MEROMORPHIC INNER FUNCTIONS, TOEPLITZ KERNELS, AND THE UNCERTAINTY PRINCIPLE

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Introduction

This paper touches upon several traditional topics of 1D linear complex analysis including distribution of zeros of entire functions, completeness problem for complex exponentials and for other families of special functions, some problems of spectral theory of selfadjoint differential operators. Their common feature is the close relation to the theory of complex Fourier transform of compactly supported measures or, more generally, Fourier-Weyl-Titchmarsh transforms associated with selfadjoint differential operators with compact resolvent.

The last part of the title is a reference to the monograph [19], which contains a large collection of results that could be described by the (informal) statement: "it is impossible for a non-zero function and its Fourier transform to be simultaneously very small." For example, if a function is supported on a small interval, then the set of zeros of its Fourier transform has to be sparse. Another example: a small amount of information about the potential of a Schrödinger operator requires a large amount of information about the spectral measure to determine the operator uniquely.

Our goal is to present a unified approach to certain problems of this type. The method is based on the reduction (complete or partial) to the injectivity problem for Toeplitz operators, which makes it possible to use the full strength of the theory of Hardy-Nevanlinna classes and Hilbert transform rather than rely on the standard techniques of the theory of entire function (Jensen type formulae, canonical products, Phragmen-Lindelöf principle). We have been able to reinterpret various classical results (often with shorter proofs and stronger conclusions) in a way that allows further generalizations. We explain the reason for such generalizations (and their nature) below. We begin by briefly describing the background.

0.1. **Complex exponentials, Paley-Wiener spaces, and Cartwright functions.** If $a > 0$ and $\Lambda \subset \mathbf{C}$, then by the very definition of the classical Fourier transform,

$$f(t) \mapsto \hat{f}(z) = \int e^{izt} f(t) dt, \qquad (0.1)$$

The first author is supported by N.S.F. Grant No. 0201893.
The second author is supported by N.S.F. Grant No. 020699.

the family of exponential functions $\mathcal{E}_\Lambda = \left\{e^{i\lambda t}: \ \lambda \in \Lambda\right\}$ is not complete in $L^2(-a,a)$ if and only if there is a non-trivial function F from the *Paley-Wiener space*

$$\mathrm{PW}_a = \{\hat{f}: \ f \in L^2(-a,a)\},$$

such that $F = 0$ on Λ. According to Paley-Wiener's theorem, see [33], PW_a can be characterized as the space of entire functions which have exponential type at most a and are square summable on the real line $\mathbb{R}$.

The *Cartwright class* $\mathrm{Cart_a}$, $a \geq 0$, consists of entire functions F of exponential type at most a satisfying a weaker integrability condition on $\mathbb{R}$:

$$\log^+|F| \in L^1(\mathbb{R}, \Pi), \qquad d\Pi(t) := \frac{dt}{1+t^2}.$$

Cartwright functions are considered in detail in the monographs [29], [6], [28], [12], [25], [19]. The following *Krein's theorem* [26] is important for the purpose of this discussion: an entire function is Cartwright if and only if it belongs to the Nevanlinna class in the lower and in the upper halfplanes. In a sense, Paley-Wiener spaces are to Cartwright spaces as the Hardy space $\mathcal{H}^2$ is to the Smirnov-Nevanlinna class $\mathcal{N}^+$. The precise meaning of this analogy, which is only true up to an arbitrarily small gap in the exponential type, is a part of the deep Beurling-Malliavin theory [4]–[5].

The growth limitations (exponential type and integrability conditions on $\mathbb{R}$) on Paley-Wiener and Cartwright functions carry with them severe limitations on the distribution of zeros. This is a central theme of the classical theory of entire functions, and numerous results have been obtained in the study of the relation between the growth and zeros. Let us mention some principle facts. For a set Λ we denote by $N(R)$ the number of points $\lambda \in \Lambda$ satisfying $|\lambda| < R$, and by $N_\pm(R)$ the number of points λ, $|\lambda| < R$, such that $\Re\lambda > 0$ and $\Re\lambda < 0$ respectively.

Density and symmetry of zeros. If F is a non-trivial Cartwright function and Λ is the set of all zeros, then the limits

$$\lim_{R\to\infty} \frac{N_\pm(R)}{R}$$

exist, are finite and equal. Moreover, there exists a finite limit

$$\text{v.p.} \sum \frac{1}{\lambda} \equiv \lim_{R\to\infty} \sum_{|\lambda|<R} \frac{1}{\lambda}.$$

See, e.g., [28], [25].

Levinson's completeness theorem. If F is a non-trivial function in PW_a and $F = 0$ on Λ (so $\mathcal{E}_\Lambda$ is not complete in $L^2[-a,a]$), then

$$\lim_{R\to\infty} \left[\int_1^R \frac{N(t)}{t} dt - \frac{2a}{\pi} R + \frac{1}{2}\log R\right] = -\infty.$$

Some other sufficient conditions for completeness are known, as well as some necessary conditions, see [38], [41]. On the other hand, finding an effective general metric criterion does not seem to be realistically possible.

Beurling-Malliavin theory. The most interesting and deep result in the completeness problem is the metric characterization of the "radius of completeness",

$$R(\Lambda) = \sup\{a : \ \mathcal{E}_\Lambda \quad \text{complete in} \quad L^2(-a,a)\},$$

obtained in [4]-[5]. Beurling and Malliavin described $R(\Lambda)$ as a certain "density" $d_{\rm BM}(\Lambda)$, which is defined in a non-trivial but completely computable way. We recall the definition of $d_{\rm BM}(\Lambda)$ and discuss the Beurling-Malliavin theory in the last part of the paper.

Let us now explain how the completeness problem can be restated in terms of Toeplitz kernels.

0.2. **Model spaces and Toeplitz operators.** By the Paley-Wiener theorem, the Fourier transform (0.1) identifies $L^2(0,\infty)$ with the Hardy space $\mathcal{H}^2$ in the upper halfplane $\mathbf{C}_+$, and therefore it identifies $L^2(\pm a,\infty)$ with $S^{\pm a}H^2$. Here and *throughout* the paper, S denotes the singular inner function

$$S(z) = e^{iz}.$$

It follows that the Paley-Wiener spaces have the following representation:

$$\mathrm{PW}_a = S^{-a}\left[\mathcal{H}^2 \ominus S^{2a}\mathcal{H}^2\right].$$

The subspace $\mathcal{H}^2 \ominus S^{2a}\mathcal{H}^2$ is the so called model space of the inner function S^{2a}. More generally, one defines model spaces

$$K_\Theta \equiv K[\Theta] = \mathcal{H}^2 \ominus \Theta\mathcal{H}^2$$

for all inner functions Θ in $\mathbf{C}_+$; these spaces play an important role in the modern function theory and also in the spectral theory, see [34], [27]. Strictly speaking, the elements of K_Θ are functions in $\mathbf{C}_+$ but if Θ is a meromorphic inner function, then every element has a meromorphic continuation to the whole complex plane. In particular, the completeness problem for exponentials is exactly the problem of describing the uniqueness sets of the model spaces $K[S^{2a}]$.

If $U \in L^\infty(\mathbb{R})$, then the *Toeplitz operator* with *symbol* U is the map

$$T_U : \mathcal{H}^2 \to \mathcal{H}^2, \qquad F \mapsto P_+(UF),$$

where P_+ is the orthogonal projection in $L^2(\mathbb{R})$ onto $\mathcal{H}^2$. A one line argument shows that $\Lambda \subset \mathbf{C}_+$ is a uniqueness set of K_Θ if and only if the kernel of the Toeplitz operator T_U is trivial, where

$$U = \bar{\Theta}B_\Lambda, \qquad B_\Lambda \quad \text{Blaschke product.}$$

(There is a similar statement for general sets $\Lambda \subset \mathbf{C}$, see Section 3.1.) The *injectivity problem* – to characterize symbols U such that ker $T_U = 0$ – is of interest in its own right as part of the spectral theory of Toeplitz operators, see [8], [35]. Compared with some other aspects of the theory such as invertibility problem, the injectivity problem has attracted relatively little attention. Let us mention the important paper [23] where the idea to use (invertibility) properties of Toeplitz operators in the study of bases of exponentials was introduced, see also [3].

We can now explain our goal more clearly. We would like to see if the classical results mentioned above could be extended to Toeplitz operators with more general symbols. More precisely, we'll be considering real-analytic symbols $U = e^{i\gamma}$, $\gamma \in$

$C^\omega(\mathbb{R})$, so that infinity is the only "singularity" of the symbol. A special case (which is in fact just as general, see [9]) is the case of symbols of the form

$$U = \bar{\Theta}J,$$

where Θ and J are meromorphic inner functions. We will show now that the injectivity problem for operators with such symbols appears as naturally as in the special case $\Theta = S^{2a}$.

0.3. **Spectral theory.** Consider the Schrödinger equation

$$-\ddot{u} + qu = \lambda u \tag{0.2}$$

on some interval (a, b) and assume that the potential $q(t)$ is locally integrable and a is a regular point, i.e. a if finite and q is L^1 at a. Let us fix some selfadjoint boundary condition at b and consider the Weyl m-function

$$m(\lambda) = \frac{\dot{u}_\lambda(a)}{u_\lambda(a)}, \qquad \lambda \notin \mathbb{R},$$

where $u_\lambda(t)$ is any non-trivial solution of (0.2) satisfying the boundary condition. We will deal only with the *compact resolvent* case, which is equivalent to saying that m extends to a meromorphic function. Then we can define the meromorphic inner function

$$\Theta = \frac{m - i}{m + i},$$

which we call the *Weyl inner function* associated with the potential and the fixed boundary condition at b. The transformation

$$f(t) \ \mapsto \ F(\lambda) = \int_a^b f(t) \frac{u_\lambda(t)}{\dot{u}_\lambda(a) + iu_\lambda(a)} \, dt \tag{0.3}$$

identifies $L^2(a, b)$ with the model space K_Θ in the same way as the classical Fourier transform (times S^a) identifies $L^2(-a, a)$ with $K[S^{2a}]$. This allows us to interprete the completeness problem for families of solutions $\{u_\lambda : \ \lambda \in \Lambda\}$ as a problem of uniqueness sets in the model space of Θ. Completeness problems of this type, particularly problems involving families of special functions, are well-known in the literature, see e.g. [20]. As we explained, they are equivalent to the invertibility problem for symbols $\bar{\Theta}J$, where Θ is a Weyl inner function.

Similar invertibility problems appear in connection with the uniqueness part of the inverse spectral problem. We discuss such applications in Section 3. Here we only mention that the results represent a rather broad generalization of such well-known facts as *Borg's two spectra theorem* [7]: two different spectra of a Schrodinger operator with compact resolvent determine the operator uniquely, and *Hochstadt-Liberman theorem* [21] that states that a regular Schrödinger operator on a finite interval is determined by its spectrum and the potential on one half of the interval. In the language of inner functions, the corresponding problems can be stated as follows.

Given a meromorphic inner function Θ and one of its factors Ψ, the problem is to decide whether this factor and the set $\{\Theta = 1\}$ determine Θ uniquely. The second problem is to describe *defining sets* of a given meromorphic inner function Φ. We say that $\Lambda \subset \mathbb{R}$ is defining if

$$\tilde{\Phi} = \Phi, \quad \arg \tilde{\Phi} = \arg \Phi \quad \text{on} \quad \Lambda \qquad \Rightarrow \qquad \tilde{\Phi} \equiv \Phi.$$

0.4. **Content of the paper.**

Section 1. *Meromorphic inner functions and spectral theory.*

1.1–1.4: We recall standard facts concerning meromorphic inner functions, their model spaces, and Weyl-Titchmarsh functions of second order selfadjoint differential operators.

1.5: We discuss the modified Fourier transform (0.3). In the case of regular operators, this is essentially the usual Weyl-Titchmarsh transform, but the construction is probably new in the (more interesting) singular case.

1.6–1.8: Basics of de Branges functions and associated spaces of Paley-Wiener and Cartwright type.

Section 2. Toeplitz kernels.

2.1-2.4: We define the kernels and state some general (mostly known) facts. As an example, we give a Toeplitz kernel interpretation of a standard asymptotic formula for solutions of a regular Schrödinger equation.

2.5–2.6: Basic criterion for non-triviality of a Toeplitz kernel with real analytic symbol. For instance, the kernel is non-trivial in the Smirnov-Nevanlinna class if and only if the argument γ of the symbol has a representation $\gamma = -\alpha + \tilde{h}$, where $\alpha \in C^\omega$ is an increasing function and $h \in L^1_\Pi$. Though this observation is very simple (and its versions in the non-analytic case are well known), the criterion turns out to be quite workable, and the rest of the paper is mostly the study and applications of this criterion.

2.7–2.8: Kolmogorov's type criterion. This is a special case where the symbol is $\bar{H}/H$, H is an outer function, and the kernel is a priory finite dimensional. This situation is typical when we explicitly know the de Branges functions. Another useful example is the twin inner function theorem: if $\{\Theta = 1\} = \{J = 1\}$, then $\ker\, T_{\bar{\Theta} J} = 0$

2.9–2.11: General form of Levinson's completeness theorem. We obtain a sufficient condition for triviality of a Toeplitz kernel that improves Levinson's theorem (and other similar results) even in the classical situation. The key ingredient of the proof is the Titchmarsh-Uly'anov theorem involving the so called A-integrals.

Section 3. *Some applications.*

3.1–3.2: Completeness and minimality problem, and uniqueness sets of the model and de Branges spaces.

3.3–3.4: Distribution of zeros of functions in Cartwright-de Branges spaces. In particular, we give a new proof of the density and symmetry result mentioned in Section 0.1, which is based on our basic criterion and the Titchmarsh-Uly'anov theorem.

3.5–3.7: Applications to the mixed data spectral problem stated in Section 0.3. For the inner function version of the Hochstadt-Liberman problem we establish some necessary and some sufficient conditions in terms of Toeplitz kernels. We also give a spectral theory interpretation of these conditions indicating stronger versions of practically all known results in this area.

3.8–3.9: Remarks on defining sets of inner functions and regular Schrödinger operators.

Section 4. *Beurling-Malliavin theory.*

4.1-4.2: Multiplier theorems. First we state a multiplier theorem for Toeplitz kernels in $\mathcal{H}^p$-spaces. Then we discuss some consequences of the Beurling-Malliavin multiplier theorem for Toeplitz kernels in the Smirnov-Nevanlinna class. We make no comments on the proof of the Beurling-Malliavin multiplier theorem itself. The presence of the Dirichlet space condition remains the most amazing feature of the theory.

4.3–4.6: Second Beurling-Malliavin and little multiplier theorems. For symbols $U = e^{i\gamma}$ with $\gamma' > -$const, we present a complete proof of the metric criterion for (non-)triviality of a Toeplitz kernel in the Smirnov-Nevanlinna class up to a gap $S^{\pm\epsilon}$. The proof is of course not totally original but our version, we believe, is better fit for generalizations.

4.7–4.9: We discuss possible generalizations of the Beurling-Malliavin theory. We mention partial results, examples, and indicate applications in the case $\gamma'(t) > -$const $|t|^a$.

1. Meromorphic inner functions and spectral theory

Function Theory in the halfplane

1.1. **Basic notations.** $\mathbf{C}_+$ is the upper half plane $\{\Re z > 0\}$. For general references concerning Hardy-Nevanlinna theory in $\mathbf{C}_+$ see [16] and [35].

We use the standard notation $\mathcal{H}^p = \mathcal{H}^p(\mathbf{C}_+)$, $0 < p \leq \infty$, for the Hardy spaces, and $\mathcal{N}^+ = \mathcal{N}^+(\mathbf{C}_+)$ for the V. I. Smirnov (or Smirnov-Nevanlinna) class in $\mathbf{C}_+$. The elements in $\mathcal{N}^+$ are ratios G/H, where $G, H \in H^\infty$ and H is an outer function. Functions in $\mathcal{N}^+$ have angular boundary values (almost everywhere) on the real line. As a general rule, we identify functions in the halfplane with their boundary values on $\mathbb{R}$. In this sense, we have

$$\mathcal{H}^p = \mathcal{N}^+ \cap L^p(\mathbb{R}),$$

and

$$F \in \mathcal{N}^+ \qquad \Rightarrow \qquad \log|F| \in L^1_\Pi \equiv L^1(\mathbb{R}, \Pi),$$

where Π is the Poisson measure

$$d\Pi(t) = \frac{dt}{1+t^2}.$$

If $h \in L^1_\Pi$ is a real-valued function, then its *Schwarz integral* is

$$\mathcal{S}h(z) = \frac{1}{\pi i}\int \left[\frac{1}{t-z} - \frac{t}{1+t^2}\right] h(t)dt.$$

The real and the imaginary parts of $\mathcal{S}h$ are the *Poisson* and the *conjugate Poisson integrals* of h:

$$\mathcal{S}h = \mathcal{P}h + i\mathcal{Q}h.$$

Outer functions are functions of the form

$$H = e^{\mathcal{S}h}, \qquad h \in L^1_\Pi;$$

note that $H \in \mathcal{N}^+$ and H has modulus e^h on $\mathbb{R}$. Every function F in $\mathcal{N}^+$ has a unique factorization $F = IH$, where H is the outer function with modulus $|F|$ on $\mathbb{R}$ and I is an *inner* function, i.e. $I \in \mathcal{H}^\infty$ and $|I| = 1$ on $\mathbb{R}$.

The *Hilbert transform* of $h \in L^1_\Pi$ is the angular limit of Qh, so the outer function $e^{\mathcal{S}h}$ is equal to $e^{h+i\tilde{h}}$ on $\mathbb{R}$. The Hilbert transform can also be defined as a singular integral:

$$\tilde{h}(x) = \frac{1}{\pi} \text{ v.p.} \int \left[\frac{1}{x-t} + \frac{t}{1+t^2} \right] h(t)dt.$$

For further references, we recall some properties of the Hilbert transform. If both h and $g = \tilde{h}$ are in L^1_Π, then $\tilde{g} = -h + \text{const}$, i.e.

$$\mathcal{S}\tilde{h} = -i\mathcal{S}h + i\mathcal{S}h(i).$$

If $h \in L^1_\Pi$, then $\tilde{h} \in L^{o(1,\infty)}_\Pi$ (the weak L^1 space), i.e.

$$\Pi\{|\tilde{h}| > A\} = o\left(\frac{1}{A}\right), \qquad A \to \infty,$$

in particular $\tilde{h} \in L^p_\Pi$ for all $p < 1$.

1.2. **Meromorphic inner functions and Herglotz functions.** A meromorphic inner function is an inner function Θ in $\mathbf{C}_+$ which has a meromorphic extension to $\mathbf{C}$. Such a function can be characterized by parameters (a, Λ) in the canonical (Riesz-Smirnov) factorization

$$\Theta = B_\Lambda S^a, \tag{1.1}$$

where $a \geq 0$, and Λ is a discrete set (possibly with multiple points) in $\mathbf{C}_+$ satisfying the Blaschke condition

$$\sum \frac{\Im\lambda}{1+|\lambda|^2} < \infty;$$

B_Λ denotes the corresponding Blaschke product and $S^a(z) = e^{iaz}$. Let us mention an obvious but important property of meromorphic inner functions:

$$\Theta = e^{i\theta} \quad \text{on } \mathbb{R}, \qquad \theta \text{ is a real analytic, increasing function.}$$

A meromorphic Herglotz function is a meromorphic functions m such that

$$\Im m > 0 \quad \text{in } \mathbf{C}_+, \qquad m(\bar{z}) = \overline{m(z)}.$$

One can establish a 1-to-1 correspondence between meromorphic inner and Herglotz functions by means of the equations

$$m = i\frac{1+\Theta}{1-\Theta}, \qquad \Theta = \frac{m-i}{m+i}. \tag{1.2}$$

Meromorphic Herglotz functions (and therefore inner functions) can be described by parameters (b, c, μ) in the Herglotz representation

$$m(z) = bz + c + iS\mu, \tag{1.3}$$

where $b \geq 0$, $c \in \mathbb{R}$, and μ is a positive discrete measure on $\mathbb{R}$ satisfying

$$\int \frac{d\mu(t)}{1+t^2} < \infty.$$

It is convenient to interprete the number πb as a point mass of μ at infinity. In the case $m = m_\Theta$, see (1.2), we call this extended measure μ_Θ the *spectral* (or *Herglotz) measure* of Θ. By definition, the (point) *spectrum* of Θ is the set

$$\sigma(\Theta) = \operatorname{supp} \mu_\Theta = \ \{\Theta = 1\} \text{ or } \{\Theta = 1\} \cup \{\infty\},$$

and by residue calculus we have

$$\mu_\Theta(t) = \frac{2\pi}{|\Theta'(t)|}, \qquad t \in \sigma(\Theta). \tag{1.4}$$

The following *equivalent* conditions are necessary and sufficient for $\mu_\Theta(\infty) \neq 0$, see e.g. [36]:

$$(i) \quad \Theta - 1 \in H^2; \qquad (ii) \quad \Theta(\infty) = 1,\ \exists\Theta'(\infty); \qquad (iii) \quad \sum \Im\lambda < \infty.$$

In (ii), $\Theta(\infty)$ and $\Theta'(\infty)$ mean the angular limit and angular derivative at infinity:

$$\Theta(\infty) = \lim_{y\to+\infty} \Theta(iy), \qquad \Theta'(\infty) = \lim_{y\to+\infty} y^2\Theta'(iy),$$

and in (iii) we also require that the singular factor is trivial.

Riesz-Smirnov and Herglotz parametrizations (1.1)-(1.3) reflect two different structures – multiplicative and convex – in the set of inner functions. These structures are related in a non-trivial and intriguing way. For example, the middle point of the segment $[\Theta_1, \Theta_2]$, i.e. the inner function such that its Herglotz measure is the average of μ_{Θ_1} and μ_{Θ_2}, is the function

$$\Theta = \frac{\Theta_1 + \Theta_2 - 2\Theta_1\Theta_2}{2 - \Theta_1 - \Theta_2},$$

and we observe that

$$\Psi|\Theta_1, \quad \Psi|\Theta_2 \qquad \Rightarrow \qquad \Psi|\Theta,$$

where the notation $\Psi|\Theta$ for two inner functions means that Ψ is a *factor* of Θ, i.e. Θ/Ψ is also an inner function.

