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Rectifiable sets and the Traveling Salesman Problem*

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§1. Introduction

Let $K \subset \mathbb{C}$ be a bounded set. In this paper we shall give a simple necessary and sufficient condition for K to lie in a rectifiable curve. We say that a set is a rectifiable curve if it is the image of a finite interval under a Lipschitz mapping. Recall that for a connected set $\Gamma \subset \mathbb{C}$, Γ is a rectifiable curve (not necessarily simple) if and only if $l(\Gamma) < \infty$, where $l(\cdot)$ denotes one dimensional Hausdorff measure. This classical result follows from the fact that on any finite graph there is a tour which covers the entire graph and which crosses each edge (but not necessarily each vertex!) at most twice. If K is a finite set then we are essentially reduced to the classical Traveling Salesman Problem (TSP): Compute the length of the shortest Hamiltonian cycle which hits all points of K. This is the same, up to a constant multiple, as asking for the infimum of $l(\Gamma)$ where Γ is a curve, $K \subset \Gamma$. (Such a Γ is called a spanning tree in TSP theory.) For infinite sets K, we cannot hope in general to have K be a subset of a Jordan curve. What we should therefore look at is connected sets which contain K.

Let $\Gamma_{\min}$ be the shortest (minimal) spanning tree. Then we cannot possibly solve our problem for sets K of infinite cardinality if we cannot find Γ, $l(\Gamma) \leqq C_0 l(\Gamma_{\min})$, for any finite set K. (Here and throughout the paper C, C_0, C_1, c_0, etc. denote various universal constants.) While there are several algorithms for computing $l(\Gamma_{\min})$, these algorithms work for finite graphs satisfying the triangle inequality, and do not use the Euclidean properties of K. (See [13] for an excellent discussion of some of these algorithms.) Therefore these methods cannot solve our problem for general infinite K. We present a method which is a minor modification of a well-known algorithm ("Farthest Insertion" – see [13]) which yields a Γ with $l(\Gamma) \leqq C_0 l(\Gamma_{\min})$. The Farthest Insertion algorithm has been extensively studied with large numerical calculations on computers, and is experimentally good in the sense that the Γ produced satisfy $l(\Gamma) \leqq C_0 l(\Gamma_{\min})$ for all examples which have

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been calculated. If K contains N points our method could be modified to calculate (up to a constant multiple) $l(\Gamma_{\rm opt})$ in time $0(N \log N)$, which is what one would expect and which is available for other algorithms [17]. We can thus say that, in a certain sense, our theorem gives a geometric "solution" to the TSP.

A square $Q \subset \mathbb{C}$ is a dyadic square if $Q = [j2^{-n}, (j+1)2^{-n}] \times [k2^{-n}, (k+1)2^{-n}]$, where $j, k, n \in \mathbb{Z}$. We denote by $l(Q) = 2^{-n}$ the sidelength of Q. For $\lambda > 0$ we denote by λQ the square with the same center as Q, with sidelength $\lambda l(Q)$, and with sides parallel to the axes. For Q a dyadic square let S_Q be an infinite strip (or line in the degenerate case) of smallest possible width which contains $K \cap 3Q$, and let $\omega(Q)$ denote the width of S_Q. We then define

$$\beta_K(Q) = \frac{\omega(Q)}{l(Q)},$$

so that $\beta_K(Q)$ measures in a scale invariant way $(0 \leqq \beta_K(Q) \leqq 3)$ the deviation of K from a straight line near Q on scale $l(Q)$. Our main result states that K is contained in a rectifiable curve Γ if and only if

$$\beta^2(K) \equiv \sum_Q \beta_K^2(Q)\, l(Q) < \infty \tag{1.1}$$

where the sum is taken over all dyadic squares (or equivalently over all dyadic squares Q with $l(Q) \leqq$ diameter (K)). Furthermore, the shortest possible Γ has length comparable to diameter (K) plus the sum in (1.1). This is an elementary (if quite lengthy) computation with the Pythagorean theorem if $\beta_K(Q) < \varepsilon_0$ for all Q (ε_0 sufficiently small) or if K already lies in a Lipschitz curve (defined in Section 2). However, if $\beta_K(Q)$ is large for many Q (K is "dispersed") there are technical difficulties to be overcome. Conversely, if Γ is a Lipschitz curve it is not hard to prove (and is already in [9]) that $\beta^2(\Gamma)$ is bounded by a multiple of $l(\Gamma)$.

Theorem 1. *If $\Gamma \subset \mathbb{C}$ is connected, then*

$$\beta^2(\Gamma) \leqq C_0\, l(\Gamma)\,. \tag{1.2}$$

Conversely, if $\beta^2(K) < \infty$ there is a connected set Γ, $K \subset \Gamma$, such that

$$l(\Gamma) \leqq (1+\delta)\,\text{diameter}\,(K) + C(\delta)\,\beta^2(K)\,. \tag{1.3}$$

Corollary 1. *If K is an analytic set and $l(K) < \infty$, then K is totally unrectifiable in the sense of Besicovitch* ([4], [5]) *if and only if*

$$\beta^2(E) = \infty$$

for every $E \subset K$ with $l(E) > 0$.

For certain purposes (see e.g. [1]) it is useful to know that the curve constructed for (1.3) can have extra properties. We will show that by taking δ large enough there is Γ, $K \subset \Gamma$, such that (1.3) holds, and

(1.4) If $z_1, z_2 \in \Gamma$ there is a connected set $\gamma \subset \Gamma$ such that $z_1, z_2 \in \gamma$ and $l(\gamma) \leqq C_0$ $|z_1 - z_2|$.

(1.5) If $K \cap Q \neq \phi$ and $l(Q) \leqq$ diameter (K), there is an infinite strip S_Q with width $\beta(Q)\, l(Q)$, with axis L_Q, $S_Q \cap K \cap Q \neq \phi$, and such that if P denotes orthogonal projection onto L_Q,

$$P(3Q) \subset P(\Gamma \cap S_Q)\,.$$

Condition (1.4) says that Γ is uniformly locally connected and (1.5) asserts that Γ "crosses" $S_Q \cap 3Q$.

