holds; then S has multipliers in the wide sense of arbitrary small type.*

Proof. We shall study an extremal problem which has as extremal function the function σ_b introduced in Theorem 3. We will not use this fact in the proof, we mention it only to explain the success of the method. The proof of this fact will be found in Remark 1 at the end of this paper.

Recall that a function f, such that $\mathcal{D}(f)<\infty$ is called a *pure potential* if there exists a positive measure ω such that

$$f = U^{\omega}.$$

Then the following is a well known and elementary fact *f is a pure potential if and only if*

$$(f\,|\,g)_{\mathcal{D}} > 0 \quad \textit{for all} \quad g > 0.$$

We have the following use of this characterization.

Lemma 3. *Let f be a given function such that*

$$\mathcal{D}(f) < \infty.$$

Let $B_f = \{g\,|\,\mathcal{D}(g) < \infty$ and $g \geqq f$ almost everywhere for the Lebesgue measure$\}$.
Denote by f^* the projection of the origin on the closed convex set B_f. Then
f^* is a pure potential: $f^* = U^{\omega}$,
$f^* = f$ almost everywhere for the measure ω.
For every pure potential h we have

$$\mathcal{D}(f^* - h) \leqq \mathcal{D}(f - h).$$

Proof. If $g > 0$ we have

$$f^* + tg \in B_f \quad \text{for all} \quad t > 0.$$

Hence

$$\mathcal{D}(f^* + tg) \geqq \mathcal{D}(f^*) \quad t > 0$$

which implies

$$(f^*|g)_{\mathcal{D}} > 0 \quad \text{for all} \quad g > 0;$$

hence f^* has to be a pure potential.

Now if $t > -1$ we have

$$(1+t)f^* \geqq (1+t)f.$$

Hence

$$f^* + t(f^* - f) \in B_f \quad \text{for} \quad t > -1,$$

which implies as before

$$(f^*|f^* - f)_{\mathcal{D}} = 0$$

or

$$\int (f^* - f)\, d\omega = 0.$$

As $\omega > 0$, $f^* > f$, this implies $f^* = f$ a.e. for the measure ω. Finally

$$\mathcal{D}(f^* - h) - \mathcal{D}(f - h) = \mathcal{D}(f^*) - \mathcal{D}(f) + (h|f - f^*)_{\mathcal{D}}.$$

Now if h is a pure potential the last integral is negative as $f-f^*<0$. From the definition of f^* we have

$$\mathscr{D}(f^*) \leqq \mathscr{D}(f)$$

and this proves the lemma.

Lemma 4. *Let σ be a given function such that*

$$\mathscr{D}(\sigma) < \infty,$$

and let ψ be an absolutely continuous positive measure given, such that the restriction of ψ to every compact set K is of finite energy.

Then there exists a measure θ, such that

$\omega = \theta + \psi$ is a positive measure;

$U^\theta \geqq \sigma$ almost everywhere for the Lebesgue measure,

$U^\theta = \sigma$ ω-almost everywhere and

$$\int (U^\theta - \sigma)\, d\psi < \infty.$$

Proof. Let K_N be an increasing sequence of compact sets, $\cup K_N$ being the positive real line. Let ψ_N be the restriction of ψ to K_N and let us apply the lemma 3 to the function

$$f_N = \sigma + U^{\psi_N};$$

denote by f_N^* the projection of the origin to the convex set B_{f_N}. Then

$$f_N^* = U^{\omega_N} \quad \omega_N > 0.$$

Let

$$\theta_N = \omega_N - \psi_N.$$

We have by Lemma 3

$$\|\theta_N\|_2^2 = \mathscr{D}(f_N^* - U^{\psi_N}) < \mathscr{D}(\sigma), \tag{18}$$

$$U^{\theta_N} \geqq \sigma \quad \text{a.e. for the Lebesgue measure}$$

$$\int (U^{\theta_N} - \sigma)\, d(\theta_N + \psi_N) = 0.$$

Let us now select a sequence θ_{N_j} which converges weakly to a measure θ. We will still have

$$U^\theta \geqq \sigma \quad \text{a.e.}$$

Also

$$\int (U^{\theta_N} - \sigma)\, d\theta_N = -\int (U^{\theta_N} - \sigma)\, d\psi_N.$$

Denote by h_N the characteristic function of K_N; this equality gives us using (18)

$$\int (U^{\theta_N} - \sigma) h_N\, d\psi < 2\mathscr{D}(\sigma).$$

As we integrate positive functions this means that

$$\int (U^\theta - \sigma)\, d\psi < 2\mathscr{D}(\sigma) < \infty.$$

Proof of Theorem 5. Let b be a positive number given; denote by ψ the measure

$$d\psi = b\frac{dx}{x}.$$

and apply Lemma 4 to (σ, ψ). Then we construct in this way a measure θ such that

$$d\theta \geqq -b\frac{dx}{x}.$$

We have

$$\int U^\theta \frac{dx}{x} = \int (U^\theta - \sigma)\frac{dx}{x} + \int \sigma \frac{dx}{x}$$

By (2) the last integral of the right hand side is convergent. Using the lemma 4 we get that the first integral is convergent and so

$$\int U^\theta \frac{dx}{x} < \infty$$

To finish we have only to remark that the fact that $d\theta > -b\frac{dx}{x}$ implies that U^θ is lower semi-continuous. With the hypothesis that $S(x)$ is continuous and $|S(x)|>1$, we get that

$$U^\theta - \sigma$$

is a lower semi continuous function, positive almost everywhere, hence everywhere positive and this proves the theorem.