1.3. **Model spaces.** The $\mathcal{H}^2$-model space of an inner function Θ,

$$K_\Theta \equiv K[\Theta] = \mathcal{H}^2 \ominus \Theta\mathcal{H}^2 = \mathcal{H}^2 \cap \Theta\bar{\mathcal{H}}^2,$$

is a Hilbert space with reproducing kernel:

$$k_\lambda^\Theta(z) = \frac{1}{2\pi i}\frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{\bar{\lambda} - z}, \qquad \lambda \in \mathbf{C}_+. \tag{1.5}$$

If Θ is meromorphic, then all elements of K_Θ are meromorphic, and one can extend (1.5) to all $\lambda \in \mathbb{R}$.

The monograph [34] provides a comprehensive study of model spaces. One of the important facts of the theory is the following Plancherel theorem (see Clark's paper [10] for the case of general inner functions): *the restriction map*

$$\mathcal{C}_\Theta : \ f \mapsto f|_{\sigma(\Theta)} \tag{1.6}$$

is a unitary operator $K_\Theta \to L^2(\mu_\Theta)$.

We also define the model spaces in the Smirnov class and in general Hardy spaces:

$$K_\Theta^+ = \{F \in \mathcal{N}^+ \cap C^\omega(\mathbb{R}) : \ \Theta \bar{F} \in \mathcal{N}^+\},$$

and

$$K_\Theta^p = K_\Theta^+ \cap L^p(\mathbb{R}).$$

If $p \geq 1$, we can drop the requirement $F \in C^\omega(\mathbb{R})$ by Morera's theorem.

Second order differential operators

1.4. **Weyl inner functions.** Meromorphic inner functions appear in the theory of 2nd order selfadjoint differential operators with *compact* resolvent. We will only discuss the case of Schrödinger operators though similar theories exist for general canonical systems. See [32] and [30] for the basics of the spectral theory.

Let q be a real locally integrable function on (a, b). We always assume that selfadjoint operators associated with the differential operation $u \mapsto -\ddot{u} + qu$ have *compact resolvent.* We suppose that a is a *regular* point but we allow b to be infinite and/or singular. Let us fix a selfadjoint boundary condition β at b; for example, β means $u \in L^2$ at b in the limit point case. The Weyl-Titchmarsh m-function of $(q; b, \beta)$ evaluated at a,

$$m(\lambda) = m^a_{b,\beta}(\lambda), \qquad \lambda \in \mathbf{C},$$

is defined by the formula

$$m(\lambda) = \frac{\dot{u}_\lambda(a)}{u_\lambda(a)},$$

where $u_\lambda(\cdot)$ is a non-trivial solution of the Schrödinger equation satisfying the boundary condition at b. It is well-known that m is a Herglotz function, and therefore we can define the corresponding inner function $\Theta^a_{b,\beta}$ by (1.2). We will call $\Theta^a_{b,\beta}$ the *Weyl* (or Weyl-Titchmarsh) *inner function* of q.

Similarly, if $b \in \mathbb{R}$ is a *regular* point and α is a selfadjoint boundary condition at $a \in [-\infty, b)$, we can consider the m-function of $(q; a, \alpha)$ evaluated at b,

$$m^b_{a,\alpha}(\lambda) = -\frac{\dot{u}_\lambda(b)}{u_\lambda(b)}$$

(mind the sign!) and define the corresponding Weyl inner function $\Theta^b_{a,\alpha}$.

Example. The Weyl inner functions of the potential $q \equiv 0$ on $[0, 1]$ with Dirichlet and, respectively, Neumann boundary conditions at $a = 0$ are

$$\Theta_D(\lambda) = \frac{\sqrt{\lambda}\cos\sqrt{\lambda} + i\sin\sqrt{\lambda}}{\sqrt{\lambda}\cos\sqrt{\lambda} - i\sin\sqrt{\lambda}}, \qquad \Theta_N(\lambda) = \frac{\sqrt{\lambda}\sin\sqrt{\lambda} - i\cos\sqrt{\lambda}}{\sqrt{\lambda}\sin\sqrt{\lambda} + i\cos\sqrt{\lambda}}. \tag{1.7}$$

(The m-functions are $m_D(\lambda) = -\sqrt{\lambda}\cot\sqrt{\lambda}$, and and $m_N(\lambda) = \sqrt{\lambda}\tan\sqrt{\lambda}$.) More generally, for $\nu \geq -1/2$ consider the potential

$$q(t) = \frac{\nu^2 - \frac{1}{4}}{t^2} \quad \text{on} \quad (0,1),$$

and let the boundary condition α at $a = 0$ be satisfied by the solution

$$u_\lambda(t) = \sqrt{t}J_\nu(t\sqrt{\lambda})$$

of the Schrödinger equation. For example, if $\nu = -1/2$ then $\alpha = (N)$, and if $\nu = 1/2$ then $\alpha = (D)$, and we have the limit point case if $\nu \geq 1$. J_ν is of course the standard notation for the Bessel function of order ν. Since

$$u_\lambda(1) = J_\nu(\sqrt{\lambda}), \qquad \dot{u}_\lambda(1) = \frac{1}{2}J_\nu(\sqrt{\lambda}) + \sqrt{\lambda}J_\nu'(\sqrt{\lambda}),$$

the corresponding Weyl inner function is

$$\Theta_\nu(\lambda) = \frac{\sqrt{\lambda}J_\nu'(\sqrt{\lambda}) + (1/2+i)J_\nu(\sqrt{\lambda})}{\sqrt{\lambda}J_\nu'(\sqrt{\lambda}) + (1/2-i)J_\nu(\sqrt{\lambda})}. \tag{1.8}$$

In particular, we have $\Theta_{-1/2} = \Theta_N$ and $\Theta_{1/2} = \Theta_D$.

One can give many other similar examples involving special functions. We will continue to discuss Bessel inner function in Sections 1.6 and 3.7. Our goal is to illustrate certain constructions in the singular case as opposed to the regular case, which is well presented in the literature.

1.5. **Modified Fourier transform.** Let $\Theta = \Theta^a_{b,\beta}$ be the Weyl-Titchmarsh inner function of a potential q defined in the previous section. We will construct a unitary operator $L^2(a,b) \to K_\Theta$, which is a modification of the Weyl-Titchmarsh Fourier transform. We modify the usual construction so that the case of a singular endpoint b could be included.

For every $z \in \mathbf{C}$ we choose a non-trivial solution $u_z(t)$ of the Schrödinger equation satisfying the boundary condition β. (For real z such a solution exists because of the compact resolvent assumption). If $z \in \mathbf{C}_+ \cup \mathbb{R}$, then the solution

$$w_z(t) = \frac{u_z(t)}{\dot{u}_z(a) + iu_z(a)}$$

does not depend on the choice of u_z, and $w_z \in L^2(a,b)$. The transform $\mathcal{W}$ is defined as follows:

$$\mathcal{W}: \ f(t) \mapsto F(z) = \int_a^b f(t)w_z(t)dt, \quad (z \in \mathbf{C}_+ \cup \mathbb{R}). \tag{1.9}$$

To state the main result we introduce the *dual reproducing kernel* of the model space K_Θ. For $\lambda \in \mathbf{C}_+ \cup \mathbb{R}$ we define

$$k^*_\lambda(z) = \frac{1}{2\pi i}\frac{\Theta(z) - \Theta(\lambda)}{z - \lambda}, \qquad (z \in \mathbf{C}_+ \cup \mathbb{R}), \tag{1.10}$$

so we have

$$\bar{\Theta}k^\Theta_\lambda = \overline{k^*_\lambda} \qquad \text{on} \quad \mathbb{R},$$

and $k^*_\lambda \in K_\Theta$. Note that if $\lambda \in \mathbb{R}$, then $k^*_\lambda = \text{const}\ k^\Theta_\lambda$.

Theorem. *The modified Fourier transform $\mathcal{W}$ is (up to a factor $\sqrt{\pi}$) a unitary operator $L^2(a,b) \to K_\Theta$. Furthermore, we have*

$$\mathcal{W}w_\lambda = \pi k^*_\lambda, \quad \mathcal{W}\bar{w}_\lambda = \pi k_\lambda \qquad (\lambda \in \mathbf{C}_+ \cup \mathbb{R}). \tag{1.11}$$

Proof: The formulae (1.11) follow from the Lagrange identity

$$(z-\lambda)\int_a^b u_\lambda u_z = u_\lambda(a)\dot{u}_z(a) - \dot{u}_\lambda(a)u_z(a).$$

(The Wronskian at b is zero because the two solutions satisfy the same boundary conditions.) The rest is straightforward:

$$(\bar{w}_\lambda, \bar{w}_\mu)_{L^2} = \int_a^b w_\mu \bar{w}_\lambda = \mathcal{W}\bar{w}_\lambda(\mu) = \pi k_\lambda(\mu) = \pi(k_\lambda, k_\mu)_{K_\Theta},$$

etc. □

Note that Weyl inner functions of Schrödinger operators have no point masses at infinity, so if $\Theta = \Theta^a_{b,\beta}$, then

$$\sigma(\Theta) = \sigma(q, D, \beta), \qquad \sigma(-\Theta) = \sigma(q, N, \beta).$$

Here $\sigma(q, D, \beta)$ means the spectrum of the Schrödinger operator with potential q, Dirichlet boundary condition at a, and boundary condition β at b. More generally, for $\alpha \in \mathbb{R}$ let α denote the following selfadjoint boundary condition at a regular endpoint a:

$$\cos\frac{\alpha}{2}u(a) + \sin\frac{\alpha}{2}\dot{u}(a) = 0. \tag{1.12}$$

Then

$$\sigma(e^{-i\alpha}\Theta) = \sigma(q, \alpha, \beta).$$

By definition, the spectral measure of the Schrödinger operator (q, α, β) is the Herglotz measure of the inner function $e^{-i\alpha}\Theta$.

Corollary. *Let $\Theta = \Theta^a_{b,\beta}$. The composition of the modified Fourier transform and the Plancherel-Clark operator* (1.6),

$$L^2(a,b) \xrightarrow{\mathcal{W}} K_\Theta \xrightarrow{\mathcal{C}_\Theta} L^2(\mu_\Theta),$$

is a unitary operator; it provides a spectral representation of the Schrödinger operator (q, D, β).

Entire functions

1.6. **de Branges functions.** Following [28] we say that an entire function E is of *Hermite-Biehler class* (HB) if E has no real zeros and

$$z \in \mathbf{C}_+ \quad \Rightarrow \quad |E(\bar{z})| < |E(z)|.$$

Every $E \in$(HB) defines a meromorphic inner function

$$\Theta_E = \frac{E^\#}{E}, \qquad E^\#(z) := \overline{E(\bar{z})}.$$

Conversely, given an inner function Θ, any $E \in (HB)$ satisfying $\Theta = \Theta_E$ is called a *de Branges function of* Θ.

It can be shown, see [12], that *every meromorphic inner function has at least one de Branges function.* In some cases one can construct de Branges functions explicitly.

Examples. (i) Let (a, b) be a finite interval and $q \in L^1(a, b)$. Given a selfadjoint boundary condition (1.12) at a, let $u_\lambda(t)$ denote the solution of the initial value problem

$$u_\lambda(a) = -\sin\frac{\alpha}{2}, \qquad \dot{u}_\lambda(a) = \cos\frac{\alpha}{2}. \tag{1.13}$$

for the Schrödinger equation. Then

$$E(\lambda) = -\dot{u}_\lambda(b) + iu_\lambda(b) \tag{1.14}$$

is a de Branges function of the Weyl inner function $\Theta = \Theta^b_{a,\alpha}$. Indeed, we have

$$\Theta(\lambda) = \frac{-\dot{u}_\lambda(b) - iu_\lambda(b)}{-\dot{u}_\lambda(b) + iu_\lambda(b)},$$

and the functions $\lambda \mapsto u_\lambda(b)$ and $\lambda \mapsto \dot{u}_\lambda(b)$ are entire because of the fixed initial conditions; clearly, they can not be both zero at the same point $\lambda \in \mathbb{R}$.

(ii) Consider now the Bessel inner functions Θ_ν, see (1.8). Note that the above construction does not apply in the singular case. From the theory of Bessel's functions we know that

$$J_\nu(z) = z^\nu G_\nu(z),$$

where G_ν is an even real entire functions and $G_\nu(0) \neq 0$. We also introduce an even real entire function

$$F_\nu(z) = zG'_\nu(z).$$

Since $zJ'_\nu = z^\nu(\nu G_\nu + F_\nu)$, we have

$$\Theta_\nu(\lambda) = \frac{F_\nu(\sqrt{\lambda}) + (1/2 + \nu + i)G_\nu(\sqrt{\lambda})}{F_\nu(\sqrt{\lambda}) + (1/2 + \nu - i)G_\nu(\sqrt{\lambda})}.$$

The function

$$E_\nu(\lambda) := F_\nu(\sqrt{\lambda}) + (1/2 + \nu - i)G_\nu(\sqrt{\lambda})$$

does not vanish at $\lambda = 0$ and therefore it has no zeros on $\mathbb{R}$. It follows that E_ν is a de Branges function of Θ_ν. Similar considerations work for other special functions.

(iii) Some more elementary (but important) examples include the following. If $a > 0$, then $E(z) = e^{-iaz}$ is a de Branges function of $\Theta = S^{2a}$. Polynomials with all roots in $\mathbf{C}_-$ are de Branges functions of finite Blaschke products.

1.7. **Spaces of entire functions.** We first define the *Cartwright-de Branges* space $B^+(E)$ of entire functions associated with a Hermite-Biehler function E:

$$B^+(E) = \left\{F : \; F/E, \; F^\#/E \in \mathcal{N}^+(\mathbf{C}_+)\right\}.$$

Proposition. $B^+(E) = EK^+[\Theta_E]$.

Proof: If $G \in K_\Theta^+$, then its meromorphic extention to $\mathbf{C}_-$ is equal to $H^\#/\Theta^\#$ for some $H \in \mathcal{N}^+(\mathbf{C}_+)$. Since $E = \Theta^\# E^\#$ in $\mathbf{C}_-$, the function

$$F = \begin{cases} E(z)\ G(z), & z \in \mathbf{C}_+ \\ E^\#(z) H^\#(z), & z \in \mathbf{C}_- \end{cases}$$

is entire, and $F \in B^+(E)$. The opposite direction is similar. $\square$

The special case $E = S^{-a}$ gives *Cartwright spaces*

$$\mathrm{Cart_a} = \mathrm{B}^+\left(\mathrm{S}^{-a}\right) = \mathrm{S}^{-a}\mathrm{K}^+\left[\mathrm{S}^{2a}\right], \qquad (\mathrm{a} \geq 0).$$

Next we define the *de Branges space* associated with $E \in (HB)$, see [12]:

$$B(E) = B^+(E) \cap L^2\left(|E(x)|^{-2}dx\right) = EK\left[\Theta_E\right].$$

The special case $E = S^{-a}$ gives the *Paley-Wiener spaces*

$$\mathrm{PW}_a = B\left(S^{-a}\right) = \mathrm{Cart_a} \cap \mathrm{L}^2(\mathbb{R}), \qquad (\mathrm{a} > 0).$$

De Branges space $B(E)$ has a natural Hilbert space structure so that the multiplication operator

$$E: \ K\left[\Theta_E\right] \to B(E)$$

is an isometry. We denote by K_λ^E, ($\lambda \in \mathbf{C}$), the reproducing kernel of $B(E)$. Theorem 1.5 has the following corollary, which is a counterpart of the Paley-Wiener theorem concerning the classical Fourier transform. Recall that $\mathcal{W}$ denotes the modified Fourier transform (1.9).

Corollary. *Let E be a de Branges function of the Weyl inner function $\Theta^a_{b,\beta}$ associated with a potential q on (a,b). Then the map*

$$\mathcal{F}: L^2(a,b) \to B(E), \qquad f \mapsto E \cdot \mathcal{W}f,$$

is a unitary operator. Furthermore, we have

$$\mathcal{F}u_\lambda = \mathrm{const}\ K_\lambda^E, \qquad \lambda \in \mathbf{C},$$

where u_λ is any non-trivial solution of the Schrödinger equation satisfying the boundary condition β.

In the *regular* case $a, b \in \mathbb{R}$ and $q \in L^1(a,b)$, the map $\mathcal{F}$ is precisely the Weyl-Titchmarsh transform

$$f \mapsto \int_a^b f(t) u_\lambda(t) dt,$$

where the solutions $u_\lambda(t)$ are normalized by initial conditions (1.13) and the de Branges function is given by (1.14).

The classical Fourier transform (0.1) originates from the 1-st order selfadjoint operator $u \mapsto -iu'$. Alternatively, it can be related to the Weyl-Titchmarsh transforms corresponding to $q \equiv 0$ by a general construction which we describe below, cf. [14].

1.8. **Square root transformation.** It is well-known that if m is a Herglotz function such that

$$0 < m < +\infty \qquad \text{on} \quad \mathbb{R}_-,$$

then $m^*(\lambda) = \lambda m(\lambda^2)$ is again a Herglotz function. If $\Theta = (m-i)/(m+i)$, then the inner function corresponding to m^* is

$$\Theta^*(z) = \frac{(z+1)\Theta(z^2) + (z-1)}{(z-1)\Theta(z^2) + (z+1)}$$

We call Θ^* the square root transform of Θ. Suppose now that $E = A + iB$ is a de Branges function of Θ, where A and B are real entire functions, and also suppose $B(0) \neq 0$. Then

$$E^*(z) = zA(z^2) + iB(z^2)$$

is a de Branges function of Θ^*.

Example. Let $q \in L^1[0,1]$ be such that the operator $L(q, D, N) \geq 0$. Consider the function

$$m(\lambda) = \frac{u(\lambda)}{\dot{u}(\lambda)} := \frac{u_\lambda(1)}{\dot{u}_\lambda(1)},$$

where u_λ is the solution of the Schrödinger equation with initial conditions $u_\lambda(0) = 0$ and $\dot{u}_\lambda(0) = 1$. (In other words, m is the Herglotz function of $-\Theta^1_{0,D}$.) Then we have $E = u + i\dot{u}$ and therefore

$$E^*(z) = zu(z^2) + i\dot{u}(z^2).$$

In particular, if $q \equiv 0$, then $E^*(z) = ie^{-iz}$, and we get the classical Paley-Wiener space.

2. Toeplitz kernels

Some generalities

2.1. **Definition of Toeplitz kernels.** Recall that to every $U \in L^\infty(\mathbb{R})$, there corresponds the Toeplitz operator $T_U : \mathcal{H}^2 \to \mathcal{H}^2$. We need to consider only the case of *unimodular* symbols

$$U = e^{i\gamma}, \qquad \gamma : \mathbb{R} \to \mathbb{R},$$

and we will concentrate on the question whether the *Toeplitz kernel*

$$N[U] = \ker\ T_U$$

is trivial or non-trivial. The best known situation is when $\gamma \in C(\mathbb{R})$ and there exist finite limits $\gamma(\pm\infty)$. (This corresponds to the case of piecewise continuous symbols in the theory on the unit circle.) If we denote

$$\delta = \gamma(+\infty) - \gamma(-\infty),$$

then

$$N[U] = 0 \quad \text{if} \quad \delta > -\pi, \qquad N[U] \neq 0 \quad \text{if} \quad \delta < -\pi.$$

(If $\delta = -\pi$, then either case is possible.)

Along with $\mathcal{H}^2$-kernels, we define Toeplitz kernels in the Smirnov class,

$$N^+[U] = \{F \in \mathcal{N}^+ \cap L^1_{\text{loc}}(\mathbb{R}) : \ \bar{U}\bar{F} \in \mathcal{N}^+\},$$

and in all Hardy spaces,

$$N^p[U] = N^+[U] \cap L^p(\mathbb{R}), \qquad (0 < p \le \infty).$$

These definitions are oriented to studying the case where ∞ is the only "singularity" of the symbol. In particular, if Θ is a meromorphic inner function, then $N^+[\bar{\Theta}] = K^+_\Theta$ and $N^p[\bar{\Theta}] = K^p_\Theta$.

We use the notation b for the Blaschke factor

$$b(z) = \frac{i-z}{i+z}.$$

The argument $2\arctan(x)$ of b increases from $-\pi$ at $-\infty$ to $+\pi$ at $+\infty$. One can characterize the dimension of a Toeplitz kernel by multiplying the symbol by integer powers of b.

Lemma. *For $n \in \mathbf{N}$,* $\dim\ N^p[U] = n+1$ *iff* $\dim\ N^p[b^n U] = 1$.

Proof: If, for instance, $\dim\ N^p[U] \ge 2$, then we can find an $F \in N^p[U]$ such that $F(i) = 0$, and so $\bar{b}F \in N^p[bU]$ and $\dim\ N^p[bU] \ge 1$. In the opposite direction, if $G \in N^p[bU]$, then both G and bG are in $N^p[U]$. □

One can also consider fractional powers of b:

$$b^s(x) = \exp\{2si \arctan x\}, \qquad (s \in \mathbb{R}).$$

The identity

$$\bar{b}^s(1-b)^s = (\bar{b}-1)^s$$

shows that $N^\infty[\bar{b}^s] \ne 0$ for $s \ge 0$. It follows that for every U and $p > 0$ there is a critical value $s_* \in \mathbb{R} \cup \{\pm\infty\}$ such that

$$N^p[\bar{b}^s U] \ne 0 \quad \text{if} \quad s > s_*, \qquad N^p[\bar{b}^s U] = 0 \quad \text{if} \quad s < s_*.$$

One can interpret s_* as a fractional and possibly negative "dimension" of the kernel. For example, "dim"$N[1] = -1/2$, but "dim"$N[\bar{\Theta}_N \Theta_D] = -1/4$ for the Dirichlet and Neumann inner functions (1.7), see Section 2.7 below.

2.2. **Basic criterion.** The following well-known observation is extremely simple but quite useful. In fact, most of our further constructions are built upon this lemma.

Lemma. $N^p[U] \ne 0$ *iff the symbol has the following representation:*

$$U = \bar{\Phi}\frac{\bar{H}}{H},$$

where $H \in \mathcal{H}^p \cap L^1_{\text{loc}}(\mathbb{R})$ is an outer function and Φ is an inner function.