To prove the first theorem we will use a result which is of some independent interest. An M Lipschitz domain of size one centered at the origin is a simply connected domain whose boundary is a Jordan curve described by $r(\theta)e^{i\theta}$, $0 \leqq \theta \leqq 2\pi$, where $\frac{1}{1+M} \leqq r(\theta) \leqq 1$, and where $|r(\theta_1) - r(\theta_2)| \leqq M\,|e^{i\theta_1} - e^{i\theta_2}|$. An M Lipschitz domain is a dilate and translate of one of the above domains. Lipschitz domains play a central rôle in modern one complex variable theory, and their behavior vis-à-vis Cauchy integrals and harmonic measure is well understood.

Theorem 2. *If Ω is a simply connected domain and $l(\partial\Omega) < \infty$, there is a rectifiable curve Γ such that $\Omega \backslash \Gamma = \bigcup_j \Omega_j$ is a decomposition of Ω into disjoint C_0 Lipschitz domains, and*

$$\sum_j l(\partial\Omega_j) \leqq C_0\, l(\partial\Omega)\,.$$

We thank Charles Pugh for pointing out that the above theorem fails if we replace C_0 by $\varepsilon_0 \ll 1$. Indeed if one tiles (in the above sense) the unit square by ε_0 Lipschitz domains Ω_j, $\sum l(\partial\Omega_j) = \infty$. Now let Γ be a bounded connected set and attach to Γ a line segment L and circle S. Applying Theorem 2 to $\mathbb{C}\backslash(\Gamma \cup L \cup S)$ we obtain

Corollary 2. *If Γ is connected there is a connected set $\tilde{\Gamma}$ such that $\Gamma \subset \tilde{\Gamma}$, $l(\tilde{\Gamma}) \leqq C_0\, l(\Gamma)$, every bounded component of $\mathbb{C}\backslash\tilde{\Gamma}$ is a C_0 Lipschitz domain, and the unbounded component of $\mathbb{C}\backslash\tilde{\Gamma}$ is the complement of a disk. Furthermore, if $x, y \in \tilde{\Gamma}$ there is a subarc $\gamma \subset \tilde{\Gamma}$ such that $x, y \in \gamma$ and $l(\gamma) \leqq C_1\,|x - y|$. The constant C_1 may be taken to be* 3.

The theorem's proof can be modified to yield further structure on Γ if (1.1) holds in some uniform sense. A curve Γ is said to be Ahlfors-David regular (AD) if

$$\sup_{\substack{z_0 \in \mathbb{C} \\ r > 0}} r^{-1}\, l(\{z \in \Gamma : |z - z_0| \leqq r\}) < \infty\,.$$

Then one can show that K lies in an AD curve if and only if

$$\sup_Q l(Q)^{-1} \sum_{Q' \subset Q} \beta_K^2(Q')\, l(Q') < \infty\,, \tag{1.6}$$

and this therefore gives us a characterization of AD curves. If Γ is an ε quasicircle passing through ∞, $\beta_\Gamma(Q) \leqq C_0 \varepsilon^{1/2}$ for all Q. Conversely, our construction can be modified to show that if $\beta_K(Q) \leqq \varepsilon < \varepsilon_0$ for all Q, K is a subset of a $\delta(\varepsilon)$ quasicircle passing through ∞.

The history behind our proof of Theorem 1 is at first glance a bit surprising, in that it was discovered by a careful study of Cauchy integral operators on Lipschitz curves. If Γ is a Lipschitz curve, a famous theorem due to A.P. Calderón [2] (for small Lipschitz constants) and Coifman, McIntosh and Meyer [3] later, (for any Lipschitz curve) asserts that the Cauchy integral operator is bounded on the Lebesgue space $L^2(\Gamma)$. By now there are myriad proofs of this result. The theorem is very old when $\Gamma = \mathbb{R}$, and is equivalent there to the L^2 boundedness of the Hilbert transform. This is essentially easy because one can apply Plancherel's theorem. The author had attempted to find a proof of the result for Lipschitz curves by using the intuitive idea that a Lipschitz curve should look like a straight line at most places on most scales. This was accomplished in [9] by noting that (1.6) holds for Lipschitz curves.

Further reflection shows that the above connection between Cauchy integrals and the TSP should not be so surprising. Let K have finite one-dimensional Hausdorff measure, $l(K) < \infty$. An old problem is to decide whether $\gamma(K)$, the analytic capacity of K, is zero. In other words, is there a non-constant, bounded holomorphic function on $\mathbb{C} \backslash K$? (See [6] or [15] for more on $\gamma(K)$.) An easy necessary condition is that $l(K) > 0$. An old conjecture is that $\gamma(K) > 0$ if and only if $\text{Fav}(K) > 0$, where $\text{Fav}(\cdot)$ is Favard length,

$$\text{Fav}(K) = \int_0^\pi l(K_\theta)\, d\theta ,$$

and where K_θ is the orthogonal projection of K onto the line $\mathbb{R}e^{i\theta}$. This is known to be false for sets where K has non sigma finite length [11, 15], but is still open when $l(K) < \infty$. The idea Murai has used [11, 15] to attack this problem is to use deep estimates on the L^2 bounds for Cauchy integrals on Lipschitz curves. Now sets where $0 < l(K) < \infty$ and $\text{Fav}(K) = 0$ play a central rôle in Besicovitch's theory of rectifiable sets (see e.g. [4, 5]). Besicovitch proved this occurs if and only if $l(K \cap \Gamma) = 0$ for every rectifiable curve Γ. Such sets are called totally irregular. Combining this with Calderón's theorem on the Cauchy integral [2], it follows that $l(K) < \infty$ and $\text{Fav}(K) > 0$ imply $\gamma(K) > 0$. This connection between Cauchy integrals, geometric measure theory, and the TSP seems all the more natural when one considers that the Cauchy integral also plays a vital part in Borsuk's proof of the Jordan curve theorem. In [1] Christopher Bishop and the author use Theorem 1 and the machinery developed for Cauchy integrals to settle some conjectures about harmonic measure.

The paper is organized as follows. In Section 2 we use an argument with conformal mappings to prove Theorem 2. Section 3 uses Theorem 2 to give a proof of (1.2). Section 4 is devoted to a construction of the curve Γ which satisfies (1.3)–(1.5). Section 5 is an appendix where we give an elementary proof that (1.2) holds for Lipschitz curves. That result (first proved by Fourier analysis in [9]) is used in Section 2.