Remark 1. We shall prove

$$\sigma_b = U^\theta$$

Obviously $\theta \in \mathscr{A}(\sigma, b)$ and therefore $\sigma_b \leqq U^\theta$. We shall denote by $\tilde{U}^\theta(z)$ the extension of U^θ at the half plane $\operatorname{Re} z>0$:

$$\tilde{U}^\theta(z) = \int_0^{+\infty} \log\left|\frac{z+t}{z-t}\right| d\theta(t), \quad \operatorname{Re} z > 0$$

Let K be the support of $\omega = \theta + b\frac{dt}{t}$, and denote by O the complement of K in $\operatorname{Re} z>0$. Then on K

$$\sigma \leqq U^{\theta_b} \leqq U^\theta = \sigma.$$

Therefore $\tilde{U}^{\theta_b - \theta}$ is a superharmonic function on O, which vanishes on ∂O, which implies $\sigma_b \geqq U^\theta$.

Remark 2. The class $\mathscr{A}(\sigma, l)$ is defined by two inequalities. We have a partition of R^+ in (K, K^c) such that, for the extremal function σ_b, on each part, one of these inequalities become an equality.

$$\begin{cases} U^{\theta_b} = \sigma & \text{on} \quad K \\ d\theta_b = -b\dfrac{dt}{t} & \text{on} \quad K^C \end{cases} \tag{19}$$

Conversely, a measure $\tilde{\theta} \in \mathscr{A}(\sigma, b)$ satisfying such equalities on a partition $\tilde{K}$, $\tilde{K}^C$ of R^+ is the extremal measure θ_b (cf. Remark 1). Therefore (19) *characterizes the extremal measure.*

This type of equalities (19) is now in frequent use in the theory of *variational inequalities.*

When σ is the potential of a measure ϱ it is possible to describe a nonlinear algorithm B which by iteration, gives the extremal measure θ_b.

Denote by χ^- the negative part of a measure χ $(\chi = \chi^+ - \chi^-)$. Introduce

$$\left(\varrho + b\frac{dt}{t}\right)^- = \tilde{\varrho}, \quad \left(\varrho + b\frac{dt}{t}\right)^+ = \hat{\varrho}.$$

Let H_1 be the support of $\hat{\varrho}$ and denote by $(\tilde{\varrho})^*$ the *swept measure* (mesure balayée) of $\tilde{\varrho}$ on H_1.

Define

$$\varrho_1 = B(\varrho) = \mathbf{1}_{H_1}\varrho - \mathbf{1}_{K_1} b\frac{dt}{t} - (\tilde{\varrho})^*.$$

where K_1 is the complement of H_1 in R^*, $\mathbf{1}_{H_1}$, $\mathbf{1}_{K_1}$ being the characteristic functions of H_1, K_1.

Then

$$U^{\varrho_1}(x) = U^{\varrho}(x), \quad x \in H_1$$

$$\left(\varrho_1 + b\frac{dt}{t}\right)\mathbf{1}_{K_1} = 0$$

$$U^{\varrho_1}(x) \geqq U^{\varrho}(x), \quad x \in R^+$$

Then

$$\int U^{\varrho_1}(x)\, d\varrho_1(x) = \int_{H_1} U^{\varrho}(x)\, d\varrho_1 - \int_{K_1} U^{\varrho_1}(x)\, b\frac{dx}{x}$$

We remark that

$$d\varrho_1 < d\varrho \quad \text{on} \quad H_1,$$

$$U^{\varrho_1} \geqq U^{\varrho} > 0.$$

Therefore

$$\mathscr{D}(\varrho_1) \leqq \mathscr{D}(\varrho). \tag{20}$$

Define

$$\varrho_2 = B(\varrho_1), \dots, \varrho_n = B(\varrho_{n-1}).$$

Then $\{\mathscr{D}(\varrho_n)\}$ is a *decreasing sequence*. Therefore we can find λ with

$$\mathscr{D}(\lambda) \leqq \mathscr{D}(\varrho)$$

such that a subsequence ϱ_{n_j} converges weakly to λ.

We remark that

$$H_n \subset H_{n-1}.$$

Therefore we have

$$d\lambda = -b\frac{dt}{t} \quad \text{on} \quad \cup K_n$$

On the other hand,

$$U^\lambda = \lim U^{\varrho_{n_j}} \quad \text{implies}$$

$$U^\lambda(x) = U^\varrho(x) \quad \text{on} \quad \cap H_n.$$

Then λ satisfies the system (19). Furthermore $B(\lambda)=\lambda$ implies $\lambda\in\mathscr{A}(\sigma, b)$. Therefore λ is the extremal measure θ_b. It is clear that this iterative scheme gives an independent construction of the multiplier in the case where σ is the potential of a measure. This last restriction can be overcome by regularization and in fact we can prove the main theorem 5 completely by this approach.

The finiteness of the Dirichlet norm of σ appears in this approach only as a technical tool which by (20) insures the convergence of the sequence of iterates $B^n(\varrho)$, and the estimate of $\int U^\lambda(x)\frac{dx}{x}$.

It is possible that the main Theorem is still valid with L^1-capacity-type of condition on σ (instead of the L^2-capacity-type of condition consisting of the finiteness of the Dirichlet norm). Then a possible start to prove that could be this iterative construction.

Received August 31, 1977

Paul Malliavin
10 Rue Saint Louis en l'Isle
75 004 Paris
France