Proof: If $UF = \bar{G}$, then $|F| = |G|$ on $\mathbb{R}$. Consider the inner-outer factorization: $F = F_i F_e$ and $G = G_i G_e$. We have $F_e = G_e$, and

$$U = (\bar{F}_i \bar{G}_i) \frac{\bar{F}_e}{F_e}.$$

The converse is obvious. $\square$

Corollary. *If $\gamma \in L^\infty \cap C(\mathbb{R})$, then $\exists p > 0$, "dim"$N^p[e^{i\gamma}] > -\infty$.*

A more precise statement is

$$\|\gamma\|_\infty < \frac{\pi}{p} \qquad \Rightarrow \qquad N^p[\bar{b}^{2/p} e^{i\gamma}] \neq 0,$$

which follows from the Smirnov-Kolmogorov estimate

$$\|h\|_\infty < \frac{\pi}{2} \qquad \Rightarrow \qquad e^{\tilde{h}} \in L^1_\Pi,$$

and from the construction of outer functions.

Of course, instead of $\|\gamma\|_\infty < \infty$ we can only require that γ be the sum of a decreasing and a BMO functions, and we don't need continuity if $p \geq 1$. It is also important to realize that p can not be arbitrary in the statement of the corollary. For instance, it is easy to construct $\gamma \in C^\omega(\mathbb{R})$ such that $\|\gamma\|_\infty = \pi/2$ but $N[\bar{b}^n e^{i\gamma}] = 0$ for all $n > 0$.

2.3. **Sufficient conditions for** $\dim N^p[u] < \infty$**.** The following statement is a version of Coburn's lemma, which states that either ker $T_U = 0$ or ker $T_{\bar{U}} = 0$.

Lemma. *If $1/p + 1/q > s$, then*

$$N^q[\bar{U}] \cap L^2_{\text{loc}}(\mathbb{R}) \neq 0 \quad \Rightarrow \quad N^p[\bar{b}^s U] \cap L^2_{\text{loc}}(\mathbb{R}) = 0.$$

Proof: Suppose both kernels are non-trivial:

$$\bar{U} F_1 = \bar{G}_1, \qquad \bar{b}^s U F_2 = \bar{G}_2$$

for some $F_1, G_1 \in \mathcal{H}^q$ and $F_2, G_2 \in \mathcal{H}^p$. Then

$$(i + z)^s F_1(z) F_2(z) = (i - \bar{z})^s \bar{G}_1(\bar{z}) \bar{G}_2(\bar{z}) \qquad \text{on} \quad \mathbb{R},$$

so we have an entire Cartwright function with at most a polynomial growth along $i\mathbb{R}$. The growth at $+i\infty$ is

$$y^s y^{-1/q} y^{-1/p},$$

so the entire function is zero if $1/p + 1/q > s$. $\square$

Corollary. *If γ is the sum of an increasing and a bounded functions, then for all $p > 0$, $\dim N^p[e^{i\gamma}] < \infty$.*

Corollary. *If $U = \bar{H}/H$ and H is an outer function such that*

$$\exists q > 0 \quad \exists N, \quad \frac{1}{H} \in L^q \left(\frac{dt}{1 + |t|^N} \right),$$

then for all $p > 0$, $\dim\ N^p[U] < \infty$.

2.4. **Trivial factors.** The following lemma is obvious.

Lemma. *If $V = \bar{H}/H$ with $H^{\pm 1} \in \mathcal{H}^\infty$, then $N^p[UV] \neq 0$ iff $N^p[U] \neq 0$.*

We will call such functions V trivial factors of the symbol. *Examples:*

(i) If B_1 and B_2 are *finite* Blaschke products of the same degree, then $N^p[U] = 0$ iff $N^p[\bar{B}_1 B_2 U] = 0$.
(ii) If we modify a smooth symbol on a compact part of $\mathbb{R}$, then this does not affect (non-)triviality of the Toeplitz kernels. Thus the injectivity property depends only on the behavior of a smooth symbol at infinity.
(iii) It is shown in [9] that up to a trivial factor every unimodular function is the ratio of two inner functions.

More relevant to the subject of the paper is our next

Example: *Weyl inner functions of regular operators.*

Let $q \in L^1[0,1]$ and let α be a *non-Dirichlet* selfadjoint boundary condition at $a = 0$. Denote by Θ the Weyl inner function of (q, α) computed at $b = 1$, i.e. $\Theta = \Theta^b_{\alpha,a}$ in the notation of Section 1.4. We want to compare Θ with the "Neumann" inner function Θ_N, see (1.7), which corresponds to the special case $q \equiv 0$ and $\alpha = (N)$.

Claim. *The ratio Θ/Θ_N is a trivial factor.*

In other words, in all problems involving Toeplitz kernels we are free to replace regular potentials with the trivial potential, and any non-Dirichlet boundary condition with the Neumann condition.

Proof: We will express the ratio of the Weyl inner functions in terms of their de Branges functions. The de Branges function of Θ_N is

$$E_N(\lambda) = \cos\sqrt{\lambda} - i\sqrt{\lambda}\sin\sqrt{\lambda},$$

and by (1.14) the de Branges function of Θ is

$$E(\lambda) = -\dot{u}(\lambda) + iu(\lambda), \qquad u(\lambda) := u_\lambda(1), \quad \dot{u}(\lambda) := \dot{u}_\lambda(1),$$

where $u_\lambda(t)$ is the solution of the Schrödinger equation with boundary condition α and initial value $u_\lambda(0) = 1$. We have

$$\frac{\Theta}{\Theta_N} = \frac{\bar{H}}{H}, \qquad H = \frac{E}{E_N}.$$

Since both de Branges functions are outer in $\mathbf{C}_+$, all we need to check is that $|E| \asymp |E_N|$ on $\mathbb{R}$. To this end we can use the standard asymptotic formulae for solutions of a regular Schrödinger equation, see e.g. [30]:

$$|u(\lambda) - \cos\sqrt{\lambda}| = O\left(\frac{1}{\sqrt{\lambda}}\right), \qquad |\dot{u}(\lambda) + \sqrt{\lambda}\sin\sqrt{\lambda}| = O(1), \qquad (\lambda \to \pm\infty).$$

For instance if $q \equiv 0$ but $\alpha \neq (N)$, then

$$u(\lambda) = \frac{\cos(\sqrt{\lambda} + \psi(\lambda))}{\cos\psi(\lambda)}, \qquad \sqrt{\lambda}\tan\psi(\lambda) = \cot\frac{\alpha}{2},$$

and the asymptotic is obvious.

If $\lambda \to +\infty$, then

$$|E(\lambda)|^2, |E_N(\lambda)|^2 \asymp [\cos\sqrt{\lambda} + O(\lambda^{-1/2})]^2 + \lambda[\sin\sqrt{\lambda} + O(\lambda^{-1/2})]^2 := I + II,$$

and we consider three cases:

if $|\sin\sqrt{\lambda}| \lesssim \lambda^{-1/2}$, then $I \asymp 1$ and $0 \le II \lesssim 1$, so $I + II \asymp 1$;

if $|\cos\sqrt{\lambda}| \lesssim \lambda^{-1/2}$, then $\sin^2\sqrt{\lambda} \approx 1$ and both $|E|^2$ and $|E_N|^2$ are $\asymp \lambda$;

if $|\sin\sqrt{\lambda}|$ and $|\cos\sqrt{\lambda}|$ are $\gg \lambda^{-1/2}$, then $I + II \asymp \cos^2\sqrt{\lambda} + \lambda\sin^2\sqrt{\lambda}$.

The estimates for $\lambda \to -\infty$ are even easier. □

Toeplitz kernels with real analytic symbols

From now on we will be considering unimodular functions with real analytic arguments, $U = e^{i\gamma}$, $\gamma \in C^\omega(\mathbb{R})$. In this case, all elements of the Toeplitz kernels are also real analytic on $\mathbb{R}$.

Lemma. *If $\gamma \in C^\omega(\mathbb{R})$, then $N^+[e^{i\gamma}] \subset C^\omega(\mathbb{R})$.*

Proof: Let $F \in N^+[U]$ and let G_- be the analytic extension of UF to $\mathbf{C}_-$. Since $U \neq 0$ in a neighborhood of $\mathbb{R}$ and $F = U^{-1}G_-$ on $\mathbb{R}$, F can be extended to a neighborhood of $\mathbb{R}$. □

2.5. Basic criterion in N^+.

Proposition. *Let $\gamma \in C^\omega(\mathbb{R})$. Then $N^+[e^{i\gamma}] \neq 0$ iff γ has a representation*

$$\gamma = -\alpha + \tilde{h},$$

where $\alpha \in C^\omega(\mathbb{R})$ is an increasing function and $h \in L^1_\Pi$.

Proof: We first observe that $N^+[u] \neq 0$ iff

$$U = \bar{\Phi}\frac{\bar{H}}{H} \qquad \text{on } \mathbb{R} \tag{2.1}$$

or some outer function $H \in C^\omega(\mathbb{R})$ that does not vanish on $\mathbb{R}$, and some meromorphic inner function Φ. Indeed, suppose $N^+[U] \neq 0$. Reasoning as in Lemma 2.2, we see that

$$U = \bar{I}\frac{\bar{F}}{F},$$

for some meromorphic inner function I and an outer function $F \in C^\omega(\mathbb{R})$. The outer function may have zeros on the real line. Suppose the zeros are simple. Take any meromorphic inner function J such that $\{J = 1\} = \{F = 0\}$. Then the outer function

$$H = \frac{F}{1 - J} \tag{2.2}$$

is zero-free on $\mathbb{R}$ and

$$U = \bar{I} \cdot \frac{1 - \bar{J}}{1 - J} \cdot \frac{\bar{H}}{H} = -\bar{I}\bar{J}\frac{\bar{H}}{H} := \bar{\Phi}\frac{\bar{H}}{H}.$$

If F has multiple zeros, then we simply repeat this reasoning taking care of the convergence.

Next we restate (2.1) in terms of the arguments of the involved functions. Since H is an outer function, it has the following representation:

$$H = e^{-(h+i\tilde{h})/2}, \qquad h \in L^1_\Pi,$$

and since H is zero free we have $\tilde{h} \in C^\omega(\mathbb{R})$. It follows that $\gamma = -\phi + \tilde{h}$, where ϕ is a continuous argument of Φ. Since ϕ is strictly increasing, this gives the "only if" part of the theorem. To prove the "if" part, we observe that given an increasing function α, we can find an inner function with argument ϕ such that

$$\beta := \alpha - \phi \in L^\infty(\mathbb{R}),$$

so

$$\gamma = -\alpha + \tilde{h} = -\phi - \beta + \tilde{h} = -\phi + \tilde{h}_1, \qquad h_1 := h + \tilde{\beta}.$$

□

2.6. **Basic criterion in $\mathcal{H}^p$.**

Proposition. *Let $U = e^{i\gamma}$ with $\gamma \in C^\omega(\mathbb{R})$. Then $N^p[U] \neq 0$ iff*

$$U = \bar{\Phi}\frac{\bar{H}}{H},$$

where H is an outer function in $\mathcal{H}^p \cap C^\omega(\mathbb{R})$, $H \neq 0$ on $\mathbb{R}$, and Φ is a meromorphic inner function. Alternatively, $N^p[U] \neq 0$ iff

$$\gamma = -\phi + \tilde{h}, \qquad h \in L^1_\Pi, \quad e^{-h} \in L^{p/2}(\mathbb{R}), \tag{2.3}$$

where ϕ is the argument of some meromorphic inner function.

To prove the statement we just repeat the previous proof using the following lemma, in which we construct the Herglotz measure of a meromorphic inner function J so that the function H in (2.2) in the previous proof belongs to $\mathcal{H}^p$.

Lemma. *If $0 < p \leq \infty$ and $F \in \mathcal{H}^p \cap C^\omega(\mathbb{R})$, then there is a finite positive measure ν supported exactly on $\{F = 0\} \cap \mathbb{R}$ such that*

$$F \cdot \mathcal{S}\nu \in \mathcal{H}^p \cap C^\omega(\mathbb{R}), \qquad \mathcal{S}\nu(z) := \int \frac{d\nu(t)}{t - z}.$$

Proof: Let $\{b_k\}$ be all real zeros of F; we assume for simplicity that the zeros are simple. Choose small positive numbers ϵ_k,

$$\sum \epsilon_k < 1, \tag{2.4}$$

such that the ϵ_k-neighborhoods of b_k are disjoint. We have

$$|F(z)| \leq C_k |z - b_k|$$

if z is in the ϵ_k-neighborhood of b_k. Take

$$\nu = \sum \nu_k \delta_{b_k}, \qquad \nu_k = C_k^{-1} 2^{-k} \epsilon_k,$$

and observe that

$$|\mathcal{S}\nu(z)| \leq \sum_k \frac{\nu_k}{|z - b_k|}.$$

If z is outside of all neighborhoods, then

$$|F\mathcal{S}\nu|(z) \leq |F(z)| \sum 2^{-k} = |F(z)|.$$

If z is in the k-th neighborhood, then

$$|F\mathcal{S}\nu|(z) \leq |F(z)| + \frac{|F(z)|\nu_k}{|z-b_k|} \leq |F(z)| + C_k\nu_k \leq |F(z)| + 1.$$

We have $F\mathcal{S}\nu \in \mathcal{H}^p$ by (2.4). $\square$

2.7. **Kolmogorov-type criterion.** The basic criterion is particularly useful in the case when we can represent γ as the argument of some explicitly given outer function, e.g., when U is the ratio of two inner functions with known de Branges functions.

Proposition. *Let $U = \bar{H}/H$, and H is an outer function real analytic and zero free on $\mathbb{R}$. Suppose* $\dim N^p[U] < \infty$. *Then*

$$N^p[U] \neq 0 \quad \Leftrightarrow \quad H \in \mathcal{H}^p.$$

Proof: If $N^p[U] \neq 0$, then $\dim N^p[b^sU] = 1$ for some $s \geq 0$. We have

$$b^sU = \bar{H}_s/H_s, \qquad H_s = (1-b)^{-s}H.$$

By Proposition 2.6 we have a representation

$$\bar{H}_s/H_s = \bar{F}/F$$

for some outer function $F \in \mathcal{H}^p \cap C^\omega(\mathbb{R})$, $F \neq 0$ on $\mathbb{R}$. Since the function H_s is also outer and zero-free, it follows that $H = \text{const}(1-b)^sF$ and so $H \in \mathcal{H}^p$. The converse is trivial. $\square$

Comments. (a) If $p = 2$ and $1/H \in H^2$, then the proof shows that the conclusion is true even without the assumption $H \in C^\omega(\mathbb{R})$. The corresponding statement in the unit disc is equivalent to Kolmogorov's minimality criterion in the theory of stationary Gaussian processes: $\{z^n\}$ is minimal in $L^2(w)$ iff $w^{-1} \in L^1$.

(b) The condition dim $N^p[u] < \infty$ is essential and related to the concept of "rigid functions", see [37]. We already mentioned two sufficient conditions for dim $N^p[u] < \infty$ in Section 2.3.

Example. This simple example is meant to illustrate the above criterion. Let Θ_D and Θ_N be the Weyl inner functions (1.7) for potential $q = 0$ on [0,1] with Dirichlet and Neumann boundary conditions at 0. We claim that the "fractional dimension" of the kernel $N[\bar{\Theta}_N\Theta_D]$ is $-1/4$, see Section 2.1.

Proof: We represent the ratio Θ_D/Θ_N in terms of de Branges functions:

$$\frac{\Theta_D}{\Theta_N} = \frac{\bar{H}}{H}, \qquad H = \frac{E_D}{E_N},$$

where

$$E_N(\lambda) = \cos\sqrt{\lambda} - i\sqrt{\lambda}\sin\sqrt{\lambda}, \qquad E_D(\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} + i\cos\sqrt{\lambda},$$

so

$$U := \frac{\bar{H}}{H}, \qquad H = \frac{1}{\sqrt{\lambda}} \cdot \frac{\sin\sqrt{\lambda} + i\sqrt{\lambda}\cos\sqrt{\lambda}}{\cos\sqrt{\lambda} - i\sqrt{\lambda}\sin\sqrt{\lambda}}.$$

Clearly, $\dim N[U] < \infty$, and by Kolmogorov's criterion we have $N[\bar{b}^s U] \neq 0$ iff $F := (1-b)^s H \in L^2(\mathbb{R})$. If $x > 0$, then

$$|H(x)|^2 = \frac{1}{x}\,\frac{\sin^2\sqrt{x} + x\cos^2\sqrt{x}}{\cos^2\sqrt{x} + x\sin^2\sqrt{x}}.$$

Let us estimate $|F|^2$ on an interval I_n about $\pi^2 n^2$ where $|\sin\sqrt{x}| \ll 1$. If we write $x = \pi^2 n^2 + s$, so

$$|\sqrt{x} - \pi n| \asymp \frac{|s|}{n}, \qquad \sin^2\sqrt{x} \asymp \frac{s^2}{n^2}, \qquad x\sin^2\sqrt{x} \asymp s^2,$$

then

$$|H(x)|^2 \asymp \frac{1}{x}\,\frac{x}{1 + x\sin^2\sqrt{x}} \asymp \frac{1}{1+s^2},$$

$$|F(x)|^2 \asymp n^{-4s}\frac{1}{1+s^2}, \qquad \int_{I_n} |F|^2 \asymp n^{-4s}.$$

Thus $F \in L^2(\mathbb{R}_+)$ iff $4s > 1$. Finally we note that

$$|H(x)| \sim \frac{1}{\sqrt{|x|}}, \qquad x \to -\infty,$$

and so $F \in L^2(\mathbb{R}_-)$ iff $s > 0$. □

Similarly one can show that "$\dim$"$N[\bar\Theta_D \Theta_N] = -5/4$. One can also compute the dimensions of kernels in other Hardy spaces. In particular, "$\dim$"$N^\infty[\bar\Theta_N \Theta_D] = 0$ and "$\dim$"$N^\infty[\bar\Theta_D \Theta_N] = -1$; moreover, $N^\infty[\bar\Theta_N\Theta_D] \neq 0$ and $N^\infty[\bar b \bar\Theta_D \Theta_N] \neq 0$.

2.8. **Twin inner functions.** We say that two meromorphic inner functions are *twins* if they have the same point spectrum (possibly including infinity). Twin functions appear in several applications (see, e.g., Section 3.5 below), and the main result is that the Toeplitz operator corresponding to their ratio is injective. This fact is quite different from Levinson's type conditions discussed later in this section.

Theorem. *Let Θ and J be twin meromorphic inner functions. Then $N[\bar\Theta J] = 0$.*

Proof: We have

$$\bar\Theta J = \frac{\bar H}{H}, \qquad H = \frac{1-\Theta}{1-J}.$$

Since $H^{\pm 1} \in C^\omega(\mathbb{R})$, we can apply Kolmogorov's criterion. (The kernel is finite dimensional because the argument of $\bar\Theta J$ is bounded.) We claim that $H \notin \mathcal{H}^2$. Indeed, if Θ has no point mass at infinity, then

$$|H| \geq \frac{1}{2}|1-\Theta| \notin L^2.$$

If both functions have a point mass at infinity, then by l'Hôpital's rule the angular limit

$$\lim_\infty \frac{1-\Theta}{1-J} = \frac{\Theta'(\infty)}{J'(\infty)}$$

exists and is non-zero (see Section 1.2), so H can not be in $\mathcal{H}^2$. □

Remarks. (a) The proof shows that $N[\bar\Theta J] = 0$ if we have $\{\Theta = 1\} = \{J = 1\}$ and $\infty \notin \sigma(\Theta)$. Moreover, it can be shown that

$$\sigma(\Theta) \subset \sigma(J) \qquad \Rightarrow \qquad N[\bar\Theta J] = 0.$$

(b) Similar technique applies to symbols of the form $u = \bar{\Theta} J \bar{H}/H$ where Θ and J are twin inner functions, and H is an outer function real-analytic and zero free on $\mathbb{R}$. If dim $N^p[\bar{H}/H] < \infty$, then

$$N^p[\bar{b}^s \bar{\Theta} J \bar{H}/H] \neq 0 \quad \Leftrightarrow \quad (1-b)^s \frac{1-\Theta}{1-J} H \in H^p.$$

General form of Levinson's completeness theorem

As we mentioned in Section 2.1, if $\exists\ \gamma(\pm\infty)$, then

$$\delta = \gamma(+\infty) - \gamma(-\infty) > -\frac{2\pi}{p} \qquad \Rightarrow \qquad N^p[e^{i\gamma}] = 0, \qquad (0 < p < \infty).$$

We extend this fact to general symbols by considering the "mean" behavior at $+\infty$ of the function

$$\delta(x) = \gamma(x) - \gamma(-x).$$

Our result has the same form as Kolmogorov's type condition in Section 3.7, but this time we don't assume a priory that the kernel is finite dimensional. The new idea is to apply the Poisson A-integral transform to the equation (2.3) in the basic criterion. (See Section 3.4 for another application of this idea.)

2.9. **Titchmarsh and Uly'anov Theorems.** Let $h \in L^1_{\text{loc}}(\mathbb{R})$ be a real valued function. For each $A > 0$ we denote

$$h^A = \begin{cases} h(x), & |h(x)| \leq A, \\ 0, & |h(x)| \geq A. \end{cases}$$

The Schwarz A-integral of h is defined by the formula

$$\mathcal{S}_{(A)} h(z) = \lim_{A\to\infty} \mathcal{S} f^A(z), \qquad z \in \mathbf{C}_+,$$

provided that the limit exists for all z. Similarly, one defined the Poisson and the conjugate Poisson A-integrals $\mathcal{P}_{(A)} h$ and $\mathcal{Q}_{(A)} h$ respectively so that

$$\mathcal{S}_{(A)} h = \mathcal{P}_{(A)} h + i\mathcal{Q}_{(A)} h.$$

Recall that if $h, \tilde{h} \in L^1_\Pi$, then $\mathcal{S}\tilde{h} = -i\mathcal{S}h + i\mathcal{S}h(i)$.