The author is grateful to Stephen Semmes for many conversations on condition (1.1). I believed at one time that I had a counter-example to (1.3). When Stephen

Semmes repeatedly asked me to write it down, I found an error, and Theorem 1 soon appeared.

§2. Proof of Theorem 2

We assume the reader is familiar with corona type constructions as presented e.g. in the book of Garnett [7]. Let $F\colon \mathbb{D} \to \Omega$ be any choice of a Riemann map of the unit disk onto Ω. Since the theorem is trivial if $l(\partial\Omega) = \infty$, we assume $l(\partial\Omega) < \infty$ so that $F'(z)$ is in the Hardy space H^1. (See [11].) Let $G(z) = (F'(z))^{1/2}$ so that $G \in H^2$, and apply either Green's formula or Plancherel to obtain

$$\int_{\mathbb{T}} |G(e^{i\theta}) - G(0)|^2 \, d\theta = \iint_{\mathbb{D}} |G'(z)|^2 \log \frac{1}{|z|} \, dxdy .$$

From this we see that if we set $F' = G^2 = e^{\varphi}$,

$$\iint_{\mathbb{D}} |F'(z)|\,|\varphi'(z)|^2 \log \frac{1}{|z|} \, dxdy \leqq 4 \int_{\mathbb{T}} |F'(e^{i\theta})| \, d\theta \tag{2.1}$$

$$\leqq 8\, l(\partial\Omega) .$$

Now it is well known that φ is in the Bloch space B and of norm at most 6, i.e.

$$|\varphi'(z)| \leqq 6(1 - |z|^2)^{-1}, \qquad z \in \mathbb{D} . \tag{2.2}$$

See e.g. [16]. Let us decompose T into dyadic arcs, i.e. arcs of length $\pi 2^{-n}$. For such a dyadic arc I we associate to it a dyadic "square"

$$Q = Q_I = \left\{ z\colon \frac{z}{|z|} \in I,\ 1 - \pi 2^{-n} \leqq |z| \leqq 1 \right\} .$$

Let $T_Q = \{ z \in Q\colon 1 - 2\pi^{-n} \leqq |z| \leqq 1 - \pi 2^{-n-1} \}$ denote the "top half" of Q. Let z_Q be the center of T_Q.

Fix a dyadic square Q and perform the following stopping time argument. If there is $z \in T_Q$ with $|\varphi(z) - \varphi(z_Q)| \geqq \varepsilon$, stop and let $\mathscr{D}_Q = T_Q$. In this case we say Q is of type 0. Otherwise let $Q_1, Q_2, \ldots$ be those dyadic squares inside Q which satisfy

$$\sup_{z \in T_{Q_j}} |\varphi(z) - \varphi(z_Q)| \geqq \varepsilon ,$$

and define

$$\mathscr{D}_Q = Q \setminus \bigcup_{j=1}^{\infty} Q_j .$$

Then if $\mathscr{D}_Q$ is not of Type 0,

$$|\varphi(z) - \varphi(z_Q)| \leqq \varepsilon, \qquad z \in \mathscr{D}_Q . \tag{2.3}$$

By the construction of $\mathscr{D}_Q$ we see that $\partial\mathscr{D}_Q \equiv \gamma_Q$ is a chord arc domain:

If $z_1, z_2 \in \gamma_Q$, there is an arc $\gamma \subset \gamma_Q$ joining z_1 to z_2 and of length $l(\gamma) \leqq 6\, |z_1 - z_2|$.

Using this procedure we tile $\mathbb{D}$ by regions $\mathscr{D}_Q$ via the usual stopping time arguments. If a region $\mathscr{D}_Q$ so formed is not of Type 0, we say $\mathscr{D}_Q$ is of Type 1 if $l(T \cap \gamma_Q) \geqq \frac{1}{2} l(T \cap \partial Q)$, and we say $\mathscr{D}_Q$ is of Type 2 otherwise. Then if $\mathscr{D}_Q$ is of Type 1 or 2,

$$|F'(z) - F'(z_Q)| \leqq 2\varepsilon |F'(z_Q)|$$

whenever $z \in \mathscr{D}_Q$.

We first bound the Type 0 regions. By normal families and (2.2),

$$\int_{\gamma_Q} |F'(z)|\, ds(z) \leqq C \iint_{T_Q} |F'(z)|\, |\varphi'(z)|^2 \log \frac{1}{|z|}\, dxdy\,.$$

whenever $\mathscr{D}_Q = T_Q$ is of Type 0. Consequently, inequality (2.1) yields

$$\sum_{\text{Type } 0} \int_{\gamma_Q} |F'(z)| ds(z) \leqq C l(\partial \Omega)\,,$$

because the regions $\mathscr{D}_Q$ are disjoint. Now let N_0 be a suitably large integer (in fact, $N_0 = 2$ will do) and divide each $\mathscr{D}_Q = T_Q \in$ Type 0 into 4^{N_0} "squares" of essentially equal sidelength. Then if $\tilde{\mathscr{D}}_Q$ is any of these "squares", (2.2) shows that $F(\tilde{\mathscr{D}}_Q)$ is a C_0 Lipschitz domain. By our last estimate,

$$\sum_{\mathscr{D}_Q \in \text{Type } 0} \sum_{\tilde{\mathscr{D}}_Q} l(\partial F(\tilde{\mathscr{D}}_Q)) \leqq C_0\, l(\partial \Omega)\,.$$

We now turn to the regions of Type 1. Since the sets $T \cap \partial Q_j$ and $T \cap \partial Q_k$ can intersect in at most two points if $j \neq k$, and since (2.3) holds,

$$\begin{aligned} &\sum_{\text{Type } 1} \int_{\gamma_Q} |F'(z)| ds(z) \\ &\leqq \sum_{\text{Type } 1} 12(1 + 2\varepsilon) \int_{T \cap \gamma_Q} |F'(z)|\, ds(z) \\ &\leqq 12(1 + 2\varepsilon) \int_T |F'(z)|\, ds(z) \\ &= 12(1 + 2\varepsilon)\, l(\partial \Omega)\,. \end{aligned} \tag{2.4}$$

The Type 2 regions are a little trickier to bound. The following is the main idea of this section: Use the L^2 bounds of (2.1) to bound the Type 2 regions. Let $I_1, I_2, \ldots$ be the horizontal line segments in $\gamma_Q \setminus \partial Q$. Then by hypothesis,

$$\sum_{j=1}^{\infty} l(I_j) \geqq \frac{1}{12} l(\gamma_Q)\,. \tag{2.5}$$

By normal families and estimate (2.2) there is $\delta > 0$ such that

$$l(\{z \in I_j \colon |\varphi(z) - \varphi(z_Q)| \geqq \delta\}) \geqq \delta l(I_j)\,. \tag{2.6}$$

(Because $\sup_{z \in T(Q_j)} |\varphi(z) - \varphi(z_Q)| \geqq \varepsilon$.) Now let ω denote harmonic measure for the region $\mathscr{D}_Q$ with respect to the base point z_Q, and let $g(z)$ denote Green's function for $\mathscr{D}_Q$ with pole at z_Q so that

$$d\omega = \frac{1}{2\pi}\frac{\partial g}{\partial n}\,ds\,.$$

Then since $\mathscr{D}_Q$ is a chord-arc domain, the results of Jerison and Kenig [8] show that ω is an A_∞ weight, i.e. there are constants $A, \eta, \eta' > 0$ so that

$$A^{-1}\left(\frac{l(E)}{l(\gamma_Q)}\right)^{\eta} \leqq \omega(E) \leqq A\left(\frac{l(E)}{l(\gamma_Q)}\right)^{\eta'} \tag{2.7}$$

for any Borel set $E \subset \gamma_Q$. (The usual definition of A_∞ uses only the second inequality, but it is well known that this is equivalent to the first [7].) Combining (2.3) with (2.5)–(2.7) we see that

$$\int_{\gamma_Q} |G(z) - G(z_Q)|^2\,d\omega \geqq c\,|F'(z_Q)|\,,$$

where $c = c(\delta, A, \eta) > 0$. Green's formula is valid for chord arc domains, so that

$$2\pi \int_{\gamma_Q} |G(z) - G(z_Q)|^2\,d\omega = \iint_{\mathscr{D}_Q} |G'(z)|^2 g(z)\,dxdy\,.$$

An elementary argument with the maximum principle shows that if $I = T \cap \partial Q$,

$$g(z) \leqq C\,l(I)^{-1} \log\frac{1}{|z|}$$

whenever $z \in \mathscr{D}_Q$ and $|z - z_Q| \geqq \frac{1}{16}\,l(I)$. Applying the maximum principle again we obtain

$$\begin{aligned}\int_{\gamma_Q} |F'(z)|\,ds(z) &\leqq C\,l(\gamma_Q) \int_{\gamma_Q} |G(z) - G(z_Q)|^2\,d\omega \\ &\leqq C'\,l(\gamma_Q) \iint_{\mathscr{D}_Q} |G'(z)|^2\, l(\gamma_Q)^{-1} \log\frac{1}{|z|}\,dx\,dy \\ &= C'' \iint_{\mathscr{D}_Q} |F'(z)|\,|\varphi'(z)|^2 \log\frac{1}{|z|}\,dx\,dy\,.\end{aligned}$$

Since the regions $\mathscr{D}_Q$ are disjoint, it follows from (2.1) that

$$\begin{aligned}\sum_{\text{Type 2}} \int_{\gamma_Q} |F'(z)|\,ds(z) &\leqq C \iint_{\mathscr{D}} |F'(z)|\,|\varphi'(z)|^2 \log\frac{1}{|z|}\,dx\,dy \\ &\leqq C'\,l(\partial\Omega)\,.\end{aligned} \tag{2.8}$$

This almost finishes the proof of the theorem, because if ε is small enough each domain $F(\mathscr{D}_Q)$ will have boundary a (7) chord-arc curve. (This follows immediately from (2.3).) Furthermore by estimates (2.4) and (2.8) on the Type 1 and Type 2 domains,

$$\sum l(\partial F(\mathscr{D}_Q)) \leqq C\,\ell(\partial\Omega)\,.$$

We thus need only show that each Type 1 or 2 region $\mathscr{D}_Q$ can be decomposed into disjoint C_0 Lipschitz domains Ω_j with $\sum_j l(\partial\Omega_j) \leqq C_0 l(\gamma_Q)$. For then $F(\Omega_j)$ will be a C_1 Lipschitz domain if ε is small enough, and

$$\sum_j l(\partial F(\Omega_j)) \leqq C\, l(\partial F(\mathscr{D}_Q)\,.$$

The construction is a little easier to describe on the upper half plane $\mathbb{R}^2_+ = \{z = x + iy\colon y > 0\}$. We are given a dyadic square $Q \subset \mathbb{R}^2_+$ with side $I \subset \mathbb{R}$. The region $\mathscr{D}_Q$ is constructed by removing from Q disjoint squares $Q_1, Q_2, \ldots$ with (disjoint) sides $I_j \subset \mathbb{R}$. Let $\tilde{I}_j$ denote the top side of Q_j. We now build a Cantor type tree T_j. Put $\tilde{I}_j$ in T_j and add on line segments on the left and right of I_j down to $\mathbb{R}$ and of length $\sqrt{2}\, l(I_j)$. We obtain a nice "tree" T^1_j with two "roots" of length $\sqrt{2}\, l(I_j)$ touching $\mathbb{R}$ at angle $\pi/4$. Let z_1 and z_2 be the two points in T^1_j of height $1/4\, l(I_j)$ so that $\operatorname{Re} z_j = x_j = k_j(1/4\, l(I_j))$, $k_j \in \mathbb{Z}, j = 1, 2$. (In other words, x_1 and x_2 are appropriate dyadic rationals.) Attach to z_1 and z_2 the line segments to $\mathbb{R}$ (of angle $\pi/4$ with $\mathbb{R}$) which are not already in T^1_j and call this T^2_j. T^2_j is obtained by sprouting the roots of T^1_j. Continue sprouting, by dropping by a factor of $1/4$ each time and obtain $T^1_j, T^2_j, \ldots, T^n_j, \ldots$ Then T^n_j has 2^n "roots" of length $\sqrt{2}\, 4^{-n}\, l(I_j)$ and each terminates at a point $k\, 4^{-n}\, l(I_j)$ where $k \in \mathbb{Z}$. Setting $T_j = \bigcup_{k=1}^{\infty} T^k_j$ we have

$$l(T_j) = (1 + 3\sqrt{2})\, l(I_j)\,.$$

It is an exercise to see that every component of $\mathscr{D}_Q \backslash \bigcup_j T_j$ is a C_0 Lipschitz domain. □

We remark that the idea of using Littlewood–Paley estimates to impose bi-Lipschitz structures on sets has been used before in [10] to give a certain quantitative version of Sard's theorem.