Theorem. *If $h \in L^1_\Pi$, then the Schwarz A-integral of $\tilde{h}$ exists, and we have*

$$\mathcal{S}_{(A)} \tilde{h}(z) = -i\mathcal{S}h(z) + i\mathcal{S}h(i), \qquad z \in \mathbf{C}_+. \tag{2.5}$$

The real part of the equation (2.5), or rather its special case

$$\mathcal{P}_{(A)} \tilde{h}(i) = 0, \tag{2.6}$$

is due to Titchmarsh, see [42], and the imaginary part of (2.5),

$$Q_{(A)} \tilde{h} = -Ph + Ph(i), \tag{2.7}$$

is the Uly'anov's theorem, see [2] for a shorter proof. Note that we use a slightly different definition of the A-transforms but the definitions are in fact equivalent because $\tilde{h} \in L^{o(1,\infty)}_\Pi$.

2.10. **A sufficient condition for $N^p[U] = 0$.** For an odd function $\delta : \mathbb{R} \to \mathbb{R}$ we define

$$L\delta(y) = \frac{2}{\pi} \int_0^\infty \left[\frac{1}{1+t^2} - \frac{1}{y^2+t^2} \right] t\delta(t)\ dt, \qquad y > 0.$$

The integral makes sense (it might be + or - ∞) if we assume that

$$\text{either} \quad \int^\infty \delta^+(t)\ t^{-3} dt < \infty \qquad \text{or} \quad \int^\infty \delta^-(t)\ t^{-3} dt < \infty, \tag{2.8}$$

where, as usual, $\delta^\pm := \max\{\pm\delta, 0\}$. Note that if $\delta \in L^1_\Pi$, then $L\delta(y) = \tilde{\delta}(iy)$.

Theorem. *Let $\gamma \in C^\omega(\mathbb{R})$ and let $\delta(x) = \gamma(x) - \gamma(-x)$ satisfy (2.8). Then*

$$N^p[e^{i\gamma}] \neq 0 \qquad \Rightarrow \qquad \exists F \in \mathcal{H}^{p/4}(\mathbf{C}_+), \quad e^{L\delta(y)} \leq |F(iy)|, \quad (y > 1).$$

Proof: By (2.3) we have

$$\gamma = -\phi + \tilde{h}_1, \qquad h_1 \in L^1_\Pi, \quad e^{-h_1} \in L^{p/2}.$$

Then

$$-\gamma(-x) = \phi(-x) + \tilde{h}_2(x), \qquad h_2 \in L^1_\Pi, \quad e^{-h_2} \in L^{p/2}.$$

It follows that

$$\delta = -\psi + \tilde{h}, \qquad h \in L^1_\Pi, \quad e^{-h} \in L^{p/4},$$

where ψ is an *odd* increasing function. We apply the Uly'anov theorem to

$$\tilde{h} = \delta_1 := \psi + \delta.$$

By (2.7) we have

$$\mathcal{Q}_A \delta_1 = \mathcal{Q}_A \tilde{h} = -\mathcal{P}h + \text{const},$$

so

$$e^{\mathcal{Q}_A \delta_1} \ = \ \text{const}\ |F|, \qquad F := e^{-\mathcal{S}h} \in \mathcal{H}^{p/4}$$

and it remains to show that

$$L\delta(y) \leq \mathcal{Q}_A \delta_1(iy).$$

Since $t\delta_1(t) \geq t\delta(t)$ for all $t \in \mathbb{R}$, and since the kernel of L is positive for $y > 1$, we have

$$\begin{aligned}
\pi \mathcal{Q}_A \delta_1(iy) &= \lim_{A\to\infty} \int_{\{|\delta_1|<A\}} \left[\frac{1}{1+t^2} - \frac{1}{y^2+t^2} \right] t\delta_1(t)\ dt \\
&\geq \lim_{A\to\infty} \int_{\{|\delta_1|<A\}} \left[\frac{1}{1+t^2} - \frac{1}{y^2+t^2} \right] t\delta(t)\ dt \\
&= \int_{\mathbb{R}} \left[\frac{1}{1+t^2} - \frac{1}{y^2+t^2} \right] t\delta(t)\ dt = \pi L\delta(y).
\end{aligned}$$

□

Remark. The proof works for any γ such that

$$\gamma = -\phi + \tilde{h}, \qquad h \in L^1_\Pi, \quad e^{-h} \in L^{p/2}.$$

We don't need to assume $\gamma \in C^\omega$ as long as we have such a representation.

2.11. **Levinson-type conditions.** We can use standard growth estimates of Hardy space functions to derive more familiar looking conditions.

(a)

$$N^p[e^{i\gamma}] \neq 0 \qquad \Rightarrow \qquad e^{L\delta(y)} = o\left(y^{-4/p}\right), \qquad y \to +\infty,$$

In other words,

$$\limsup_{y\to\infty} \left[L\delta(y) + \frac{4}{p}\log y\right] > -\infty \qquad \Rightarrow \qquad N^p[u] = 0. \tag{2.9}$$

Example. A simple computation shows

$$[L\mathrm{sign}](y) = \frac{2}{\pi}\log y.$$

It follows that $N[e^{i\gamma}] = 0$ if $\gamma(x) \geq \gamma(-x) - \pi$ for $x \geq x_0$. Indeed, we have $\delta \geq -\pi$ for large x, and therefore

$$L\delta(y) \geq -\pi[L\mathrm{sign}](y) + O(1) = -2\log y + O(1).$$

(b)

$$N^p[e^{i\gamma}] \neq 0 \qquad \Rightarrow \qquad \int^\infty e^{\frac{p}{4}L\delta(y)}dy < \infty.$$

For example,

$$L\delta(y) \geq -\frac{4}{p}\log y - \log\log y \qquad \Rightarrow \qquad N^p[e^{i\gamma}] = 0.$$

A more general statement is

$$N^p[e^{i\gamma}] \neq 0 \qquad \Rightarrow \qquad \int_0^\infty e^{\frac{p}{4}L\delta(y)}d\nu(y) < \infty$$

for any Carleson measure ν in $\mathbf{C}_+$.

(c) For each $y > 1$, the quantity

$$\frac{L\delta(y)}{[L\mathrm{sign}](y)} = \frac{\pi}{2}\,\frac{L\delta(y)}{\log y}$$

is some sort of "averaging" of δ near $+\infty$. The meaning of (2.9) is that the "mean" value of δ in infinity has to be "less" than $-2\pi/p$ for the Toeplitz kernel $N^p[e^{i\gamma}]$ to be non-trivial. Here is a different way to express the same idea.

Corollary. *Suppose $\delta(x)$ satisfies the integrability conditions (2.8). Then $N^p[e^{i\gamma}] = 0$ if*

$$\int_0^x \frac{\delta(t)}{t}dt \geq -\frac{2\pi}{p}\log x + O(1), \qquad x \to +\infty.$$

Proof: Integration by parts shows: if

$$\int_0^x \frac{a(t)}{t}dt \leq \log x + O(1), \qquad x \to \infty,$$

then

$$\int_0^\infty \frac{y^2}{y^2+t^2}\cdot\frac{a(t)dt}{t} \leq \log y + O(1), \qquad y \to \infty.$$

(If the latter integral is not converging, then we understand it as $\limsup \int_0^x .$) □

3. Some applications

Completeness and minimality problem

3.1. **Restatement in terms of uniqueness sets and Toeplitz kernels.** We will study the following situation. Consider the Schrödinger equation (0.2) on an interval (a, b) with potential q and some fixed selfadjoint boundary condition β at b. We assume that the endpoint a is regular. For each $\lambda \in \mathbf{C}$ we choose a non-trivial solution u_λ satisfying the boundary condition; this solution is unique up to a constant. If $\Lambda \subset \mathbf{C}$, we say that the family $\{u_\lambda\}_{\lambda\in\Lambda}$ is *complete* if

$$\operatorname{span}\{u_\lambda : \lambda \in \Lambda\} = L^2(a, b),$$

and is *minimal* if

$$\forall \lambda_0 \in \Lambda, \quad u_{\lambda_0} \notin \operatorname{span}\{u_\lambda : \lambda \in \Lambda \setminus \lambda_0\}.$$

We will use the notation of Sections 1.4–1.6: $\Theta = \Theta^a_{b,\beta}$ is the Weyl inner function, and E is a de Branges function of Θ. For $\lambda \in \mathbf{C}_+ \cup \mathbb{R}$ we have reproducing kernels k_λ and dual reproducing kernels k^*_λ of K_Θ, see (1.10). For $\lambda \in \mathbf{C}$, let K^E_λ denote the reproducing kernel in $B(E)$. Finally, we represent

$$\Lambda = \Lambda_+ \cup \Lambda_-, \qquad \Lambda_+ := \Lambda \cap (\mathbf{C}_+ \cup \mathbb{R}), \quad \Lambda_- := \Lambda \cap \mathbf{C}_-.$$

Lemma. *The following assertions are equivalent:*

(i) the family $\{u_\lambda\}_{\lambda\in\Lambda}$ is complete (minimal) in $L^2(a, b)$;

(ii) the family $\{K^E_\lambda\}_{\lambda\in\Lambda}$ is complete (minimal) in $B(E)$;

(iii) the family $\{k_{\bar\lambda}\}_{\lambda\in\Lambda_-} \cup \{k^*_\lambda\}_{\lambda\in\Lambda_+}$ is complete (minimal) in K_Θ.

Proof: This follows from Theorem 1.5 and and Corollary 1.7. □

We say that $\Lambda \subset \mathbf{C}$ is a *uniqueness set* of $B(E)$ if there is no non-trivial function $F \in B(E)$ such that $F = 0$ on Λ. The equivalence $(i)\Leftrightarrow(ii)$ in the above lemma means that $\{u_\lambda\}_{\lambda\in\Lambda}$ is complete if and only if Λ is a uniqueness set of $B(E)$, and that $\{u_\lambda\}_{\lambda\in\Lambda}$ is minimal if and only if $\Lambda \setminus \lambda_0$ is not a uniqueness set for any $\lambda_0 \in \Lambda$.

We define uniqueness sets Λ of K_Θ in a similar way; in this case $\Lambda \subset \mathbf{C}_+ \cup \mathbb{R}$. The definition obviously extends to *divisors*, i.e. sets of points with assigned multiplicities.

Lemma. *Let $\Lambda \subset \mathbf{C}_+ \cup \mathbb{R}$ and let $M \subset \mathbf{C}_+$. Then the family*

$$\{k_\lambda\}_{\lambda\in\Lambda} \cup \{k^*_\mu\}_{\mu\in M}$$

is complete in K_Θ if and only if $\Lambda \cup M$ is a uniqueness divisor.

Proof: Suppose the family is not complete, so there is a non-trivial $F \in K_\Theta$ orthogonal to all k_λ and k^*_μ. Let $H \in K_\Theta$ be defined by the relation

$$\bar{\Theta} F = \bar{H} \quad \text{on} \quad \mathbb{R}.$$

Then we have $F = 0$ on Λ and $H = 0$ on M. (Recall that $\bar{\Theta} k_\mu = k^*_\mu$.) From the latter fact we have a representation

$$H = B_M G, \qquad G \in \mathcal{H}^2,$$

where B_M is the Blaschke product, and therefore

$$\bar{\Theta}(B_M F) = \bar{G}.$$

The function $B_M F$ is in K_Θ and is zero on $\Lambda \cup M$. The opposite direction is similar. $\square$

The equivalence $(i) \Leftrightarrow (iii)$ in the first lemma now means that $\{u_\lambda\}_{\lambda \in \Lambda}$ is complete in $L^2(a, b)$ if and only if $\Lambda_+ \cup \overline{\Lambda_-}$ is a uniqueness divisor of $B(E)$. The characterization of minimality is similar.

It is very easy to characterize uniqueness sets (or divisors) Λ of K_Θ in the terms of Toeplitz kernels in the case $\Lambda \subset \mathbf{C}_+$. (The case $\Lambda \subset \mathbb{R}$ is discussed in the next subsection.) A necessary and sufficient condition for uniqueness is the following:

$$N[\bar{\Theta} B_\Lambda] = 0.$$

Indeed, $f \in K_\Theta$ is zero on Λ if and only if $g = \bar{B}_\Lambda f \in N[\bar{\Theta} B_\Lambda]$.

Let us summarize the above discussion.

Theorem. *Let $\Lambda = \Lambda_+ \cup \Lambda_-$, $\Lambda_\pm \subset \mathbf{C}_\pm$, and let B denote the Blaschke product corresponding to the divisor $\Lambda_+ \cup \overline{\Lambda_-}$. The family $\{u_\lambda\}_{\lambda \in \Lambda}$ is complete in $L^2(a, b)$ iff $N[\bar{\Theta} B] = 0$ and is minimal iff $N[\bar{b} \bar{\Theta} B] \neq 0$. The family is complete and minimal if and only if*

$$\dim\ [\bar{b} \bar{\Theta} B] = 1.$$

The "dimension" of the kernel, which may be negative, see Section 2.1, can be interpreted as the excess/deficiency of the family.

3.2. **Uniqueness sets of K^p_Θ.** We will consider the case of an arbitrary $p > 0$. As we explained, if $\Lambda \subset \mathbf{C}_+$, then

$$\Lambda \text{ is a uniqueness set of } K^p_\Theta \quad \Leftrightarrow \quad N^p[\bar{\Theta} B] = 0. \tag{3.1}$$

Now we concentrate on the case $\Lambda \subset \mathbb{R}$.

Proposition. *Let Θ be a meromorphic inner function and $\Lambda \subset \mathbb{R}$. Then Λ is a uniqueness set of K^p_Θ if and only if $N^p[\bar{\Theta} J] = 0$ for every meromorphic inner function J such that $\{J = 1\} = \Lambda$.*

Proof: Suppose we have a non-trivial function $F \in K^p_\Theta$ which is zero on Λ. By Lemma 2.6, we can find an inner function J with $\{J = 1\} = \Lambda$ such that

$$G = \frac{F}{1-J} \in \mathcal{H}^p.$$

Then $G \in K^p_\Theta$, and we have

$$\bar{\Theta} J G = \bar{\Theta}(G - F) = \bar{\Theta} F \frac{J}{1-J} = -\frac{\bar{\Theta} F}{1 - \bar{J}} \in \bar{\mathcal{H}}^p,$$

so the Toeplitz kernel is non-trivial.

Conversely, if G is a non-trivial element of $N^p[\bar{\Theta} J]$, then $F = (1-J)G \in K^p_\Theta$, and so Λ is not a uniqueness set. Indeed, since $G \in N^p[\bar{\Theta} J]$, we have $JG \in K^p_\Theta$, and therefore $G \in K^p_\Theta$ and $G - JG \in K^p_\Theta$. □

Compared to the case $\Lambda \subset \mathbf{C}_+$, the condition in the last lemma seems to be less useful since it involves an infinite set of inner functions J. Nevertheless, we can overcome this difficulty and restate the criterion in both cases $\Lambda \subset \mathbf{C}_+$ and $\Lambda \subset \mathbb{R}$ in a unified way.

If $\Lambda \subset \mathbf{C}_+$, then combining (3.1) and (2.3) we see that Λ is not a uniqueness set of K^p_Θ iff the function

$$\gamma = \arg B_\Lambda - \arg \Theta \tag{3.2}$$

has a representation

$$\gamma = -\phi + \tilde{h}, \qquad h \in L^1_\Pi, \quad e^{-h} \in L^{p/2},$$

where ϕ is the argument of a meromorphic inner function.

In the case $\Lambda \subset \mathbb{R}$, we will use the *counting function* n_Λ of Λ instead of the argument function of B_Λ in (3.2). By definition,

$$n_\Lambda = \sum n_\lambda, \qquad n_\lambda = \begin{cases} \chi(\lambda, +\infty), & \lambda > 0, \\ -\chi(-\infty, \lambda), & \lambda < 0. \end{cases} \tag{3.3}$$

Theorem. *Let Θ be a meromorphic inner function and $\Lambda \subset \mathbb{R}$. Then Λ is not a uniqueness set of K^p_Θ if and only if the function*

$$\gamma = 2\pi n_\Lambda - \arg \Theta$$

has a representation

$$\gamma = -\phi + \tilde{h}, \qquad h \in L^1_\Pi, \quad e^{-h} \in L^{p/2}, \tag{3.4}$$

where ϕ is the argument of a meromorphic inner function.

Proof: If Λ is not a uniqueness set, then by the last proposition there is a meromorphic inner function J such that $\{J = 1\} = \Lambda$ and such that the kernel $N^p[\bar{\Theta} J]$ contains an *outer* function G which has no zeros on $\mathbb{R}$. We have

$$\bar{\Theta} J = -\bar{\Phi} \frac{\bar{G}}{G}$$

for some inner function Φ, and if we denote $F = (1-J)G$, then

$$\Theta = \Phi \frac{F}{\bar{F}} = \Phi \, \exp\{2i(\log|F|)^\sim\} = \Phi \, \exp\{i\,[2(\log|F|)^\sim + 2\pi n_\Lambda]\}.$$

The key observation is that the function in the square brackets is continuous – this follows from the fact that F is an outer function with zero set Λ, and from the identity $(\log|x|)^{\sim} = \pi\chi_{\mathbb{R}_-}$. We conclude that

$$\gamma = -\phi + \tilde{h}, \qquad h := -2\log|F|,$$

which proves one half of the statement. Reversing the argument we get the second half. $\square$

Example. Λ is a uniqueness set of $K^p[S^a]$ if $\gamma(t) = 2\pi n_\Lambda(t) - at$ has a representation (3.4). As in Section 2.10 consider the function

$$\delta(t) = \gamma(t) - \gamma(-t) = 2\pi N(t) - 2at, \qquad N(t) = \#[\Lambda \cap (-t,t)].$$

Applying Corollary 2.11 (and also Remark 2.10) we get a sufficient condition

$$\int_1^x \frac{N(t)}{t}dt \geq \frac{a}{\pi}x - \frac{1}{p}\log x + O(1), \qquad (x \to +\infty).$$

In fact, Theorem 2.10 implies the latter condition in the lim sup sense, as well as some other sufficient conditions. Similar results can be stated for various families of special functions.

Zero sets of entire functions

3.3. **Exact zero sets of $B^+(E)$-functions.** We will be considering general Cartwright-de Branges spaces, see Section 1.7. A set $\Lambda \subset \mathbf{C}$ is said to be an *exact zero set* of $B^+(E)$ if there exists an entire function $F \in B^+(E)$ such that $F = 0$ exactly on Λ. We'll restrict the discussion to the case $\Lambda \subset \mathbb{R}$.

Recall that $B^+(E) = EK^+(\Theta)$ where $\Theta = \Theta_E = E^\#/E$, so $\Lambda \subset \mathbb{R}$ is also an exact zero set of $K^+(\Theta)$. The following description is essentially our basic criterion in Proposition 2.5. In the next subsection we will see that in the special case $E = S^{-a}$, this description contains main results of the theory of Cartwright's functions. As usual, S denotes the singular inner function e^{iz}.

Theorem. *$\Lambda \subset \mathbb{R}$ is an exact zero set of $B^+(E)$ if and only if*

$$2\pi n_\Lambda - \arg\ \Theta_E = -bx + \tilde{h}, \qquad h \in L^1_\Pi, \quad b \geq 0.$$

Proof: We need to show that a function $F \in B^+(E)$ with zero set Λ exists if and only

$$\Theta = JS^b\frac{H}{\bar{H}} \tag{3.5}$$

for some inner function J with $\{J = 1\} = \Lambda$, some $b \geq 0$, and some outer function $H \in C^\omega(\mathbb{R})$ which has no zeros on $\mathbb{R}$.

$\Leftarrow$: The function

$$F(z) = \begin{cases} (1 - J(z))\ H(z)E(z)S^b(z), & z \in \mathbf{C}_+ \\ (J^\#(z) - 1)\ H^\#(z)E^\#(z), & z \in \mathbf{C}_- \end{cases}$$

is in $B^+(E)$ and vanishes exactly on Λ.

$\Rightarrow$: Suppose $F \in B^+(E)$ vanishes exactly on Λ. Since $B^+(E) = EK^+_\Theta$, there is $G \in K^+_\Theta$ such that $F = EG$ in $\mathbf{C}_+$. Clearly, we can assume that G is an outer function. Define $G_- \in N^+(\mathbf{C}_+)$ by the equation

$$\bar{\Theta} G = \bar{G}_- \quad \text{on} \quad \mathbb{R}.$$

Since

$$F(z) = E^\#(z)\overline{G_-(\bar{z})}, \qquad z \in \mathbf{C}_-,$$

G_- has no zeros in C_+, and therefore its inner-outer factorization has the form

$$G_- = S^b G, \qquad b \geq 0. \tag{3.6}$$

We now take any meromorphic inner function J with $\{J = 1\} = \Lambda$ and define

$$H = \frac{G}{1-J}.$$

This is an outer function with the stated properties. We have

$$G = H(1-J), \qquad G_- = H(1-J)S^b,$$

and from (3.6) we get the representation (3.5). $\square$

Example: $\Lambda \subset \mathbb{R}$ is an exact zero set of a Cartwright function if and only

$$n_\Lambda(x) = cx + \tilde{h}, \qquad c \geq 0,, \quad h \in L^1_\Pi. \tag{3.7}$$

3.4. **Zeros of Cartwright's functions.** We need the following elementary lemma.

Lemma. *Suppose $g \in L^{o(1,\infty)}_\Pi$ and $g' \geq -\text{const}$. Then*

$$g(x) = o(x), \qquad x \to \pm\infty.$$

Proof: If $g(x_*) \geq cx_*$, $x_* \gg 1$, then for $x > x_*$ we have (assuming $g' \geq -1$)

$$g(x) \geq g(x_*) - (x - x_*) \geq (1+c)x_* - x,$$

and so $g(x) \gtrsim x_*$ on an interval $[x_*, (1+\delta)x_*]$. The Poisson measure of this interval is $\asymp 1/x_*$, which contradicts the weak L^1-condition. $\square$

Corollary. *Let $\Lambda \subset \mathbb{R}$ be the exact zero set of some Cartwright function. Then*

$$\exists c \geq 0, \qquad \lim_{x\to-\infty} \frac{n_\Lambda(x)}{x} = \lim_{x\to+\infty} \frac{n_\Lambda(x)}{x} = c.$$

Proof: From (3.7) we conclude that $g(x) = n_\Lambda(x) - cx$ satisfied the conditions of the lemma. $\square$

Corollary. *If $\Lambda \subset \mathbb{R}$ is the exact zero set of a Cartwright function, then there exists a limit*

$$\text{v.p.} \sum_{\lambda \in \Lambda} \frac{1}{\lambda}.$$

Proof: For every $B > 0$ we define the function

$$\eta_B(t) = \begin{cases} B, & t > B, \\ t, & t \in [-B, B], \\ -B, & t < -B. \end{cases}$$

Using the notation (3.3), we have

$$\sum_{|\lambda| \le B} n_\lambda = n_\Lambda \circ \eta_B.$$

We also introduce an elementary function

$$g(\lambda) = \int n_\lambda d\Pi = \frac{\pi}{2}\operatorname{sign}(\lambda) - \arctan\lambda = \frac{1}{\lambda} + O\left(\frac{1}{\lambda^2}\right), \quad \lambda \to \pm\infty.$$

By the previous corollary we have

$$\exists \quad \text{v.p.} \sum \frac{1}{\lambda} \qquad \Leftrightarrow \qquad \exists \quad \text{v.p.} \sum g(\lambda).$$

By (3.7), $n_\Lambda = cx + f$, where we write f for $\tilde{h}$, and therefore

$$\sum_{|\lambda| \le B} g(\lambda) = \int n_\Lambda \circ \eta_B \ d\Pi = \int f \circ \eta_B \ d\Pi + \text{const.}$$

The only properties of f that we use in the rest of the proof are: $f(x) = o(x)$ as $x \to \infty$, and Titchmarsh's theorem (2.6),

$$\lim_{A \to \infty} \int f^A d\Pi = 0, \qquad f^A = f \cdot (\chi_{[-A,A]} \circ f).$$

Since $f(x) = o(x)$, we have

$$\sup_{[-B,B]} |f| = o(B), \qquad B \to \infty.$$

It follows that there is a function $A = A(B)$ such that $A = o(B)$, $A(\infty) = \infty$, and

$$f = f^{A(B)} \quad \text{on} \quad [-B, B].$$

Then we also have

$$|f \circ \eta_B|, \ |f^{A(B)}| \ \le \ A(B) \quad \text{on} \quad \mathbb{R},$$

and therefore

$$\int \left| f \circ \eta_B - f^{A(B)} \right| \ d\Pi \le A(B) \int_{\mathbb{R} \setminus [-B,B]} d\Pi \ \to \ 0, \qquad B \to \infty.$$

□

We leave it to the reader to state relevant results concerning zeros of functions from more general Cartwright-de Branges spaces $B^+(E)$. For example, the key property $\tilde{h} = o(t)$ is valid if

$$\lim_{\delta \to 0} \limsup_{x \to \infty} \frac{\theta(x + \delta x) - \theta(x)}{x} = 0,$$

where θ is the argument of Θ_E.

3.5. **Abstract Hochstadt-Liberman problem.** We will be considering the following problem concerning general meromorphic inner functions. In the next section we will explain its relation to Hochstadt-Liberman's theorem [21].

Let Φ and Ψ be meromorphic inner function and $\Theta = \Psi\Phi$. As usual, $\sigma(\Theta)$ denotes the (point) spectrum of Θ, see Section 1.2; recall that $\sigma(\Theta)$ may include ∞. We say that *the data* $[\Psi, \sigma(\Theta)]$ *determine* Θ if for any inner function $\tilde{\Phi}$,

$$\tilde{\Theta} = \Psi\tilde{\Phi}, \quad \sigma(\tilde{\Theta}) = \sigma(\Theta) \qquad \Rightarrow \qquad \Theta = \tilde{\Theta}.$$

Alternatively, we can say that Ψ and $\sigma(\Phi\Psi)$ determine Φ. Given Φ and Ψ, the problem is to decide if this is the case.

The set of Herglotz measures of inner functions $\tilde{\Theta}$ satisfying $\Psi|\tilde{\Theta}$ and $\sigma(\tilde{\Theta}) = \sigma(\Theta)$ is convex, see Section 1.2. We will refer to the dimension of this set as the dimension of the set of solutions.

Example. Suppose Θ is a *finite Blaschke product.* Then

$$[\Psi, \sigma(\Theta)] \quad \text{determine} \quad \Theta \qquad \Leftrightarrow \qquad 2\deg\Psi > \deg\Theta.$$

The proof is elementary; it also follows from the results below. As an illustration consider the simplest case $\Theta = b^2$, $\Psi = b$. Then $\sigma(\Theta) = \{0, \infty\}$, and the data $[\Psi, \sigma(\Theta)]$ does not determine Θ. In fact, the set of solutions is one-dimensional; the solutions are given by the formula

$$\tilde{\Phi}(z) = \frac{z - ia}{z + ia}, \qquad (a > 0).$$

We will state some conditions in terms of the Toeplitz kernels with symbol $U = \bar{\Phi}\Psi$. The rough meaning of these conditions is the following: for the data $[\Psi, \sigma(\Theta)]$ to determine Θ, the known factor Ψ of the inner function has to be "bigger" than the unknown factor Φ.

Proposition. *If $N^\infty[\bar{\Phi}\Psi] \neq \{0\}$, then the data $[\Psi, \sigma(\Theta)]$ does not determine Θ.*

Proof: Take

$$a \in N^\infty[\bar{\Theta}\Psi^2], \qquad \|a\|_\infty < \frac{1}{2}.$$

Then

$$\bar{\Theta}\Psi^2 a = \bar{b},$$

and

$$\bar{\Theta}\Psi^2(a + b) = \bar{a} + \bar{b}.$$

Denote

$$g = a + b, \qquad f = \Psi g,$$

so that

$$\bar{\Theta}f = \bar{f}, \qquad \Psi|f, \qquad \|f\|_\infty < 1.$$

Then the function

$$\tilde{\Theta} = \frac{f + \Theta}{f + 1} \tag{3.8}$$

is inner:

$$|\tilde{\Theta}|^2 = \frac{(f+\Theta)(\bar{f}+\bar{\Theta})}{(f+1)(\bar{f}+1)} = \frac{(f+\Theta)(\bar{\Theta}f+\bar{\Theta})}{(f+1)(\bar{\Theta}f+1)} = 1,$$

and $\Psi|\tilde{\Theta}$. Let us check that $\sigma(\Theta) = \sigma(\tilde{\Theta})$. This is clear for finite eigenvalues, and we also note that $\infty \in \sigma(\Theta)$ iff $\infty \in \sigma(\tilde{\Theta})$ because $\Theta - 1 \in \mathcal{H}^2$ iff

$$\tilde{\Theta} - 1 = \frac{\Theta - 1}{f+1} \in \mathcal{H}^2.$$

□

Here is a partial converse. We write $N^p_\Pi[U]$ for $N^+[U] \cap L^p_\Pi$.

Proposition. *If $N^p_\Pi[\bar{\Phi}\Psi] = \{0\}$ for some $p < 1$, then $[\Psi, \sigma(\Theta)]$ determine Θ.*

Proof: Suppose $\sigma(\Theta_1) = \sigma(\Theta)$ and $\Psi|\Theta_1$. Then the function

$$f = \frac{\Theta_1 - \Theta}{1 - \Theta_1}$$

is in $\mathcal{H}^p_\Pi \cap C^\omega(\mathbb{R})$ for all $p < 1$ because $(1-\Theta_1)^{-1}$ has positive real part in $\mathbf{C}_+$ and $\Theta_1 = \Theta$ on $\{\Theta_1 = 1\}$. Observe that

$$\bar{\Theta}f = \bar{\Theta}\frac{\Theta_1 - \Theta}{1 - \Theta_1} = \frac{\Theta_1\bar{\Theta} - 1}{1 - \Theta_1} = \frac{\bar{\Theta} - \bar{\Theta}_1}{\bar{\Theta}_1 - 1} = \bar{f}.$$

Since $\Psi|f$, we can define

$$g = \bar{\Psi}f \in \mathcal{H}^p_\Pi \cap C^\omega(\mathbb{R}).$$

We have

$$\bar{\Phi}\Psi g = \bar{\Theta}\Psi^2 g = \Psi\bar{\Theta}f = \Psi\bar{f} = \bar{g},$$

and so $g \in N^p_\Pi[\bar{\Phi}\Psi] = \{0\}$ and $\Theta_1 = \Theta$. □

How big is the gap between the conditions in the above statements? As we will see in Section 4.1, the gap is just finite dimensional if the inner function Φ is not very "wild". Namely, if the argument of Φ has a polynomial growth at infinity, then $N^p_\Pi[\bar{\Phi}\Psi] = \{0\}$ implies $N^\infty[b^n\bar{\Phi}\Psi] = \{0\}$ for some $n < \infty$.

We now demonstrate a different way to state a partial converse of the first statement. We get precisely the converse statement up to a factor which is the ratio of two twin inner functions, see Section 2.8.

Proposition. *If the data $[\Psi, \sigma(\Theta)]$ don't determine Θ, then there are inner functions Θ_1 and J such that $\{\Theta_1 = 1\} = \{J = 1\}$ and*

$$N^\infty[(\bar{\Theta}_1 J)\bar{\Phi}\Psi] \neq 0.$$

Proof: Suppose we have $\sigma(\Theta_1) = \sigma(\Theta)$, $\Theta_1 = \Phi_1\Psi$. Then the function $\Phi - \Phi_1$ is in $K^\infty[\Phi\Phi_1]$ and is zero on $\{\Theta_1 = 1\}$. By Proposition 3.2, there is an inner function J such that $J = 1$ exactly on $\{\Theta_1 = 1\}$ and $N^\infty[\bar{\Phi}_1\bar{\Phi}J)] \neq 0$. Finally, we note

$$\bar{\Phi}\bar{\Phi}_1 J = \bar{\Phi}\Psi(\bar{\Theta}_1 J).$$

□

We can apply our results on Toeplitz kernels (Sections 2 and 4) to obtain various necessary or sufficient conditions in the Hochstadt-Liberman problem. Here is a simple example that extends the original Hochstadt-Liberman theorem, which we will recall in the next subsection.

Corollary. *Suppose* $\Theta = \Psi^2$ *and* $\infty \notin \sigma(\Theta)$. *Then the set of solutions is exactly one-dimensional:* $\tilde{\Theta}$ *satisfies* $\Psi|\tilde{\Theta}$, $\sigma(\tilde{\Theta}) = \sigma(\Theta)$ *iff*

$$\exists r \in (-1,1), \qquad \tilde{\Theta} = \Psi \frac{r+\Psi}{1+r\Psi}. \tag{3.9}$$

Proof: We have $\Phi = \Psi$, so $N^\infty[\bar{\Phi}\Psi] \neq 0$ and by the first proposition the dimension is at least 1. On the other hand, the dimension can not be ≥ 2, for otherwise by the last proposition we would have $N^\infty[b\bar{\Theta}_1 J] \neq 0$ and therefore $N[\bar{\Theta}_1 J] \neq 0$ for some inner functions Θ_1 and J such that $\{J = 1\} = \{\Theta_1 = 1\}$ and $\sigma(\Theta_1) = \sigma(\Theta)$. By assumption, $\infty \notin \sigma(\Theta_1)$, so and we get a contradiction with the twin function theorem in Section 2.8. The formula (3.9) now follows from the construction (3.8) in the proof of the first proposition. □

Remark. One can show that the statement is true even without assumption $\infty \notin \sigma(\Theta)$. Also, one can state the following "one-sided" version (see Remark (a) in Section 2.8):

Let $\Theta = \Psi^2$. *Then* $\tilde{\Theta}$ *satisfies* $\Psi|\tilde{\Theta}$, $\sigma(\tilde{\Theta}) \subset \sigma(\Theta)$ *iff*

$$\exists r \in [-1,1], \qquad \tilde{\Theta} = \Psi \frac{r+\Psi}{1+r\Psi}.$$

In other words, the (convex) set of all such $\tilde{\Theta}$'s is the segment $[-\Psi, \Psi]$. Once we know that the dimension is one, the formula follows from the obvious fact that $\pm\Psi$ are extreme points of this set.

3.6. Spectral theory interpretation: Hochstadt-Liberman and Khodakovski theorems.

Consider a Schrödinger operator $L = (q, \alpha, \beta)$ on (a,b), where $q \in L^1_{\mathrm{loc}}(a,b)$ and α, β are selfadjoint boundary conditions at a and b respectively; the endpoints can be infinite and/or singular. We assume that L has compact resolvent. As usual, $\sigma(L)$ denotes the spectrum of L.

Suppose $a < c < b$. We will write q_- for the restriction of q to (a,c) and q_+ for the restriction of q to (c,b). We say that the data $(q_-, \alpha, \sigma(L))$ *determines* L if for any other Schrödinger operator $\tilde{L} = (\tilde{q}, \tilde{\alpha}, \tilde{\beta})$,

$$q_- = q, \ \alpha = \tilde{\alpha}, \ \sigma(\tilde{L}) = \sigma(L) \qquad \Rightarrow \qquad \tilde{q}_+ = q_+, \ \tilde{\beta} = \beta.$$

Let Θ_- denote the Weyl inner function of (q_-, α) computed at c and Θ_+ the Weyl inner function of (q_+, β) computed at c.

Lemma. $\sigma(L) = \sigma(\Theta_- \Theta_+)$.

Proof: The equation $\Theta_-(\lambda)\Theta_+(\lambda) = 1$ is equivalent to the statement

$$m_+(\lambda) + m_-(\lambda) = 0 \quad \text{or} \quad m_-(\lambda) = m_+(\lambda) = \infty$$

for the corresponding m-functions. The latter means that we have the matching

$$\frac{\dot{u}_-(c,\lambda)}{u_-(c,\lambda)} = \frac{\dot{u}_+(c,\lambda)}{u_+(c,\lambda)}$$

for any two non-trivial solutions $u_-(\cdot,\lambda)$ and $u_+(\cdot,\lambda)$ of the Schrödinger equation with boundary conditions α and β respectively, which is possible if and only if λ is an eigenvalue of L. □

Corollary. *$(q_-,\alpha,\sigma(L))$ determine L if the data $(\Theta_-,\sigma(\Theta_-\Theta_+))$ determine Θ_+.*

Here we of course rely on the fundamental uniqueness theorem of Borg and Marchenko [Borg], [M1]: the m-function (and therefore the Weyl inner function) determines both the potential and the boundary condition.

Remark. We would have an "iff" statement if we considered the problem in some class of canonical systems with a one-to-one correspondence between the systems and inner functions such as the class of Krein's "strings", see [12], [14]. The effective characterization of inner functions of Schrödinger operators is an open problem, so we will use our general results to state only sufficient conditions for Schrödinger operators. To obtain necessary condition one has to use more specific techniques of the Schrödinger operator theory, see [7], [22].

Let us apply the above corollary to the situation described at the end of the last subsection.

Example 1. *Let L be a Schrödinger operator on $\mathbb{R}$ with compact resolvent and limit point boundary conditions at $\pm\infty$. Suppose the potential $q(x)$ is an even function:*

$$q(-x)=q(x), \qquad (x>0).$$

Then $q|_{\mathbb{R}_-}$ and $\sigma(L)$ determine L.

Proof: By Everitt's theorem [15] (we recall it in the next subsection), all the inner functions $(r+\Psi)/(1+r+\Psi)$ in (3.9) with $r\neq 0$ are not Weyl inner functions corresponding to a Schrödinger operator. □

This result is a special case of Khodakovski's theorem [24], where only the equality $q(-x)\leq q(x)$ for $x>0$ is assumed. The full version of Khodakovski's theorem requires a slightly different approach which we describe in the next subsection. Similarly, from the remark at the end of Section 2.5 we derive the following statement.

Let L be as above, and let $\tilde{L}$ be another Schrödinger operator on $(-\infty,b)$, $b\geq 0$. If

$$q=\tilde{q}\quad\text{on}\quad\mathbb{R}_-\qquad\text{and}\qquad\sigma(\tilde{L})\subset\sigma(L),$$

then either $\tilde{L}=L$ or $b=0$ and $\tilde{L}$ is the operator with potential q_- and Dirichlet or Neumann condition at 0.

Example 2. *Let L be a regular selfadjoint Schrödinger operator on $[a,b]$ with non-Dirichlet boundary conditions α and β at a and b respectively. If $c=(a+b)/2$, then $(q_-,\alpha,\sigma(L))$ determine L.*

The statement is also true if one or both boundary conditions are Dirichlet, see next subsection. This is a stronger version of the Hochstadt-Liberman theorem [21], see also [17] which states that if *both* L and $\tilde{L}$ are regular, and $\tilde{q}_-=q_-$, $\tilde{\alpha}=\alpha$, $\sigma(\tilde{L})=\sigma(L)$, then $\tilde{L}=L$. We *do not* require $\tilde{L}$ to be regular. Also, we can replace $\sigma(\tilde{L})=\sigma(L)$ with $\sigma(\tilde{L})\subset\sigma(L)$.

3.7. **Everitt's class of inner functions.** We need the following well-known fact, see [15]:

If $m(z)$ is an m-function of a Schrödinger operator, then

$$m(z) = i\sqrt{z} + o(1), \qquad (z \to \infty,\ z \in i\mathbb{R}_+).$$

It follows that if $\Psi(z)$ is a Weyl inner function of a Schrödinger operator, then

$$\Psi(z) = 1 - \frac{2}{\sqrt{z}} + \frac{2}{z} + o\left(\frac{1}{z}\right), \qquad (z \to \infty,\ z \in i\mathbb{R}_+). \tag{3.10}$$

This motivates the following definitions. We say that a meromorphic inner function Ψ belongs to the class (Ev) if it satisfies the asymptotic relation (3.10). (Note though that (3.10) is by no means a full characterization of Weyl inner functions of Schrödinger operators.)

Definition. Let $\Psi, \Phi \in$ (Ev). We say that $[\Psi, \sigma(\Psi\Phi)]$ determine Φ *in the class* (Ev) if

$$\tilde{\Phi} \in (\mathrm{Ev}), \quad \sigma(\Psi\Phi) = \sigma(\Psi\tilde{\Phi}) \qquad \Rightarrow \qquad \tilde{\Phi} = \Phi.$$

Proposition. *Let $\Psi, \Phi \in$ (Ev). Suppose*

$$\exists p < 1, \quad \dim\ N^p[\bar{\Phi}\Psi] < \infty.$$

Furthermore, suppose we have a representation $\bar{\Phi}\Psi = \bar{H}/H$, where H is an outer function such that $H^{\pm 1} \in C^\omega(\mathbb{R})$, and

$$H \neq o\left(\frac{1}{\sqrt{|z|}}\right), \qquad (z \to \infty,\ z \in i\mathbb{R}_+).$$

*Then $(\Psi, \sigma(\Psi\Phi))$ determine Φ in the class (*Ev*).*

Proof: We first argue as in the proof of the second proposition in Section 3.5. The function

$$G = \frac{\tilde{\Phi} - \Phi}{1 - \tilde{\Phi}\Psi} \in \mathcal{H}^p_\Pi \cap C^\omega(\mathbb{R})$$

satisfies

$$\bar{\Phi}\Psi G = \bar{G}. \tag{3.11}$$

We also derive from (3.10) that

$$G(z) = o\left(\frac{1}{\sqrt{z}}\right), \quad (z \to \infty,\ z \in i\mathbb{R}_+).$$

Since the dimension of the Toeplitz kernel is finite, G has at most finitely many zeros on $\mathbb{R}$ and its inner factor is a finite Blaschke product (if any). Thus we can assume that G is an outer function zero free on $\mathbb{R}$ satisfying (3.11). But in this case $H = G$, and we have a contradiction. □

(Remark. One can show that the statement is true for all $p \leq 2$.)

Examples. (i) If $\Phi = \Psi \in$ (Ev), then $(\Psi, \sigma(\Psi\Phi))$ determine Φ in the class (Ev).
Proof: $H = 1 \neq o(|z|^{-1/2})$. □

(ii) If $\Psi = \Theta_D$ and $\Phi = \Theta_N$, see (1.7), then Ψ and $\sigma(\Psi\Phi)$ determine Φ in (Ev), and if $\Psi = \Theta_N$ and $\Phi = \Theta_D$, then Ψ and the spectrum $\sigma(\Psi\Phi)$ minus any one point determine (in an obvious sense) Φ in the class (Ev). In particular, we have the Hochstadt-Liberman type result for all regular operators with arbitrary boundary conditions.

Proof: Reasoning as in Example 2.7, in the first case we have $H = E_D/E_N$ and so $|H| \sim |z|^{-1/2}$. In the second case, $|H| = |E_N/E_D| \sim |z|^{1/2}$. □

Example: Bessel inner functions. This is an extension of the previous example. We want to show that the Hochstadt-Liberman phenomenon occurs not only for regular potentials.

We consider the Bessel inner functions Θ_ν, $\nu \geq -1/2$, see (1.8). Recall that

$$E_\nu(\lambda) = (1 + i/2 + i\nu)G_\nu(\sqrt{\lambda}) + iF_\nu(\sqrt{\lambda})$$

is a de Branges function of Θ_ν, where $G_\nu(z) = z^{-\nu}J_\nu(z)$ and $F_\nu(z) = zG'_\nu(z)$, see Section 1.6. If $\Psi = \Theta_{\nu_1}$ and $\Phi = \Theta_{\nu_2}$, then we have a representation

$$\bar{\Phi}\Psi = \frac{\bar{H}}{H}, \qquad H = \frac{E_{\nu_1}}{E_{\nu_2}}.$$

Lemma.
$$\left|\frac{E_{\nu_1}(\lambda)}{E_{\nu_2}(\lambda)}\right| \sim |\lambda|^{\frac{\nu_2-\nu_1}{2}}, \qquad \lambda \in i\mathbb{R}_+,\ \lambda \to \infty.$$

Proof: It is known that $|J_\nu(z)| \sim |J_0(z)|$ as $z \to \infty$ in any Stolz angle in $\mathbf{C}_+$. Therefore,

$$|G_\nu(z)| \sim |z^{-\nu}J_0(z)|,$$

and we also have

$$|F_\nu(z)| \sim |G_{\nu-1}(z)| \sim |z^{1-\nu}J_0(z)|.$$

(The first relation follows from the identity $zJ'_\nu(z) = zJ_{\nu-1}(z) - \nu J_\nu(z)$, which implies $F_\nu = G_{\nu-1} - 2\nu G_\nu$.) □

We can now apply Proposition. In particular, we get the following result.