§3. Proof of (1.2)

We may assume that $l(\Gamma) < \infty$ and the conclusions of Corollary 2 hold for Γ, because if $\Gamma \subset \tilde{\Gamma}$, $\beta^2(\Gamma) \leqq \beta^2(\tilde{\Gamma})$. We denote by Ω_j the components of $\mathbb{C} \backslash \Gamma$ and set $\Gamma_j = \partial\Omega_j$, $d_j =$ diameter (Γ_j). For a dyadic square Q let $\mathscr{F}(Q) = \{\Gamma_j\colon \Gamma_j \cap 4Q \neq \phi, d_j \geqq l(Q)\}$ and let $G(Q) = \{\Gamma_j\colon \Gamma_j \cap 5Q \neq \phi, d_j < l(Q)\}$. Also let Q^* be the dyadic double of Q.

Lemma 3.1.

$$\beta^2_\Gamma(Q) \leqq C_1 \sum_{\mathscr{F}(Q)} \beta^2_{\Gamma_j}(Q^*) + C_1\, l(Q)^{-2} \sum_{G(Q)} \operatorname{Area}(\Omega_k)\,.$$

Proof. The lemma is immediate if $\mathscr{F}(Q) = \phi$, for then either $\beta(Q) = 0$, or $\sum_{G(Q)} \operatorname{Area}(\Omega_k) \geqq 9 l(Q)^2$. So suppose $\Gamma_0 \in \mathscr{F}(Q)$. By scaling we may assume $l(Q) = 1$.

Let L be a line such that

$$d \equiv \sup_{z \in \Gamma_0 \cap 5Q} \text{distance}(z, L) \leqq \beta_{\Gamma_0}(Q^*)\, l(Q^*) .$$

Let $z_0 \in \Gamma \cap 3Q$ maximize distance (z, Γ_0) and let distance $(z_0, \Gamma_0) = d_0$. Denote by z_1 the closest point in Γ_0 to z_0, and let $I = [z_0, z_1]$. We define $z_2 = 1/2(z_0 + z_1)$ to be the midpoint of I.

Case 1. $\mathscr{F}(Q)$ contains three or more curves. Then there is $\Gamma_j \in \mathscr{F}(Q)$ such that $\beta_\Gamma^2(Q^*) \geqq C_1$, so

$$\sum_{\mathscr{F}(Q)} \beta_{\Gamma_j}^2(Q^*) \geqq C_1 .$$

Case 2. $\mathscr{F}(Q)$ contains only Γ_0. Then $B\left(z_2, \frac{d_0}{2}\right) \subset \bigcup_{G(Q)} \bar{\Omega}_k$. Consequently,

$$\begin{aligned} \beta_\Gamma^2(Q) &\leqq (d + d_0)^2 \\ &\leqq 2\beta_{\Gamma_0}^2(Q^*) + 2d_0^2 \\ &\leqq 2\beta_{\Gamma_0}^2(Q^*) + \frac{8}{\pi} \sum_{G(Q)} \text{Area}(\Omega_k) . \end{aligned}$$

Case 3. $\mathscr{F}(Q) = \{\Gamma_0 \Lambda\, \Gamma_j\}$.

We may as well assume $\beta_{\Gamma_j}(Q^*) < \varepsilon_0, j = 0, 1$, for otherwise there is nothing to prove. Let $d_1 = \sup_{z \in \Gamma_1 \cap 4Q} \text{distance}(z, \Gamma_0)$. Then if ε_0 is small enough $\Gamma \cap 3Q$ is trapped between Γ_0 and Γ_1, and

$$\beta_\Gamma(Q) \leqq \beta_{\Gamma_0}(Q^*) + \beta_{\Gamma_1}(Q^*)) + d_1 .$$

Since Γ_1 is a C_0 Lipschitz curve there is $z_3 \in 9/2Q \backslash (\Omega_0 \cup \Omega_1)$ such that distance $(z_3, \Gamma_j) \geqq c_0 d_1$, $j = 0, 1$. But then

$$\begin{aligned} \beta_\Gamma^2(Q) &\leqq C_0(\beta_{\Gamma_0}^2(Q^*) + \beta_{\Gamma_1}^2(Q^*)) + C_0 d_1^2 \\ &\leqq C_0(\beta_{\Gamma_0}^2(Q^*) + \beta_{\Gamma_0}^2(Q^*)) + C_1 \sum_{G(Q)} \text{Area}(\Omega_k) . \end{aligned}$$

It is now an easy matter to finish the proof of (1.2). By the results of [9] (or our section 5),

$$\sum_Q \beta_{\Gamma_j}^2(Q)\, l(Q) \leqq C_1\, l(\Gamma_j)$$

for any C_0 Lipschitz curve. By Lemma 3.1 it is enough to estimate the sum

$$\sum_Q l(Q)^{-1} \sum_{G(Q)} \text{Area}(\Omega_k) .$$

Now for each $n \in \mathbb{Z}$ such that $d_j < 2^{-n}$ there are at most C_1 squares Q such that $l(Q) = 2^{-n}$ and $\Gamma_j \in G(Q)$. Consequently, we may estimate the above sum by reversing the order of summation to obtain

$$\sum_{\Gamma_k} \text{Area}(\Omega_k) \sum_{\substack{Q \\ \Omega_k \in G(Q)}} l(Q)^{-1}$$

$$\leqq C \sum_{\Gamma_k} \text{Area}(\Omega_k) \sum_{m=0}^{\infty} (2^m d_k)^{-1}$$

$$\leqq 2C \sum_{\Gamma_k} \text{Area}(\Omega_k)\, d_k^{-1}$$

$$\leqq C' \sum l(\Gamma_k) \leqq 2C'\, l(\Gamma)\,.$$

§4. Construction of Γ. (Proof of (1.3)–(1.5).)