Theorem. *Let L be the Schrödinger operator with potential $q(t) = 2t^{-2}$ on $[0, 2]$ and with Dirichlet boundary condition at $t = 2$. Then $q|_{(0,1)}$ and the spectrum $\sigma(L)$ determine L in the class of Schrödinger operators.*

Proof: In our usual notation we have

$$\Psi = \Theta_{3/2}, \qquad \bar{\Phi}\Theta_{1/2} = \frac{\bar{F}}{F}, \quad F^{\pm 1} \in H^\infty,$$

see Section 2.4. Therefore,

$$\begin{aligned}\bar{\Phi}\Psi &= (\bar{\Phi}\Theta_{1/2})\bar{\Theta}_{1/2}\Theta_{3/2} \\ &= \frac{\bar{F}}{F}\,\frac{E_{1/2}}{\bar{E}_{1/2}}\,\frac{\bar{E}_{3/2}}{E_{3/2}} \;=\; \frac{\bar{H}}{H},\end{aligned}$$

where

$$H = \frac{FE_{3/2}}{E_{1/2}}$$

is an outer function such that $H^{\pm 1} \in C^\omega(\mathbb{R})$ and $H(iy) \neq o(1/\sqrt{y})$ as $y \to +\infty$. We can apply the proposition because the argument of the unimodular function

$\bar{\Phi}\Psi$ is bounded, which is a consequence of the well-known asymptotic for the zeros of Bessel's functions. $\square$

Defining sets

3.8. **Defining sets of inner functions.** Let $\Phi = e^{i\phi}$ be a meromorphic inner function, and let $\Lambda \subset \mathbb{R}$. We say that Λ is a *defining set* for Φ if

$$\tilde{\Phi} = e^{i\tilde{\phi}}, \quad \tilde{\phi} = \phi \quad \text{on} \quad \Lambda \qquad \Rightarrow \qquad \Phi \equiv \tilde{\Phi}.$$

In this definition we tacitly assume $\phi(\pm\infty) = \pm\infty$. In the "one-sided" case, say if $\phi(-\infty) > -\infty$ and $\phi(+\infty) = +\infty$, one should modify the definition in an obvious way.

One can extend this definition to divisors. For instance, if all points in $\Lambda \subset \sigma(\Phi)$ are double, then the equality $\tilde{\Phi} = \Phi$ on Λ means that the spectral measures the inner functions coincide on Λ.

Let us mention several special cases.

(a) *Two spectra problem.* This corresponds to the case

$$\Lambda = \{\Phi = 1\} \cup \{\Phi = -1\}.$$

The meaning of the following statement is that Λ is defining for Φ with deficiency one (in the case $\phi(\pm\infty) = \pm\infty$, to be accurate). Various related statements are of course well-known, see e.g. [7].

Let Φ be a meromorphic inner function. Then a meromorphic inner function $\tilde{\Phi}$ satisfies $\{\tilde{\Phi} = 1\} = \{\Phi = 1\}$ and $\{\tilde{\Phi} = -1\} = \{\Phi = -1\}$ iff

$$\tilde{\Phi} = \frac{\Phi - c}{1 - c\Phi}, \qquad c \in (-1, 1). \tag{3.12}$$

The easiest way to see this is use Krein's shift construction: since

$$\Re\left[\frac{1}{\pi i}\log\frac{\tilde{\Phi}+1}{\tilde{\Phi}-1}\right] = \chi_e \qquad \text{on} \quad \mathbb{R},$$

where $e = \{\Im\Phi > 0\}$, we have

$$\frac{1}{\pi i}\log\frac{\tilde{\Phi}+1}{\tilde{\Phi}-1} = \mathcal{S}\chi_e + \text{const.} \qquad \square$$

This argument also shows that *given any two intertwining discrete sets $\Lambda_\pm$ of real numbers there is a meromorphic inner function Φ such that*

$$\{\Phi = \pm 1\} = \Lambda_\pm.$$

Let us also mention that the statement (3.12) can be derived from the twin inner function theorem, see Section 2.8. For instance, suppose $I := \tilde{\Phi}\Phi$ has no point mass at infinity. We want to show that $\tilde{\Phi} - \Phi = c(1 - I)$. It is enough to check that the dimension of the set of functions $F \in K_I^\infty$ vanishing on $\{I = 1\}$ is at most one. (Obviously, both $\tilde{\Phi} - \Phi$ and $1 - I$ belong to this set.) If not, we would have dim $N^\infty[\bar{I}J] \geq 2$ for some J vanishing on $\{I = 1\}$. But then $N^\infty[b\bar{I}J] \neq 0$, and $N[\bar{I}J] \neq 0$, which is impossible by the twin inner function theorem. $\square$

(b) *General mixed data spectral problem.* The Hochstadt-Liberman problem for inner functions that we discussed above can be viewed as a special case of the defining sets problem. It is easy to see that if (assuming $\arg\Theta(\pm\infty) = \pm\infty$) $\Theta = \Psi\Phi$ and $\Lambda = \sigma(\Theta)$, then

$$(\Psi, \sigma(\Theta)) \quad \text{determine} \quad \Theta \qquad \Leftrightarrow \qquad \Lambda \quad \text{is defining for} \quad \Phi.$$

This can be generalized in the following way. Let $\Theta = \Psi\Phi$ be a given meromorphic inner function and let $\{\lambda_n\}$ be the set of its eigenvalues numbered in the increasing order. Given $M \subset \mathbf{Z}$ we denote

$$\sigma_M(\Theta) = \{\lambda_n : \ n \in M\}.$$

The question is whether the factor Ψ and the partial spectrum $\sigma_M(\Theta)$ determine Θ, i.e. whether

$$\tilde{\Theta} = \Psi\tilde{\Phi}, \quad \tilde{\lambda}_n = \lambda_n \ (n \in M) \qquad \Rightarrow \qquad \tilde{\Theta} \equiv \Theta.$$

Once again, this is equivalent (assuming $\phi(\pm\infty) = \pm\infty$) to saying that $\Lambda = \sigma_M(\Theta)$ is a defining set for Φ. The spectral theory meaning was explained in Section 3.6 (but now we consider eigenvalues from different spectra), and the partial spectral problem for Schrödinger operators and Jacobi matrices appeared in several publications, e.g. [17], [18].

(c) *A version for spectral measures.* Given a meromorphic inner function Θ and a factor $\Psi|\Theta$, and also given a part of the spectrum $\Lambda = \sigma_M(\Theta)$, the question is whether there is another inner function $\tilde{\Theta} \neq \Theta$ such that $\Psi|\tilde{\Theta}$ and the spectral measures $\mu = \mu_\Theta$ and $\tilde{\mu} = \mu_{\tilde{\Theta}}$ coincide on Λ:

$$\tilde{\lambda}_n = \lambda_n, \quad \tilde{\mu}\{\lambda_n\} = \tilde{\mu}\{\lambda_n\}, \qquad (n \in M).$$

Claim: If $\Theta = \Psi\Phi$, then Ψ and the spectral measure on $\Lambda = \sigma_M(\Theta)$ determine Θ iff the divisor $2\chi_\Lambda$ is defining for Φ.
Indeed, if $\tilde{\Theta} = \Psi\tilde{\Phi}$, and

$$\arg\tilde{\Theta}(\lambda_n) = \arg\tilde{\Theta}(\lambda_n) = 2\pi n, \qquad (n \in M),$$

then

$$\arg\tilde{\Phi} = \arg\Phi \qquad \text{on} \quad \Lambda.$$

The relation

$$\mu\{\lambda\} = \tilde{\mu}\{\lambda\}, \qquad \lambda \in \Lambda$$

then implies, see (1.4), $\tilde{\Theta}'(\lambda) = \Theta'(\lambda)$, so

$$\Psi'(\lambda)\tilde{\Phi}(\lambda) + \Psi(\lambda)\tilde{\Phi}'(\lambda) = \Psi'(\lambda)\Phi(\lambda) + \Psi(\lambda)\Phi'(\lambda), \qquad (\lambda \in \Lambda),$$

and

$$(\arg\tilde{\Phi})' = (\arg\Phi)' \qquad \text{on} \quad \Lambda. \qquad \square$$

Again, the spectral theory interpretation is the same as above: we know some part of a differential operator and some part of its spectral measure and we want to know if this information determines the operator uniquely.

As usual we can consider the problem in a restricted class of inner functions. Here is the simplest example.

Example. Let $\Theta = \Psi\Phi$ be a finite Blaschke product. Then Ψ and $\Lambda \subset \sigma(\Theta)$ determine Θ iff $\#\Lambda > 2\deg\ \Phi$ in the class of Blaschke products of a fixed degree. Similarly, Ψ and the spectral measure on Λ determine Θ iff $\#\Lambda > \deg\ \Phi$. This extends in an obvious way to the cases where only Φ or Ψ has a finite degree. These facts follow for instance from the statements in the next section, also cf. [18].

3.9. **Relation to uniqueness sets.**

Proposition. *Λ is not defining for Φ if there is a non-constant function $G \in K^\infty_\Phi$ such that*

$$G = \bar{G} \quad \text{on} \quad \Lambda. \tag{3.13}$$

Proof: We can assume $\|G\|_\infty < 1$. Define $F \in H^\infty$ by the equation $\bar{\Phi}G = \bar{F}$ on $\mathbb{R}$, and consider

$$\tilde{\Phi} = \frac{\Phi + F}{1 + G}.$$

Then $\tilde{\Phi}$ is an inner function because it is in $\mathcal{N}^+$ and

$$|\Phi + F| = |\Phi + \Phi\bar{G}| = |1 + G| \qquad \text{on} \quad \mathbb{R}.$$

Also, $\tilde{\Phi} \neq \Phi$ because otherwise we would have $F = G\Phi$, which together with $F = \Phi\bar{G}$ implies $G = \bar{G}$, so $G = \text{const}$. Finally, we have

$$\tilde{\Phi} = \Phi\frac{1 + \bar{\Phi}F}{1 + G} = \Phi\frac{1 + \bar{G}}{1 + G} = \Phi \qquad \text{on} \quad \Lambda,$$

and since

$$\|\arg\ \tilde{\Phi} - \arg\ \Phi\|_{L^\infty(\mathbb{R})} < 2\pi$$

by construction, we get $\arg\ \tilde{\Phi} = \arg\ \Phi$ on Λ. □

Remark. The condition (3.13) is very close to the condition that Λ is a not a uniqueness set for $K^\infty[\Phi^2]$. The precise relation between the two statements is an interesting question, which we will not discuss here. We only mention that if $p \in (1, \infty)$, then

$$\exists G \in K^p[\Phi], \quad G \not\equiv \text{const}, \quad G = \bar{G} \quad \text{on} \quad \Lambda,$$

iff

$$\exists F \in K^p[\Phi^2], \qquad F \not\equiv 0, \quad F = 0 \quad \text{on} \quad \Lambda.$$

The above proposition gives a necessary condition for a set Λ to be defining for Φ. To get sufficient conditions one can use the following obvious observation.

Lemma. *If $\tilde{\Phi} = \Phi$ on Λ and $F = \tilde{\Phi} - \Phi$, then*

$$F \in K^\infty[\tilde{\Phi}\Phi], \qquad F = 0 \quad \text{on} \quad \Lambda.$$

If we also have $\arg\ \tilde{\Phi} = \arg\ \Phi$ on Λ (as in the definition of defining sets), then we can estimate the argument of $\tilde{\Phi}\Phi$ in terms of the data (Φ, Λ), so we can apply our results concerning uniqueness sets.

3.10. **Defining sets of regular operators. Horváth theorem.** We now consider the defining sets problem in some restricted classes of inner functions. We will use the spectral theory language. For $r \geq 1$ let $\mathrm{Schr}(L^r, D)$ denote the class of selfadjoint Schrödinger operators on $[0,1]$ with an L^r potential and Dirichlet boundary condition at 0.

We say that $\Lambda \subset \mathbb{R}$ is a *defining set for the class* $\mathrm{Schr}(L^r, D)$ if for any two operators in $\mathrm{Schr}(L^r, D)$ with potentials q and $\tilde{q}$, the equality $\tilde{\Theta} = \Theta$ on Λ implies $\tilde{q} \equiv q$, where $\tilde{\Theta}$ and Θ are the corresponding Weyl inner functions.

We have a similar definition for the classes $\mathrm{Schr}(L^r, N)$ of Schrödinger operators with Neumann boundary condition at 0.

Let Θ_D denote the standard inner function (1.7), i.e. the Weyl inner function in the case $q \equiv 0$. From Lemma 3.9 we immediately conclude that

Λ *is defining in the class* $\mathrm{Schr}(L^1, D)$ *if* Λ *is a uniqueness set of* $K^\infty[\Theta_D^2]$.

This sufficient condition is not optimal because for regular operators, the function $\tilde{\Phi} - \Phi$ (see the statement of Lemma 3.9) has some extra smoothness at infinity as follows from the standard asymptotic formulae (see the end of the section), which are getting more precise if we require more regularity of the potential, in particular if we consider the case $q \in L^r$ with $r > 1$.

In an interesting paper [22], Horváth gives a complete description of defining sets in terms of uniqueness sets of certain model spaces (or, equivalently, in terms of the completeness problem for exponential functions). The description involves the spaces $\mathcal{F}L^r \equiv \mathcal{F}L^r(-2,2)$, where $\mathcal{F}$ stands for the classical Fourier transform (0.1). Recall that

$$PW_2 = \mathcal{F}L^2 \subset \mathcal{F}L^1 \subset \mathrm{Cart}_2 \cap \mathrm{L}^\infty(\mathbb{R}).$$

Here's a selection of Horváth' results. We use the following notation: $\sqrt{\Lambda} = \{z : z^2 \in \Lambda\}$, and $\sqrt{\Lambda} \cup \{*, *\}$ means $\sqrt{\Lambda}$ plus any two points.

(i) Λ *is defining in the class* $\mathrm{Schr}(L^r, D)$ *iff* $\sqrt{\Lambda} \cup \{*, *\}$ *is a uniqueness set of* $\mathcal{F}L^r$*;*

(ii) Λ *is defining in* $\mathrm{Schr}(L^r, N)$ *if* $\sqrt{\Lambda}$ *is not a zero set of* $\mathcal{F}L^r$.

(In the second case, the "only if" part of Horváth' theorem comes with some additional condition.)

Let us explain how to prove the "if" parts of these statements using the methods of this paper. We prove for example (ii).

Proposition. Λ *is defining in the class* $\mathrm{Schr}(L^2, D)$ *if* $\sqrt{\Lambda} \cup \{*, *\}$ *is a uniqueness set of* PW_2.

Proof: Let $q, \tilde{q} \in L^2(0,1)$. Without loss of generality we will assume that the corresponding Schrödinger operators with boundary conditions (D) at 0 and (N) at 1 are positive. Otherwise, we simply add a large positive constant a to both potentials, and using the transformation

$$F(z) \mapsto F(\sqrt{z^2 + a^2})$$

for even entire functions we observe that $\sqrt{\Lambda}$ is a uniqueness set iff $\sqrt{\Lambda + a}$ is.

Let Θ^* and $\tilde\Theta^*(z)$ be the square root transforms of Θ and $\tilde\Theta$, the Weyl functions taken with sign minus, see Section 1.8. From the standard asymptotic formula for solutions of a regular Schrödinger equation we obtain

$$\frac{\Theta^*}{S^2}=\frac{\bar H}{H}\quad \text{on}\quad \mathbb{R},\qquad H^{\pm 1}\in H^\infty, \tag{3.14}$$

and

$$x[\Theta^*(x)-\tilde\Theta^*(x)]\in L^2(\mathbb{R}). \tag{3.15}$$

(For convenience we reproduce the standard argument at the end of the proof .)

If $\tilde\Theta=\Theta$ on Λ, then since $\tilde\Theta^*(0)=\Theta^*(0)$, we have

$$\Theta^*=\tilde\Theta^*\qquad \text{on}\quad \{0\}\cup\sqrt{\Lambda},$$

where we regard Θ^* and $\tilde\Theta^*$ as meromorphic functions in the whole plane. By (3.15),

$$(z-1)(\Theta^*-\tilde\Theta^*)\in K[\Theta^*\tilde\Theta^*],$$

so $\sqrt{\Lambda}\cup\{0,1\}$ is a zero set of some $K[\Theta^*\tilde\Theta^*]$ function, and therefore by (3.14) a zero set of some function in $K[S^4]$ or PW_2 . (For zeros in $\mathbf{C}_-$ we can use the argument with dual reproducing kernels as in Section 3.1.) $\square$

Proof of (3.14)–(3.15). If $s>0$, then the solution $u_s(t)$ of the IVP

$$-\ddot u+qu=s^2u,\qquad u(0)=0,\quad \dot u(0)=1,$$

satisfies the integral equation

$$u_s(x)=\sin sx+\frac{1}{s}\int_0^x\cos s(x-t)\ q(t)\ u_s(t)\ dt.$$

Iterating, we have

$$u_s(1)=\sin s+\frac{F(s)}{s}+\frac{R(s)}{s^2},$$

where

$$F(s)=\int_0^1\cos s(1-t)\ \sin st\ q(t)\ dt,$$

and

$$R(s)=\int_0^1\cos s(1-x)\ q(x)\ dx\int_0^x\cos s(x-t)\ q(t)\ u_s(t)\ dt.$$

We have an elementary a priori bound

$$|u_s(t)|\le C,\qquad (s>0,\ t\in[0,1]),$$

so

$$\forall s,\quad |R(s)|\le \text{const}.$$

On the other hand, F is basically the Fourier transform of a function on $(-1,1)$, and

$$q\in L^2\quad\Rightarrow\quad F\in L^2(\mathbb{R}).$$

We also get the corresponding estimates of $\dot u_s(1)$. The resulting estimates of Θ imply both statements.

Multiplier theorems

4.1. **Tempered inner functions.** A meromorphic inner function $\Theta = e^{i\Theta}$ is called *tempered* if Θ' has at most polynomial growth at $\pm\infty$:

$$\exists N, \quad \Theta'(x) = O\left(|x|^N\right), \quad x \to \infty.$$

Theorem. *Suppose Θ is a tempered inner function. Then for any meromorphic inner function J and any $p > 0$,*

$$N^p[\bar{\Theta}J] \neq 0 \qquad \Rightarrow \qquad \exists n, \quad N^\infty[\bar{b}^n\bar{\Theta}J] \neq 0.$$

Note that the opposite is trivial: if $q > p$ then $N^q[\bar{\Theta}J] \neq 0$ implies $N^p[\bar{b}^n\bar{\Theta}J] \neq 0$ with $n = n(p, q)$. Questions of this type were studied by Dyakonov [13] who was the first to observe the analogy with the Beurling-Malliavin multiplier theorem.

The proof for $p = 2$ is elementary: if $F \in N[\bar{\Theta}J]$, then $JF \in K_\Theta$ and by (1.6) and (1.4),

$$\sum_{\lambda \in \sigma(\Theta)} \frac{|F(\lambda)|^2}{|\Theta'(\lambda)|} \asymp \|F\|^2.$$

Thus $|F(\lambda)| \lesssim |\Theta'(\lambda)|$, and this is true for all $\lambda \in \mathbb{R}$ because we can replace Θ with $e^{-i\alpha}\Theta$. It follows that $(z + i)^{-n}F(z) \in N^\infty[\bar{b}^n\bar{\Theta}J]$.

We will derive the theorem from the following special case of Carleson's type embedding theorem of Treil and Volberg [40]. For a given meromorphic inner function Θ denote

$$d(x) = \mathrm{dist}\{x, \{|\Theta| = 0.5\}), \qquad (x \in \mathbb{R}).$$

Claim: *the measure $\nu_x = d(x)\delta_x$ is a Carleson measure for K^p_Θ, i.e.*

$$K^p_\Theta \subset L^p(\nu),$$

where the norm of the embedding depends only on p.

In other word, if $F \in K^p_\Theta$, then

$$d(x)\ |F(x)|^p \leq \mathrm{const}, \qquad (x \in \mathbb{R}). \tag{4.1}$$

Lemma. *If Θ is tempered, then there is an N such that*

$$\mathrm{dist}(x, \{|\Phi| = 0.5\})\ \gtrsim\ (1 + |x|)^{-N}.$$

Proof: We can only consider the case of Blaschke products. If

$$\Theta = B_\Lambda = \prod \epsilon_\lambda \frac{z - \lambda}{z - \bar{\lambda}},$$

then we have

$$\Theta'(x) = \sum \frac{2\Im\lambda}{|x - \lambda|^2} \tag{4.2}$$

and

$$-\log|\Theta(z)| \asymp \sum \frac{\Im\lambda \cdot \Im z}{|z-\bar\lambda|^2}. \tag{4.3}$$

Indeed,

$$\begin{aligned}
-\log|\Theta(z)|^2 &= \sum \log\left|\frac{z-\bar\lambda}{z-\lambda}\right|^2 \\
&\asymp \sum\left[1-\frac{|z-\lambda|^2}{|z-\bar\lambda|^2}\right] \qquad \text{(see below)} \\
&= \sum \frac{(z-\bar z)(\lambda-\bar\lambda)}{|z-\bar\lambda|^2}
\end{aligned}$$

We want to show that if $z = x+iy$ and $y \ll |x|^{-N}$, then (4.3)< const. For each x consider the Stolz sector of some fixed angle at x of height x^{-N} and observe that this sector does not contain any λ, for otherwise, the corresponding term in (4.2) would be of the order $1/\Im\lambda \geq x^N$. This justifies the "see below" item, and makes the estimate of (4.3) in terms of (4.2) obvious. $\square$

Proof of Theorem. Suppose $N^p[\bar\Theta J] \neq 0$, so

$$\bar\Theta J F = \bar G, \qquad F, G \in \mathcal{H}^p \cap C^\omega(\mathbb{R}).$$

We have

$$JF \in K^p_\Theta$$

so by the lemma and by (4.1) we have

$$|F(x)| \lesssim 1 + |x|^N$$

for all $x \in \mathbb{R}$ and some N. It follows that $(z+i)^{-N}F(z) \in N^\infty[\bar b^N \bar\Theta J]$. $\square$

Corollary. *Let $U = \bar\Theta J = e^{i\gamma}$ and let Θ be tempered. If γ is the sum of a bounded and a decreasing functions, then $N^\infty[\bar b^n U] \neq 0$ for some n.*

Note that in the case of a general bounded γ, we can not multiply down to $\mathcal{H}^\infty$ elements of $N[U]$ even by using factors like $\bar S$.