We build sets $\mathscr{L}_0, \mathscr{L}_1, \ldots, \mathscr{L}_n \subset K$ with the following properties:

$$|z_j - z_k| \geqq 2^{-n}, \qquad z_j, z_k \in \mathscr{L}_n, \qquad j \neq k\,.$$

and

$$\inf_{z_j \in \mathscr{L}_n} |z - z_j| \leqq 2^{-n}, \qquad z \in K\,.$$

These sets may need to be slightly perturbed at various stages of the construction, but the two properties listed above will still hold. We may assume $K \subset [0,1]^2$, and by scaling we may also assume $\mathscr{L}_0 = \{z_0, z_1\}$ where $|z_0 - z_1| = \sup_{z, w \in K} |z - w|$. We define $\Gamma_0 = [2z_0 - z_1, 2z_1 - z_0]$ to be a line segment containing z_0, z_1, and extending beyond those points. We let $A \geqq 1$ be a constant to be fixed later, $A = 2^{k_0}$, and then let $\varepsilon_0 > 0$ be small. The value of ε_0 is determined later. Suppose by induction that $\Gamma_0, \Gamma_1, \ldots, \Gamma_{n-1}$ have been formed and let $x_0 \in \mathscr{L}_n \backslash \mathscr{L}_{n-1}$. If $x_0 \in \Gamma_{n-1}$ we do no construction about x_0. Let Q be that dyadic cube containing x_0 with $l(Q) = A2^{-n}$. We call the collection of all such cubes $\mathscr{D}_n$. By a translation and rotation we may assume $x_0 \geqq 0$ and its nearest neighbor in $\mathscr{L}_{n-1}$ is the origin. Let $W = \{z\colon 0 < |z| \leqq A2^{-n+1}, |\arg z| \leqq 2\pi/3\}$, and let $W^* = \{z\colon 0 < |z| \leqq A2^{-n+1}, |\pi\text{-}\arg z| \leqq 2\pi/3\}$, so that $W \cup W^* = \mathscr{D} = \{z\colon 0 < |z| \leqq A2^{-n+1}\}$. We assume by induction that the following properties hold:

(P1) Let $\{y_1, \ldots, y_N\} = \mathscr{L}_{n-1} \cap \{\mathscr{D} \cup \{0\}\}$ and arrange the points so that $\text{Re}\, y_1 \leqq \text{Re}\, y_2 \leqq \ldots \leqq \text{Re}\, y_N$. Then Γ_{n-1} contains the segments $[y_j, y_{j+1}]$, $1 \leqq N-1$.

(P2) If $x_0 \notin \Gamma_{n-1}$ and $\mathscr{L}_{n-1} \cap W = \phi$ there is θ, $|\theta| \leqq 2\pi/3$, such that $[0, A2^{-n+1} e^{i\theta}] \subset \Gamma_{n-1}$. If $x_0 \notin \Gamma_{n-1}$ and $\mathscr{L}_{n-1} \cap W^* = \phi$, there is ψ, $|\psi - \pi| \leqq 2\pi/3$, such that $[0, A2^{-n+1} e^{i\psi}] \subset \Gamma_{n-1}$.

Case 1. $\beta(Q) \geqq \varepsilon_0$. Connect x_0 to all points in $\mathscr{L}_n \cap \{|x - x_0| \leqq C2^{-n}\}$ by (straight) line segments. Also add on the line segment $[0, 4Ax_0]$. Then the amount of length added to Γ_{n-1} is bounded by

$$C \sum_{Q \in \mathcal{D}_n} \ell(Q) \leqq C' \sum_{Q \in \mathcal{D}_n} \beta^2(Q)\, l(Q)$$

because each Case 1 Q has $\beta(Q)^2 \geqq \varepsilon_0^2$.

For the rest of the cases we assume $\beta(Q) < \varepsilon_0$.

Case 2. $\mathcal{L}_{n-1} \cap W \neq \phi$, $\mathcal{L}_{n-1} \cap W^* \neq \phi$. Let $y_1 \in W$ minimize $|z|$, $z \in \mathcal{L}_{n-1} \cap W$, and let $y_{-1} \in W^*$ minimize $|z|$, $z \in \mathcal{L}_{n-1} \cap \mathcal{W}^*$. Let $\{x_1, \ldots, x_M\}$ be those points in $\mathcal{L}_n \cap W$ such that $\mathrm{Re} y_{-1} \leqq \mathrm{Re} x_j \leqq \mathrm{Re} y_1$, and label the points so that $\mathrm{Re} x_1 \leqq \mathrm{Re} x_2 \leqq \ldots \leqq \mathrm{Re} x_M$. Replace the segments $[y_{-1}, 0]$, $[0, y_1] \subset \Gamma_{n-1}$ with $[x_1, x_2], \ldots, [x_{M-1}, x_M]$. Then by the Pythagorean theorem, the amount of length added to Γ_n is bounded by

$$C\beta^2(Q)\, l(Q)\,.$$

Since $\beta(Q) < \varepsilon_0$, properties (P1) and (P2) are maintained.

Case 3. $\mathcal{L}_{n-1} \cap W \neq \phi \cong \mathcal{L}_{n-1} \cap W^* = \phi$, $\mathcal{L}_n \cap W^* \cap \{|z| \leqq 2^{-n+1}\} = \phi$. Let $y_1 \in \mathcal{L}_{n-1} \cap W$ minimize $|z|$, $z \in \mathcal{L}_{n-1} \cap W$. Then $[0, y_1] \subset \Gamma_{n-1}$. Let $\{0, x_1, x_2, \ldots, x_M, y_1\}$ be the points in $\mathcal{L}_n \cap W$ between 0 and y_1, arranged by increasing real parts. Replace $[0, y_1]$ by $[0, x_1]$, $[x_1, x_2], \ldots, [x_M, y_1]$. Then by Pythagoras the amount of length added is bounded by

$$C\beta^2(Q)\, l(Q)\,,$$

and properties (P1) and (P2) are maintained at $x_1, \ldots, x_M$ because $\beta(Q) < \varepsilon_0$. We now add on some line segments near 0 and y_1 to assure properties (P1) and (P2) hold there. First suppose $x_{-1} \in K \cap W^*$ maximizes $|z|$, $z \in K \cap W^* \cap \{|z| \leqq A2^{-n}\}$. If $|x_{-1}| < 8A^{-1}2^{-n}$ add to Γ_n the line segment $[-2^{-n+1}, 0]$. If $|x_{-1}| \geqq 8A^{-1}2^{-n}$, add to Γ_n the line segment $\left[2^{-n+1}\frac{x_{-1}}{|x_{-1}|}, 0\right]$. In either case the amount of length added is bounded by 2^{-n+1}. Let $z_0, z_1, \ldots, z_N = x_1$ be the points in $\mathcal{L}_{n+k_0} \cap \{|z| \leqq 2^{-n}\}$, arranged by increasing real parts, where $2^{k_0} = 2A^2$. We add on to Γ_n the line segments $[z_0, z_1], \ldots, [z_{N-1}, z_N]$, so that at stages $n+1$, $n+2, \ldots, n+k_0-1$ no constructions need be performed in $\{|z| \leqq 2^{-n}\}$. Since $\beta(Q) < \varepsilon_0$, properties (P1), (P2) are preserved for $z_1, \ldots, z_N$ at stage $n+k_0$ by the choice of $[-2^{-n+1}, 0]$ $\left[\text{respectively } 2^{-n+1}\frac{x_{-1}}{|x_{-1}|}, 0\right]$. A similar construction is performed at y_1.

Let $E = [0, A2^{-n+1}e^{i\theta}] \subset \Gamma_{n-1} \cap W^*$ be the line segment assured by hypothesis (P2) and let

$$I_Q = [A2^{-n-1}e^{i\theta}, A2^{-n}e^{i\theta}]\,.$$

Then the construction will show that I_Q is not altered at any future stage, $I_Q \subset \bigcap_{k=0}^{\infty} \Gamma_{n+k}$. If A is large enough,

(4.1) The amount of length added is bounded by $1/2\, l(I_Q)$.

Furthermore, if I_Q and $I_{Q'}$ are any intervals so formed at any stage of the construction,

$$I_Q \cap I_{Q'} = \phi . \tag{4.2}$$

Case 4. $\mathscr{L}_{n-1} \cap W \neq \phi$, $\mathscr{L}_{n-1} \cap W^* = \phi$, $\mathscr{L}_n \cap W^* \cap \{|z| \leqq 2^{-n+1}\} \neq \phi$. First suppose $\mathscr{L}_n \cap W^* \cap \{|z| \leqq 2^{-n+1}\} = \{x_{-1}\}$. We may assume, by changing $\mathscr{L}_n$ if necessary, that x_{-1} maximizes $|z|$, $z \in K \cap W^* \cap \{|z| \leqq 2^{-n+1}\}$. The construction is as in Case 3, but we add the segment $[2x_{-1}, 0]$. Then estimates (4.1) and (4.2) hold and as in the argument of Case 3 properties (P1) and (P2) will hold at future stages. If $\mathscr{L}_n \cap W^* \cap \{|x| \leqq 2^{-n+1}\}$ contains two points x_{-1}, x_{-2}, we may assume x_{-1} maximizes $|z|$, $z \in K \cap W^* \cap \{|z| \leqq 2^{-n+1}\}$. Then we add $[2x_{-1}, x_{-1}]$, and $[x_{-1}, x_{-2}]$, $[x_{-2}, 0]$. Estimates (4.1) and (4.2) hold as do properties (P1) and (P2) at future stages. The case where $\mathscr{L}_n \cap W^* \cap \{|z| \leqq 2^{-n+1}\}$ contains two points is treated similarly.

Case 5. $\mathscr{L}_{n-1} \cap W = \phi \cap \mathscr{L}_{n-1} \cap W^* \neq \phi$. We assume x_0 maximizes $|z|$, $z \in K \cap W \cap \{|z| \leqq 2^{-n+1}\}$. Let $\{y_1, y_2, \ldots, y_N = x_0\}$ be the points in $(\mathscr{L}_n \cap W^*) \cup \{|z| \leqq 2^{-n+1}\}$ arranged by increasing real parts, and as in Case 3 replace arcs of Γ_{n-1} in that region by $[y_1, y_2], \ldots, [y_{N-1}, y_N]$. As in Case 3 we add on $[x_0, 2x_0]$ and line segments in $W \cap \{|z| \leqq 2^{-n+1}\}$ so that $\mathscr{L}_{n+k_0} \cap W \cap \{|z| \leqq 2^{-n+1}\} \subset \Gamma_n$. We also choose I_Q as in Case 3. Then (4.1) and (4.2) hold and as in Case 3, (P1) and (P2) are preserved.

Case 6. $\mathscr{L}_{n-1} \cap W = \phi$, $\mathscr{L}_{n-1} \cap W^* = \phi$. Let $\{y_1, \ldots, y_N\}$ be the points in $\mathscr{L}_{n+k_0} \cap \{|z| \leqq 2^{-n+1}\}$ arranged by increasing real part. We may assume either $y_1 = 0$ or y_1 maximizes $|z|$, $z \in K \cap W^* \cap \{2^{-(n+k_0)} \leqq |z| \leqq 2^{-n+1}\}$, and we may assume y_N maximizes $|z|$, $z \in K \cap W \cap \{|z| \leqq 2^{-n+1}\}$. Add on the line segments $[2y_1, y_1]$, $[2y_N, y_N]$ and $[y_j, y_{j+1}]$, $1 \leqq j \leqq N-1$. Let I_Q be as in Case 3. Then (4.1) and (4.2) hold and (P1) and (P2) are preserved.

Remark. By the choice of Γ_0 only Case 1 or Case 2 constructions can happen at z_0 and z_1 until stage k_0, $2^{k_0} = A$. Therefore (P1) and (P2) will always hold at z_0, z_1.

To conclude the proof we note that by Cases 1–6 and estimate (4.1) the quantity $l(\Gamma_n) - l(\Gamma_{n-1})$ is bounded by

$$C \sum_{Q \in \mathscr{D}_n} \beta^2(Q)\, l(Q) + \frac{1}{2} \sum_{Q \in \mathscr{D}_n} l(I_Q) .$$

Summing from $n = 1$ to N we obtain

$$\begin{aligned} l(\Gamma_N) - l(\Gamma_0) &\leqq C \sum_Q \beta^2(Q)\, l(Q) + \frac{1}{2} \sum_{l(Q) \leqq 2^{-N}} l(I_Q) \\ &\leqq C \sum_Q \beta^2(Q)\, l(Q) + \frac{1}{2} l(\Gamma_N) , \end{aligned}$$

the final inequality being a consequence of (4.2). Therefore

$$l(\Gamma_N) \leqq 2 l(\Gamma_0) + C \sum_Q \beta^2(Q)\, l(Q) ,$$

and taking limits we obtain the first part of the theorem. We note that the segments $[2z_0 - z_1, z_0]$, $[z_1, 2z_1 - z_0] \subset \Gamma_0$ are never altered at any stage of the construction. By throwing them away and taking A large enough we could build Γ so that

$$l(\Gamma) \leqq (1 + \delta)\,\text{diameter}(K) + C_\delta \sum_Q \beta_K^2(Q)\, l(Q),$$

but then (1.5) would not hold for $Q = [0, 1]^2$.