Also note that in the statement of the theorem one can give explicit bounds on n in terms of the growth of $|\Theta'(x)|$. For example, if $U = \bar\Theta J$ and $\Theta' \leq 1$, then $N[U] \subset N^\infty[U]$.

4.2. **Beurling-Malliavin multiplier theorem.**

Theorem. *Suppose Θ is a meromorphic inner function satisfying $|\Theta'| \leq$ const. Then for any meromorphic inner function J, we have*

$$N^+[\bar\Theta J] \neq 0 \qquad \Rightarrow \qquad \forall\epsilon, \quad N^\infty[\bar S^\epsilon \bar\Theta J] \neq 0.$$

This theorem follows from the Beurling-Malliavin multiplier theorem [4]: if W is an outer function, then

$$z^{-1}\log W(z) \in \mathcal{D}(\mathbf{C}_+) \qquad \Rightarrow \qquad W \in (\mathrm{BM}). \tag{4.4}$$

Here $\mathcal{D}(\mathbf{C}_+)$ is the notation for the usual Dirichlet space in the halfplane, and by definition, W is a *Beurling-Malliavin multiplier*, or $W \in (\mathrm{BM})$, if

$$\forall \epsilon > 0,\ \exists G \in K[S^\epsilon], \qquad WG \in L^2(\mathbb{R}).$$

Note that if $|W| \le |W_1|$ and $W_1 \in (\mathrm{BM})$, then $W \in (\mathrm{BM})$.

Lemma. *If $W \in C^\omega(\mathbb{R})$, $|W| \ge 1$, and $(\arg W)' \le \mathrm{const}$ on $\mathbb{R}$, then $W \in (\mathrm{BM})$.*

Proof: We will use some ideas from the proof of Theorem 64 in [12]. We can assume $|W(0)| = 1$. Otherwise, multiply W by $(z+i)$ to get $W(\infty) = \infty$. In this case there's a global minimum, which we can take for 0.
Denote

$$z^{-1}\log W(z) = u(z) + iv(z),$$

where u and v are real-valued functions. Then we have

$$x^{-1}u(x) \in L^1(\mathbb{R}), \qquad x^{-1}u(x) \ge 0 \quad \text{on} \quad \mathbb{R}. \tag{4.5}$$

Since $\arg W \in \tilde{L}^1_\Pi$ and $(\arg W)' \le \mathrm{const}$ on $\mathbb{R}$, by Lemma 3.4 we have

$$\arg W(x) = o(|x|), \qquad x \in \mathbb{R}, \quad x \to \infty, \tag{4.6}$$

and it follows that v is a bounded function in $\mathbf{C}_+$. For $r > 0$ let $D(r)$ denote the semidics $\{|z| < r\} \cap \mathbf{C}_+$. We have

$$\|u + iv\|^2_{\mathcal{D}} = \lim_{r\to\infty} \int_{\partial D(r)} u dv,$$

and

$$\int_{\partial D(r)} u dv = \int_{-r}^{r} uv' dx - uv\Big|_{-r}^{r} + rI'(r), \qquad I(r) := \frac{1}{2}\int_0^\pi v^2\left(re^{i\theta}\right) d\theta.$$

By (4.5) and (4.6), the integrals $\int uv'dx$ are uniformly bounded from above:

$$\int_{-r}^{r} uv' dx = \int_{-r}^{r} \frac{u(x)}{x}(\arg W)'(x) dx - \int_{-r}^{r} \frac{u(x)}{x} \cdot \frac{\arg W(x)}{x} dx < \mathrm{const}.$$

It remains to show that

$$\liminf_{r\to\infty} A(r) < \infty, \qquad A(r) := rI'(r) - uv\Big|_{-r}^{r}.$$

Suppose $A(r) \ge 1$ for all $r \gg 1$. Then since v is bounded, we have

$$I'(r) \ge \frac{1}{r} - \mathrm{const}\frac{u(r) + u(-r)}{r}.$$

By (4.5), this contradicts the uniform boundedness of $I(r)$. □

Corollary. *If $W \in K^+_\Theta$ and $(\arg\Theta)' \le \mathrm{const}$, then $W \in (\mathrm{BM})$.*

Proof: We have $W\bar\Theta = \bar H$ for some $H \in \mathcal{N}^+$. Define

$$W_1 = WH + \Theta.$$

Clearly, $W_1 \in \mathcal{N}^+$, and since

$$\bar\Theta^2 W_1 = \bar\Theta W \bar\Theta H + \bar\Theta = \bar H \bar W + \bar\Theta = \bar W_1,$$

we have $W_1 \in K^+[\Theta^2]$. Notice that

$$|W_1| = |W\bar W \Theta + \Theta| = 1 + |W|^2 \geq 1.$$

Also, $|W| \leq |W_1|$. Since $(\arg\Theta)' \leq \text{const}$, from the equation $\bar\Theta^2 W_1 = \bar W_1 \bar\Phi$, where Φ is an inner function, we obtain $(\arg W_1)' \leq \text{const}$. By the previous lemma, $W_1 \in (\text{BM})$, and therefore $W \in (\text{BM})$. $\square$

Proof of Theorem. Take an outer function $W \in N^+[\bar\Theta J]$. Then $W \in K^+_\Theta$ and by the last corollary, $W \in (\text{BM})$, and therefore $WG \in \mathcal{H}^2$ for some $G \in N^+[\bar S^\epsilon]$. It then follows that

$$WG \in N^+[\bar S^\epsilon \bar\Theta J] \cap \mathcal{H}^2 = N[\bar S^\epsilon \bar\Theta J].$$

It remains to multiply down to $\mathcal{H}^\infty$, which is possible by Theorem 4.1. $\square$

$\mathcal{N}^+$-part of the Beurling-Malliavin theory

In this part of the section we will assume

$$\gamma' \geq -\text{const}. \tag{4.7}$$

We'll give a metric criterion for (non)triviality of the Toeplitz kernel $N^+[e^{i\gamma}]$ up to a gap $S^{\pm\epsilon}$. Recall the basic criterion: $N^+[e^{i\gamma}] \neq 0$ iff

$$\gamma = -\alpha + \tilde h \tag{4.8}$$

for some increasing function $\alpha \in C^\omega(\mathbb{R})$ and some $h \in L^1_\Pi$. As usual we denote $U = e^{i\gamma}$.

Lemma. *Suppose* $\gamma' \geq -\text{const}$. *Then* $\forall \epsilon > 0$, $N^+[US^\epsilon] = 0$ *unless* $\gamma(\mp\infty) = \pm\infty$.

Proof: If $N^+[US^\epsilon] \neq 0$, then by (4.8) $\gamma + \epsilon x + \alpha = \tilde h$, so $(\tilde h)' \geq -\text{const}$ and of course $\tilde h \in L^{o(1,\infty)}_\Pi$. By Lemma 3.4 it then follows that $\tilde h(x) = o(x)$ as $x \to \pm\infty$, so $\gamma + \epsilon x + \alpha = o(x)$, which is possible only if $\gamma(\mp\infty) = \pm\infty$. $\square$

4.3. **Beurling-Malliavin intervals.** Suppose a continuous function $\gamma : \mathbb{R} \to \mathbb{R}$ satisfies

$$\gamma(-\infty) = +\infty, \qquad \gamma(+\infty) = -\infty. \tag{4.9}$$

The family $\mathcal{BM}(\gamma)$ is defined as the collection of the components of the open set $\{\gamma^\star \neq \gamma\}$, where

$$\gamma^\star(x) = \max_{[x,+\infty]} \gamma.$$

For an interval $l = [a,b] \subset \mathbb{R}_+$ or $\subset \mathbb{R}_-$ we write $|l|$ for the Euclidian length, and $\delta(l)$ for the distance from the origin. A family of finite disjoint intervals $\{l\}$ is called *long* if

$$\sum_{\delta(l)\geq 1} \frac{|l|^2}{\delta(l)^2} = \infty.$$

Otherwise, we call the family *short*.

Theorem. *Suppose* $\gamma' > -\mathrm{const}$.

(i) If $\gamma \notin (4.9)$*, or if* $\gamma \in (4.9)$ *but the family* $\mathcal{BM}(\gamma)$ *is long, then*

$$\forall \epsilon > 0, \quad N^+[S^\epsilon U] = 0.$$

(ii) If $\gamma \in (4.9)$ *and* $\mathcal{BM}(\gamma)$ *is short, then*

$$\forall \epsilon > 0, \quad N^+[\bar{S}^\epsilon U] \neq 0.$$

The first statement corresponds to the "second Beurling-Malliavin theorem" [5], and statement (ii) to the so called "little multiplier theorem", see [25], [19].

4.4. **Second Beurling-Malliavin theorem.** The first part of Theorem 4.3 follows from a more general fact; we don't need to assume (4.7). For an interval $l \subset \mathbb{R}$ we denote

$$\Delta_l^*[\gamma] = \inf_{l''} \gamma - \sup_{l'} \gamma,$$

where l' and l'' are the left and the right adjacent intervals of length $|l|$.

Theorem. *If* $\exists c > 0$ *and a long family of intervals* $\{l\}$ *such that*

$$\Delta_l^*[\gamma] \geq c|l|, \tag{4.10}$$

then $N^+[U] = 0$.

A simple standard argument shows that we can assume without loss of generality that all intervals satisfy the inequality $10|l| < \delta(l)$, and that the multiplicity of the covering $\{10l\}$ is finite. (The interval $10l$ is concentric with l and has length $10|l|$.)

The idea of the proof is quite simple. According to the basic criterion (4.8) we have to exclude the possibility

$$\gamma + \alpha = \tilde{h}, \qquad \alpha \ \uparrow, \quad h \in L^1_\Pi.$$

Since α is increasing we have

$$\Delta_l^*[\tilde{h}] \gtrsim |l|.$$

Suppose we can localize this estimate to each function $\tilde{h}_l$, where h_l is the restriction of h to the interval $10l$. Choosing $A \asymp |l|$ and applying the weak type inequality, we have

$$\frac{|l|}{\delta(l)^2} \lesssim \Pi\{|\tilde{h}_l| > A\} \lesssim \frac{\|h_l\|_\Pi}{A}, \tag{4.11}$$

so

$$\frac{|l|^2}{\delta(l)^2} \lesssim \|h_l\|_\Pi. \tag{4.12}$$

Summing up over l's we arrive to a contradiction:

$$\infty = \sum \frac{|l|^2}{\delta(l)^2} \lesssim \sum \|h_l\|_\Pi \lesssim \|h\|_\Pi < \infty.$$

Proof of Theorem. For an interval l we denote by Q_l its Carleson square:

$$Q_l = \{z : x \in l, \ |l| < y < 2|l|\}.$$

Also, denote

$$H(z) = \int_{\mathbb{R}} \frac{h^-(t)dt}{(t-z)^2}, \qquad (z \in \mathbf{C}_+),$$

where as usual $h_- = \max\{0, -h\}$. We claim that if the estimate (4.12) does not hold for some interval l in our family, then

$$|H| \gtrsim 1 \qquad \text{on} \quad Q_l. \tag{4.13}$$

To see this we observe that the argument (4.11) is valid unless there is an interval l_1, $l \subset l_1 \subset l \cup l' \cup l''$, such that

$$\Delta_{l_1} \tilde{f} \geq (c/2)|l|, \qquad f := h - h_l.$$

Let's assume (for simplicity of notation) $l_1 = l$, so $\Delta \tilde{f} \equiv \Delta_l \tilde{f} \gtrsim l$. Represent $f = f^+ - f^-$, $f^\pm \geq 0$. The functions $\tilde{f}^\pm$ are decreasing on l:

$$\tilde{f}'_\pm(x) = -\frac{1}{\pi} \int_{\mathbb{R}\setminus(10l)} \frac{f^\pm(t)dt}{(t-x)^2} > 0, \qquad (x \in l).$$

It follows that $-\Delta \tilde{f}^- \gtrsim |l|$, and so there is a point $x_* \in l$ such that

$$\frac{1}{\pi} \int \frac{f^-(t)dt}{(t-x_*)^2} = -\tilde{f}'_-(x_*) = -\frac{\Delta \tilde{f}^-}{|l|} \gtrsim 1.$$

If $z \in Q_l$, then

$$\left| \int \frac{f^-(t)dt}{(t-z)^2} \right| \geq \int \Re \left[\frac{1}{(t-z)^2} \right] f^-(t)dt \asymp \int \frac{f^-(t)dt}{(t-x_*)^2} \gtrsim 1.$$

On the other hand, if $z \in Q_l$ and if (4.12) is not true, then

$$\left| \int \frac{h_l^-(t)dt}{(t-z)^2} \right| \lesssim \frac{1}{|l|^2} \int_{(10l)} |h| \asymp \frac{\delta(l)^2}{|l|^2} \int_{(10l)} |h| d\Pi \ll \frac{\delta(l)^2}{|l|^2} \cdot \frac{|l|^2}{\delta(l)^2} = 1,$$

and we get

$$|H(z)| = \left| \int \frac{f^-(t) + h_l^-(t)}{(t-z)^2} dt \right| \geq \left| \int \frac{f^-(t)dt}{(t-z)^2} \right| - \left| \int \frac{h_l^-(t)dt}{(t-z)^2} \right| \gtrsim 1.$$

To finish the proof of the theorem it remains to show that

$$\sum_{l \in (*)} \frac{|l|^2}{\delta(l)^2} < \infty,$$

where we write $l \in (*)$ if (4.13) holds for l. Denote $\psi = \sum_{l \in (*)} |l| \chi_l$, so

$$\sum_{l \in (*)} \frac{|l|^2}{\delta(l)^2} \asymp \int \frac{\psi(t)dt}{1+t^2}.$$

Then we have

$$\int \frac{\psi(t)}{1+t^2} dt \lesssim \int_1^\infty \frac{dA}{A^3} \int_{-A}^{A} \psi(t)dt.$$

Fix $C \gg 1$ such that

$$\int_{|t| \geq C} \frac{h^-(t)dt}{1+t^2} \ll 1.$$

We claim that

$$\int_{-A}^{A} \psi(t)dt \lesssim \int_{-CA}^{CA} h^-(t)dt,$$

so we have

$$\int \frac{\psi(t)}{1+t^2}dt \lesssim \int_1^\infty \frac{dA}{A^3} \int_{-CA}^{CA} h^-(t)dt \lesssim \|h\|_\Pi$$

(and we are done). To prove the claim, fix A and consider the 2D Hilbert transform

$$H_A(z) = \int_{-CA}^{CA} \frac{h^-(t)dt}{(t-z)^2}.$$

If $l \subset (-A, A)$ and $l \in (*)$, then

$$|H_A(z)| \asymp |H(z)| \gtrsim 1, \qquad z \in Q_l,$$

by the choice of C. Applying the weak-L^1 estimate for the Hilbert transform, we get

$$\int_{-A}^{A} \psi(t)dt \lesssim \text{Area}(|H_A| \gtrsim 1) \lesssim \int_{-CA}^{CA} h^-(t)dt. \qquad \square$$

4.5. **Little multiplier theorem.** We now turn to the proof of the second part of Theorem 4.3. We assume (4.7) and $\gamma(\mp\infty) = \pm\infty$. The function $\gamma^\star$ and the family $\mathcal{BM}(\gamma)$ were defined in section 4.3. Note that $\gamma^\star$ is decreasing and $\gamma^\star - \gamma \geq 0$.

Lemma. *If the family $\mathcal{BM}(\gamma)$ is short, then $\gamma^\star - \gamma \in L^1_\Pi$.*

Proof:

$$\int_l \frac{\gamma^\star - \gamma}{1+x^2}dx \lesssim \frac{1}{\delta(l)^2} \int_0^{|l|} tdt \asymp \frac{|l|^2}{\delta(l)^2}.$$

$\square$

The proof of the little multiplier is extremely simple if we settle for a slightly weaker statement:

$$\sum_{\mathcal{BM}(\gamma)} \frac{|l|^2 \log^+ |l|}{\delta(l)^2} < \infty \qquad \Rightarrow \qquad N^+[U] \neq 0.$$

Indeed, the last computation shows that in this case we have

$$\int (\gamma^\star - \gamma) \log^+ (\gamma^\star - \gamma) \; d\Pi < \infty,$$

which is a (necessary and) sufficient condition for $(\gamma^\star - \gamma)\tilde{} \in L^1_\Pi$, see [42]. We have a representation

$$\gamma = \gamma^\star + (\gamma - \gamma^\star),$$

where the first term is decreasing and the second one is in $\tilde{L}^1_\Pi$, so $N^+[U] \neq 0$ by the basic criterion.

Main Lemma. *If the family $BM(\gamma)$ is short, then for any given $\epsilon > 0$ there is a function β such that $\beta' \leq \epsilon$ near $\pm\infty$ and*

$$\gamma^\star - \gamma + \beta \in \tilde{L}^1_\Pi.$$

The little multiplier theorem now follows immediately:

$$\gamma(x) - \epsilon x = (\gamma - \gamma^\star - \beta) + (\beta - \epsilon x) + \gamma^\star.$$

The first term is in $\tilde{L}^1_\Pi$, and the last two terms are decreasing at infinities.

Proof of Main Lemma. Denote $f = \gamma^\star - \gamma$, so f is a non-negative function in L^1_Π, $f' \le \text{const}$, and the family $\mathcal{L} = \mathcal{BM}(\gamma)$ of the components of $\{f \neq 0\}$ satisfies

$$\sum_{l \in \mathcal{L}} \frac{|l|^2}{\delta(l)^2} < \infty.$$

These are the only properties of f that will be used. We will also need the following notation: for an interval $l = [a, b]$ we define its "tent" function

$$T_l(x) = \begin{cases} x - a, & a \le x \le (a+b)/2, \\ b - x, & (a+b)/2 \le x \le b, \\ 0, & x \in \mathbb{R} \setminus l. \end{cases}$$

Note that

$$\|T_l\|_\Pi \gtrsim \frac{|l|^2}{(\delta(l) + |l|)^2}. \tag{4.14}$$

We will construct disjoint intervals l_n such that $\{f \neq 0\} \subset \cup\, l_n$,

$$\sum_n \frac{|l_n|^2}{\delta(l_n)^2} < \infty, \tag{4.15}$$

and

$$\forall n \quad \exists \epsilon_n \in [0, \epsilon], \qquad \int_{l_n} (f - \epsilon_n T_{l_n})\, d\Pi = 0. \tag{4.16}$$

Let us show that the existence of such intervals l_n implies the main lemma. We will use the easier part of the atomic decomposition technique of Hardy spaces, see [11].

We define

$$\beta = -\sum_n \epsilon_n T_{l_n}, \qquad g = f + \beta,$$

so β is in $\mathrm{Lip}(\epsilon)$, and all we need to check is that $\tilde{g} \in L^1_\Pi$ or, in other words, that g belongs to the *real* Hardy space $\mathcal{H}^1_\Pi(\mathbb{R})$. We can represent g as follows:

$$g = \sum \chi_n g = \sum \lambda_n \frac{\chi_n g}{\lambda_n} := \sum \lambda_n A_n,$$

where χ_n is the characteristic function of l_n, and we choose

$$\lambda_n = \Pi(l_n)\ \|\chi_n g\|_\infty.$$

It is clear that the functions $A_n = \lambda_n^{-1}(\chi_n g)$ are "atoms":

$$\int A_n d\Pi = \frac{1}{\lambda_n} \int_{l_n} g d\Pi = 0 \qquad \text{by} \quad (4.16),$$

and

$$\|A_n\|_\infty = \frac{\|\chi_n g\|_\infty}{\lambda_n} = \frac{1}{\Pi(l_n)}.$$

By (4.15) we also have

$$\sum \lambda_n \lesssim \sum \Pi(l_n)|l_n| \asymp \sum \frac{|l_n|^2}{\delta(l_n)^2} < \infty.$$

It follows that $\sum \lambda_n A_n \in \mathcal{H}^1_\Pi(\mathbb{R})$, see [11].

To finish the proof it remains to describe the construction of the intervals $\tilde{l}_n$ and the slopes ϵ_n. We consider the case where all intervals $l \in \mathcal{L}$ are in $(1, +\infty)$. The construction for intervals in $[-\infty, -1)$ is similar.

Suppose the left endpoint $a_n = a$ of l_n has been constructed, and a is also the left endpoint of some interval $l = (a, b(l)) \in \mathcal{L}$. (To start the induction we take the leftmost endpoint for a_1.) Consider the function

$$F(b) \equiv F_\epsilon(b) = \int_a^b \left[f - \epsilon T_{(a,b)}\right] d\Pi,$$

and define

$$b_n = \min\{b \geq b(l) : \ f(b) = 0, \ F(b) \leq 0\}.$$

For example, if we already have $F(b(l)) \leq 0$, then $b_n = b(l)$. Since $f \in L^1_\Pi$, we have $F(+\infty) = -\infty$ and so $b_n < \infty$. We also define ϵ_n from the equation

$$\int_{a_n}^{b_n} \left[f - \epsilon_n T_{(a_n,b_n)}\right] d\Pi = 0.$$

Finally, we define a_{n+1} as the leftmost endpoint of all intervals $l \in \mathcal{L}$ to the right of l_n.

It is clear from the construction that the intervals l_n cover $\{f \neq 0\}$, that all ϵ_n's are $\leq \epsilon$, and that we have (4.16). Let us check (4.15). We consider three types of intervals l_n:

(a) $F(b_n) < 0$ but $\exists l \in \mathcal{L}$ such that $l \subset l_n$ and $|l|/\delta(l) \asymp |l_n|/\delta(l_n)$,

(b) $F(b_n) = 0$,

(c) other intervals.