To show property (1.5) holds, we must add some line segments to Γ to form a new curve $\tilde{\Gamma}$. Fix a dyadic cube Q with $\beta(2Q) < \varepsilon_0$ and first suppose that there are points $x_0 \in K \cap Q$ and $x_1, x_2 \in K \cap (5Q \backslash 3Q)$ with the angle between $[x_0, x_1]$ and $[x_0, x_2]$ greater than $\pi/2$. Then the construction yields a subcurve of Γ which connects x_1 to x_2 in S_Q. For the other case (where $x_1, x_2 \in K \cap (5Q\backslash 3Q)$ implies the angle between $[x_0, x_1]$ and $[x_0, x_2]$ is less than $\pi/2$), the construction shows there is an arc $I_Q \subset \Gamma \cap 3Q$ such that distance $(I_Q, K) \geqq C\, l(Q)$. Add to Γ a line segment J_Q crossing $5Q$ in S_Q. Then $l(J_Q) \leqq Cl(I_Q)$ and since the I_Q are essentially disjoint from each other,

$$\sum_Q l(J_Q) \leqq C' \sum_Q l(I_Q) \leqq C'\, l(\Gamma).$$

Consequently $l(\tilde{\Gamma}) \leqq C\, l(\Gamma)$.

To show that (1.4) holds, apply Corollary 2 to Γ and obtain $\tilde{\tilde{\Gamma}}$ such that $l(\tilde{\tilde{\Gamma}}) \leqq C_0 + C_0 \beta^2(K)$ and such that the conclusions of Corollary 2 hold. Let $\mathbb{C} \backslash \tilde{\tilde{\Gamma}} = \bigcup \Omega_j$, and suppose $x, y \in \tilde{\tilde{\Gamma}}$. Let $I = [x, y]$ and let $I = E \bigcup \bigcup_j I_j$ be a decomposition of I into $E = I \cap \tilde{\tilde{\Gamma}}$ and open intervals I_j which lie in $\Omega_{k(j)}$. Setting $I_j = [x_j, y_j]$, we have $x_j, y_j \in \partial\Omega_{k(j)}$, and consequently there is an arc $\gamma_j \subset \partial\Omega_{k(j)}$ connecting x_j to y_j such that $l(\gamma_j) \leqq C_0 |x_j - y_j|$. Then if $\gamma = E \bigcup \bigcup_j \gamma_j$, γ is connected, $\gamma \subset \tilde{\tilde{\Gamma}}$, and $l(\gamma) \leqq l(E) + \sum_j l(\gamma_j) \leqq l(E) + C_0 \sum_j |x_j - y_j| \leqq C_0 |x - y|$.

§5. Appendix

In this section we show that (1.2) holds when Γ is a C_0 Lipschitz curve. By using a dilation, we may assume that Γ is given by the parametrization $\psi(\theta) = r(\theta) e^{i\theta}$, where $C_0^{-1} \leqq r(\theta) \leqq 1$, and $|r(\theta_1) - r(\theta_2)| \leqq C_0 |\theta_1 - \theta_2|$. Let Γ_n be the polygon obtained from the line segments

$$J_j^n = [\psi(j 2^{-n+1}\pi), \psi(j + 1) 2^{-n+1}\pi)], \qquad 0 \leqq j \leqq 2^n.$$

Then J_j^n splits into two intervals $J_{2j}^{n+1}, J_{2j+1}^{n+1}$ at stage $n + 1$. Define

$$\delta_{n,j} = 2^{-n} \sup_{z \in J_{2j}^{n+1} \cup J_{2j+1}^{n+1}} \text{distance}(z, J_j^n).$$

Then by elementary geometry,

$$l(J_{2j}^{n+1}) + l(J_{2j+n}^{n+1}) - l(J_j^n) \geqq C(\delta_{n,j})^2\, 2^{-n}$$

Summing from $n = 1$ to ∞ we obtain

$$c \sum_{n,j} (\delta_{n,j})^2 2^{-n} \leqq l(\Gamma) . \tag{5.1}$$

$$\sum_{k=n}^{\infty} \sum_{J_m^k \subset \theta_j^n} (\delta_{k,m})^2 2^{-k} \leqq C 2^{-n} .$$

Here we define $\Gamma_j^n = \Gamma \cap \{ j 2^{-n+1} \pi \leqq \theta \leqq (j+1) 2^{-n+1} \pi \} \equiv \Gamma \cap \theta_j^n$. Our result will follow if we can show that

$$\tilde{\beta}(\Gamma_j^n) = 2^{-n} \sup_{z \in \Gamma_j^n} \text{distance}\,(z, J_j^n)$$

satisfies

$$\sum_{n,j} \tilde{\beta}(\Gamma_j^n)^2 2^{-n} \leqq C_1 , \tag{5.2}$$

for then we may rotate the dyadic grid through $[0, 2\pi]$ to obtain quantities $\tilde{\beta}_\theta(\Gamma_j^n)$ and note that

$$\sum_{l(Q) = 2^{-n-2}} \beta_\Gamma(Q)^2 \, l(Q) \leqq$$

$$C \int_0^{2\pi} \left\{ \sum_j \tilde{\beta}_\theta(\Gamma_j^n)^2 2^{-n} \right\} d\theta .$$

To prove (5.2) notice that

$$\tilde{\beta}(\Gamma_j^n) \leqq C_1 \sum_{k=1}^{\infty} \sup_{J_m^{n+k} \subset \theta_j^n} 2^{-k} \delta_m^{n+k} .$$

Then (5.2) follows from the above inequality, Minkowski's inequality (or Cauchy–Schwarz), and (5.1).

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