Property (4.15) is obvious for the group (a). For the group (b), we use (4.14):

$$\sum_{(b)} \frac{|l_n|^2}{(\delta(l_n) + |l_n|)^2} \lesssim \frac{1}{\epsilon} \int_{\cup_{(a)} l_n} f d\Pi < \infty.$$

The argument for the group (c) is similar as long as we can show that the slopes are $> \epsilon/2$, i.e. $F_{\epsilon/2}(b_n) > 0$. Since $F(b_n) < 0$, b_n is by construction the right endpoint of some interval $l \in \mathcal{L}$, and since $l_n \notin (a)$ we have $|l| \ll |l_n|$. Let c be the left endpoint of the above l; by construction, $F(c) > 0$. We have

$$F_{\epsilon/2}(b_n) = \left(\int_{a_n}^c + \int_c^{b_n}\right) \left[f - \frac{\epsilon}{2} T_{(a,b)}\right] d\Pi > \int_{a_n}^c \left[f - \epsilon T_{(a,c)}\right] d\Pi = F(c) > 0.$$

4.6. **Radius of completeness.** Let $\Lambda \subset \mathbb{R}$ and let $\mathcal{E}_\Lambda$ denote the family of exponential functions $\{e^{i\lambda t}: \ \lambda \in \Lambda\}$. By definition, the *radius of completeness* of the family $\mathcal{E}_\Lambda$ is the number

$$R(\Lambda) = \sup\left\{a : \ \mathcal{E}_\Lambda \quad \text{is complete in} \quad L^2(0,a)\right\}.$$

In terms of Toeplitz kernels, by Proposition 3.2 we have

$$R(\Lambda) = \sup\left\{a : \ N[\bar{S}^a J_\Lambda] = 0\right\},$$

where J_Λ denotes some/any meromorphic inner function J such that $\{J = 1\} = \Lambda$. By the Beurling-Malliavin multiplier theorem we also have

$$R(\Lambda) = \sup\left\{a : \ N^+[\bar{S}^a J_\Lambda] = 0\right\}.$$

The combination of the second Beurling-Malliavin and the little multiplier theorems, see Theorem 4.3, then gives the following metric characterization of $R(\Lambda)$. By definition, the *Beurling-Malliavin density* of Λ is the number

$$d_{\rm BM}(\Lambda) = \inf\{a : \gamma_a(\pm\infty) = \mp\infty \quad \text{and} \quad \mathcal{BM}[\gamma_a] \quad \text{is short}\}, \qquad \gamma_a := 2\pi n_\Lambda - at.$$

Corollary. $R(\Lambda) = d_{\rm BM}(\Lambda)$.

4.7. **General transition parameter problems.** Let M be a unimodular function with non-decreasing continuous argument; we call it a *gap function.* For a given unimodular function $U = e^{i\gamma}$ we want to compute the critical exponent

$$\sup\left\{a : \ N[\bar{M}^a U] = 0\right\}.$$

This number of course can be $\pm\infty$. The Beurling-Malliavin density theorem corresponds to the case $M = S$ and $U = J$ (or $\bar{S}^c J$). More generally, Theorems 4.2 and 4.3 allow us to compute the transition parameter in the case $M = S$ and $U = \bar{\Theta} J$ where $|\Theta'| \le \text{const}$.

Of course, one can state similar problems concerning families of Toeplitz kernels in other functional spaces. Theorem 4.1 states that the critical exponent is the same in all $\mathcal{H}^p$-spaces if both M and Θ are tempered and the "gap" is wide enough: $N^\infty[b^N \bar{M}^\epsilon] \neq 0$ for all $\epsilon > 0$ and $\forall N$.

One could try to generalize the Beurling-Malliavin theory ($M = S$) to arbitrary gap functions. As a first step it is natural to consider the "*standard*" gap functions $S^{(\alpha)}$ and $S_+^{(\alpha)}$, $\alpha > 0$, defined as follows:

$$S^{(\alpha)}(x) = \begin{cases} S(x^\alpha), & x > 0, \\ S(-|x|^\alpha), & x < 0, \end{cases} \qquad S_+^{(\alpha)}(x) = \begin{cases} S(x^\alpha), & x > 0, \\ S(0), & x < 0. \end{cases}$$

Note that these functions satisfy identities like

$$S^{(\alpha)}(kx) = \left[S^{(\alpha)}(x)\right]^{k^\alpha}, \qquad (k > 0).$$

For each standard gap function, one would expect to have some kind of a *Beurling-Malliavin theory*, i.e. a combination of theorems that allow to express the transition parameter in terms of the Beurling-Malliavin intervals under an appropriate growth restriction on γ.

Here is a typical example of a transition parameter problem with a standard gap function.

Example 1. Let $\mathrm{Ai}(x)$ denote the usual Airy function: it is a solution of the equation $y'' = xy$, which is L^2 at $+\infty$. It is well-known that the Airy function is entire and also

$$\mathrm{Ai}(x) = \sqrt{x}F\left(\frac{2}{3}ix^{3/2}\right)$$

for some solution F of the Bessel equation of order $1/3$. If $x = -\lambda$ and $\lambda > 0$, then we have

$$\mathrm{Ai}(-\lambda) = i\sqrt{\lambda}F\left(\frac{2}{3}\lambda^{3/2}\right) \sim \lambda^{-1/4}\cos\left(\frac{2}{3}\lambda^{3/2} + \mathrm{const}\right), \qquad \lambda \to +\infty. \tag{4.17}$$

Given $\Lambda \subset \mathbb{R}$ we ask if the family of shifts

$$\mathcal{E}_\Lambda = \{\mathrm{Ai}(t-\lambda) :\ \lambda \in \Lambda\}$$

is complete in $L^2(\mathbb{R}_+)$. Note that $u_\lambda(t) = \mathrm{Ai}(t-\lambda)$ is an L^2-solution of the Schrödinger equation

$$-\ddot{u} + tu = \lambda u, \qquad t \in \mathbb{R}_+,$$

so we can apply our general approach, see Section 3.1.

Claim. Up to a finite dimensional gap, the family $\mathcal{E}_\Lambda$ is complete in $L^2(\mathbb{R}_+)$ iff

$$N[\bar{M}J_\Lambda] = 0, \qquad M = \left[S_+^{(3/2)}\right]^{2/3}.$$

Proof: According to Section 3.1 and Theorem 4.1, the criterion for completeness (up to a finite dimensional gap) is $N[\bar{\Theta}J_\Lambda] = 0$, where Θ is the Weyl inner function. We can replace Θ by M because of the asymptotic formula (4.17) for $u_\lambda(0) = \mathrm{Ai}(-\lambda)$ and a similar formula for $\dot{u}_\lambda(0)$. □

Remark. This completeness problem, which involves shifts of a given function in $L^2(\mathbb{R}_+))$ is different from the well-known Wiener problem concerning shifts in $L^2(\mathbb{R})$. The latter is essentially the problem concerning exponential families in weighted L^2-spaces; it can also be restated in terms of Toeplitz kernels.

Let us now state the transition parameter problem. Given Λ, denote by $\mathcal{E}_\Lambda^{(a)}$ the family $\{u_\lambda : \lambda \in \Lambda\}$ of L^2-solutions of the Schrödinger equation with potential $q(t) = t/a$, $(a > 0)$. We want to compute the "radius of completeness" $R(\Lambda)$, i.e. the critical value of a such that $\mathcal{E}_\Lambda^{(a)}$ is complete in $L^2(\mathbb{R}_+)$. In terms of Toeplitz kernels we have

$$R(\Lambda) = \sup\left\{a :\ N[\bar{M}^aJ_\Lambda] = 0\right\}, \qquad M = \left[S_+^{(3/2)}\right]^{2/3}.$$

This is similar to the Beurling-Malliavin situation, which can be reformulated as the completeness problem for the solutions of the Schrödinger equation with $q \equiv 0$ on $[0, a]$:

$$R(\Lambda) = \sup\left\{a :\ N[\bar{M}^aJ_\Lambda] = 0\right\}, \qquad M = S_+^{(1/2)}.$$

In the Airy situation, the parameter a characterizes the "size" of the singularity, which plays the same role as the length of the interval in the regular case.

Transition parameter problems arise also in connection with the spectral theory problems that we discussed in Section 3.

Example 2. Let L is a regular Schrödinger operator on [0,1] and let $\Lambda\subset\sigma(L)$. We want to characterize the numbers c such that

(i) the potential on $[0,c]$ and the partial spectrum Λ determine L,

(ii) the potential on $[0,c]$ and the spectral measure on Λ determine L,

see Section 3.8. According to the results of Section 3.9 (and Theorem 4.1), these questions are equivalent to the transition parameter problems with $M=S_+^{(1/2)}$ – to find the values of

$$\inf\left\{c:\ N\left[\bar M^{2(1-c)}J_\Lambda\right]=0\right\},\qquad \inf\left\{c:\ N\left[\bar M^{2(1-c)}J_\Lambda^2\right]=0\right\}$$

in the cases (i) and (ii) respectively.

Indeed, in the case (i), Λ has to be defining for Φ, the Weyl function corresponding to the restriction of the potential to $[c,1]$, which means that $N[\bar\Phi^2J_\Lambda]=0$ within the admissible gap. Also, we can replace Φ by $M^{(1-c)}$. In the case (ii), we use the same argument for the divisor $2\chi_\Lambda$.

4.8. **Square root transformation.** The following construction and its corollaries in this and the next subsections are meant to give some isea of what the Beurling-Malliavin theories of standard gap functions may look like.

Let $U=e^{i\gamma}$ be a unimodular function such that $\gamma=0$ on $\mathbb{R}_-$, and let $U_*=e^{i\gamma_*}$ be a unimodular function with an *odd* argument related to γ by the equation

$$\gamma_*(t)=\gamma(t^2),\qquad (t\ge 0).$$

Proposition.

$$N^\infty[U]\neq 0\quad\Leftrightarrow\quad N^\infty[U_*]\neq 0.$$

Proof: $\Leftarrow$ Suppose we have $U_*H=\bar G$ on $\mathbb{R}$ for some $H,G\in\mathcal{H}^\infty$. Then we also have $U_*H^\flat=\bar G^\flat$, where we use the notation

$$H^\flat(z)=\overline{H(-\bar z)}.$$

(Note $U_*=U_*^\flat$.) Thus we have

$$U_*F=\bar F^\flat,\qquad F:=H+G^\flat.$$

Consider now the functions $f,g\in H^\infty(\mathbf{C}_+)$,

$$f(z)=F(\sqrt z),\qquad g(z)=\bar F(\sqrt{\bar z}),$$

where the square root denotes the conformal map $\mathbf{C}\setminus\mathbb{R}_+\to\mathbf{C}_+$. Let us check that $Uf=\bar g$ on $\mathbb{R}$. If $t>0$, then $f(-t^2)=F(it)$ and $g(-t^2)=\bar F(it)$, so

$$\frac{\bar g(-t^2)}{f(-t^2)}=1=U(-t^2).$$

On the other hand, $f(t^2)=F(t)$ and $g(t^2)=\bar F(-t)=F^\flat(t)$, and therefore

$$\frac{\bar g(t^2)}{f(t^2)}=\frac{\bar F^\flat(t)}{F(t)}=U_*(t)=U(t^2).$$

$\Rightarrow$ Suppose $Uf = \bar{g}$ on $\mathbb{R}$ for some $f, g \in H^\infty$. Since $U \equiv 1$ on $\mathbb{R}_-$, the analytic functions $f \in H^\infty(\mathbf{C}_+)$ and $g^\# \in H^\infty(\mathbf{C}_-)$ match on $\mathbb{R}_+$ and therefore define a function in $\mathbf{C}\backslash\mathbb{R}_+$. Applying the square root transformation, we get $F \in H^\infty(\mathbf{C}_+)$,

$$F(z) = \begin{cases} f(z^2), & \Re\, z > 0 \\ g^\#(z^2), & \Re\, z < 0 \end{cases}$$

Let us check that $U_* F = \bar{F}^\flat$ on $\mathbb{R}$. If $x > 0$, then $F(x) = f(x^2)$, $\bar{F}^\flat(x) = F(-x) = \bar{g}(x^2)$, and

$$\frac{\bar{F}^\flat(x)}{F(x)} = \frac{\bar{g}(x^2)}{f(x^2)} = U(x^2) = U_*(x).$$

On the other hand, if $x < 0$, then $F(x) = \bar{g}(x^2)$, $\bar{F}^\flat(x) = F(-x) = f(x^2)$, and

$$\frac{\bar{F}^\flat(x)}{F(x)} = \frac{f(x^2)}{\bar{g}(x^2)} = \frac{1}{U(x^2)} = U_*(x).$$

□

The square root transformation makes it possible to derive the Beurling-Malliavin theory of the gap function $M = S_+^{(1/2)}$ from Theorems 4.2-4.3 (we also use Theorem 4.1).

Corollary. *Let $U = e^{i\gamma} = \bar{\Theta} J$ and suppose that both Θ and J have bounded arguments at $-\infty$. Suppose in addition*

$$(\arg \Theta)'(t) \le \frac{\text{const}}{\sqrt{t}}, \qquad t \to +\infty.$$

(i) If $\gamma(t) \not\to -\infty$ as $t \to +\infty$, or if $\gamma(+\infty) = -\infty$ but the family $\mathcal{BM}(\gamma)$ is long, then

$$\forall \epsilon > 0, \qquad N\left[\left(S_+^{(1/2)}\right)^\epsilon U\right] = 0.$$

(ii) If $\gamma(+\infty) = -\infty$ and the family $\mathcal{BM}(\gamma)$ is short, then

$$\forall \epsilon > 0, \qquad N\left[\left(\bar{S}_+^{(1/2)}\right)^\epsilon U\right] \ne 0.$$

Here we consider only the Beurling-Malliavin intervals in $\mathbb{R}_+$, and the meaning of the terms "long" and "short" is the same as in Section 4.3.

Example. Let L be the Schrödinger operator on $\mathbb{R}$ with potential $q(t) = t^2/4$ ("quantum harmonic oscillator") and let $\Lambda \subset \sigma(L) = \mathbf{N} - \frac{1}{2}$. We want to find the critical value c_* of real numbers c such that q on $(-\infty, -c)$ and the spectral measure on Λ (including the numbering of eigenvalues) determine L, see Section 3.8. We can explicitely compute c_* in terms of Beurling-Malliavin intervals if the set Λ satisfies the inequality

$$\#(\Lambda \cap l) \ge \frac{|l|}{2} - \text{const} \tag{4.18}$$

for all intervals $l = [a, b] \subset \mathbb{R}_+$ such that $b \le a + \sqrt{a}$. *Claim:*

$$c_* = \inf\{c : \gamma_c(+\infty) = -\infty \quad \text{and} \quad \mathcal{BM}[\gamma_c] \quad \text{is short}\}, \qquad \gamma_c := 2n_\Lambda(t) - t - 2c\sqrt{t}.$$

Sketch of Proof: Let Θ be the Weyl inner function corresponding to the restriction of the potential to $\mathbb{R}_+$, so Θ^2 is essentially the same (i.e. up to a finite dimensional gap) as S_+. Then the Weyl inner function corresponding to $(-c, \infty)$ is essentially the same as $\Phi = \Theta M^c$, where $M = S_+^{(1/2)}$. The data determine L if the divisor $2\chi_\Lambda$ is defining for $K[\Phi^2]$, i.e. (again up to a finite dimensional gap)

$$N[\bar{M}^{2c}U] = 0, \qquad U = \bar{S}_+ J_\Lambda^2.$$

Note that

$$\gamma(t) \equiv \arg U(t) = 2n_\Lambda(t) - t + O(1), \qquad (t > 0).$$

To apply the corollary, we need to make sure that $\gamma(t)$ does not drop by more than a constant on each interval l described in the statement. This is exactly our condition (4.18). $\square$

Let us state another corollary, which gives a necessary condition for the nontriviality of a Toeplitz kernel in the "one-sided" situation.

Corollary. *Let $U = e^{i\gamma}$ and $\gamma = 0$ on $\mathbb{R}_-$. If $N^\infty[U] \neq 0$, then*

$$\int_{\{\gamma > A\}} \frac{dt}{t\sqrt{t}} = \frac{o(1)}{A}, \qquad A \to +\infty, \tag{4.19}$$

in particular

$$\gamma^+ \in L^p\left(\frac{dt}{1 + t\sqrt{t}}\right), \qquad (0 < p < 1).$$

Proof: Consider the square root transform U_*. By the basic criterion, $N^+[U_*] \neq 0$ implies

$$\Pi\{\gamma_* > A\} = \frac{o(1)}{A}, \qquad A \to +\infty.$$

$\square$

One can state similar results for tempered γ's satisfying $\gamma = O(1)$ at $-\infty$: if $N^p[U] \neq 0$, then we have (4.19) at $+\infty$.

4.9. **Final thoughts.** Except for the "two-sided" case $M = S$ and the "one-sided" case $M = S_+^{(1/2)}$, the complete Beurling-Malliavin theory of the standard gap functions is not known. Here we mention some preliminary considerations.

Two facts seem to be certain. First, the theories should be different in the subexponential case ($\alpha < 1$ for two-sided gap functions and $\alpha < 1/2$ for one-sided functions) and superexponential case ($\alpha > 1$ and $\alpha > 1/2$ respectively). Second, the role of the Smirnov class $\mathcal{N}^+$ in the case $S^{(\alpha)}$, $\alpha \neq 1$, is not the same as in the classical case $M = S$. One should probably introduce appropriate "Smirnov classes" for all $\alpha < 1$, e.g. the preimage of $\mathcal{N}^+$ under the square root transformation in the case $\alpha = 1/2$, cf. the last corollary.

Subexponential case. We have the following partial results. From the last corollary we immediately derive

Corollary. *Let $M = S^{(1/2)}$ and $U = e^{i\gamma}$. Suppose*

$$\gamma'(x) \geq -\frac{C}{\sqrt{|x|}}, \qquad x \to \pm\infty.$$

If $\gamma(\mp\infty) = \pm\infty$ and the family $\mathcal{BM}(\gamma)$ is short, then

$$\forall \epsilon > 0, \qquad N^\infty\left[\bar{M}^\epsilon U\right] \neq 0.$$

Applying the square root transformation one more time we get

Corollary. *Let $M = S^{(1/4)}$ and $U = e^{i\gamma}$. Suppose*

$$\gamma'(x) \geq -C|x|^{-3/4}, \qquad x \to \pm\infty.$$

If $\gamma(\mp\infty) = \pm\infty$ and the family $\mathcal{BM}(\gamma)$ is short, then

$$\forall \epsilon > 0, \qquad N^{\infty}\left[\bar{M}^{\epsilon} U\right] \neq 0.$$

These facts should extend to all $S^{(\alpha)}$ with $\alpha < 1$. The "converse" statements (if true) should follow from Theorem 4.4 in appropriate "Smirnov classes".

Superexponential case. We only discuss the gap function S_+, which corresponds to the two-sided case $M = S^{(2)}$. The proof of Thorem 4.3 applies verbatim to give the following criterion in the class $\mathcal{N}^+$.

Proposition. *Let $U = e^{i\gamma}$. Suppose γ is bounded on $\mathbb{R}_-$ and $\gamma' \geq -\mathrm{const}$ on $\mathbb{R}_+$.*
(i) If $\gamma \not\to -\infty$ at $+\infty$, or if $\gamma(+\infty) = -\infty$ but $\mathcal{BM}(\gamma)$ is long, then

$$\forall \epsilon > 0, \qquad N^{+}\left[S_+^{\epsilon} U\right] = 0.$$

(ii) If $\gamma(+\infty) = -\infty$ and $\mathcal{BM}(\gamma)$ is short, then

$$\forall \epsilon > 0, \qquad N^{+}\left[\bar{S}_+^{\epsilon} U\right] \neq 0.$$

On the other hand, it is *false* that we have $N^{\infty}\left[\bar{S}_+^{\epsilon} U\right] \neq 0$ in the case (ii). (Of course, we still have $N^{\infty}\left[u\bar{S}^{\epsilon}\right] \neq 0$.) The criterion in $\mathcal{H}^{\infty}$ should involve a different definition of long families, probably the following:

$$\sum_{l \in \mathcal{BM}(\gamma)} \frac{|l|^2}{\delta(l)^{3/2}} = \infty. \tag{4.20}$$

Let us state a partial result in the language of $S^{(2)}$. Note that under the square root transformation, the condition (4.20) becomes

$$\sum_{l \in \mathcal{BM}(\gamma)} \frac{|l|^2}{\delta(l)} = \infty.$$

We get the exactly this condition if we apply the non-rigorous argument preceeding the proof of Theorem 4.4.

Proposition. *Let $M = S^{(2)}$ and $U = e^{i\gamma}$. Suppose*

$$\gamma'(x) \geq -C|x|, \qquad x \to \pm\infty.$$

(i) If $\gamma(x) \not\to \pm\infty$ as $x \to \mp\infty$, or if $\gamma(\mp\infty) = \pm\infty$ but

$$\sum_{l \in \mathcal{BM}(\gamma)} \frac{|l|^2}{\delta(l)(1 + \log^+ |l|)} = \infty,$$

then

$$\forall \epsilon > 0, \qquad N^{+}\left[M^{\epsilon} U\right] = 0.$$

(ii) If $\gamma(+\infty) = -\infty$ and

$$\sum_{\mathcal{BM}(\gamma)} \frac{|l|^2 \log^+ |l|}{\delta(l)} < \infty,$$

then

$$\forall \epsilon > 0, \qquad N^+ \left[\bar{M}^\epsilon U\right] \neq 0.$$

Example. Let L be the harmonic oscillator considered in Example 4.9, and let $\Lambda \subset \sigma(L)$. The first part of the last proposition provides a sufficient condition for the fact that q on $\mathbb{R}_-$ and the spectral measure on Λ determine L. This condition is less precise than the one stated in Example 4.9 because we have a larger gap, which characterizes the singularity at infinity rather than the extra length of the known part of the spectrum. On the other hand, the condition in the above proposition does not require any extra assumptions on Λ, cf. (4.18).

In conclusion, we mention that the study of gap functions which are less regular than the standard ones (or even arbitrary) is an interesting and probably difficult problem.

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