THREE REVOLUTIONS IN THE KERNEL ARE WORSE THAN ONE

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Abstract. An example is constructed of a purely unrectifiable measure μ for which the singular integral associated to the kernel $K(z) = \frac{\bar{z}}{z^2}$ is bounded in $L^2(\mu)$. The singular integral fails to exist in the sense of principal value μ-almost everywhere. This is in sharp contrast with the results known for the kernel $\frac{1}{z}$ (the Cauchy transform).

1. Introduction

Let $B(z, r)$ denote the closed disc in $\mathbb{C}$ centred at z with radius $r > 0$. A finite Borel measure μ is said to be 1-dimensional if $\mathcal{H}^1(\text{supp}(\mu)) < \infty$, and there exists a constant $C > 0$ such that $\mu(B(z, r)) \leq Cr$ for any $z \in \mathbb{C}$ and $r > 0$.

For a kernel function $K : \mathbb{C}\backslash\{0\} \to \mathbb{C}$, and a finite measure μ, we define the singular integral operator associated to K by

$$T_\mu(f)(z) = \int_{\mathbb{C}} K(z - \xi)f(\xi)d\mu(\xi), \text{ for } z \notin \text{supp}(\mu).$$

A well-known problem in harmonic analysis is to determine geometric properties of μ from regularity properties of the operator T_μ, see for instance the monograph of David and Semmes [DS]. This paper concerns the question of characterizing those functions K with the following property:

$$(*) \qquad \begin{array}{l} \textit{Let } \mu \textit{ be a 1-dimensional measure. Then} \\ \|T_\mu(1)\|_{L^\infty(\mathbb{C}\setminus \text{supp}(\mu))} < \infty \textit{ implies that } \mu \textit{ is rectifiable.} \end{array}$$

The property that $\|T_\mu(1)\|_{L^\infty(\mathbb{C}\setminus \text{supp}(\mu))} < \infty$ is equivalent to the boundedness of T_μ as an operator in $L^2(\mu)$, see for instance [NTV]. A measure μ is rectifiable if $\text{supp}(\mu)$ can be covered (up to an exceptional set of $\mathcal{H}^1$ measure zero) by a countable union of rectifiable curves. A measure μ is purely unrectifiable if its support is purely unrectifiable, that is, $\mathcal{H}^1(\Gamma \cap \text{supp}(\mu)) = 0$ for any rectifiable curve Γ.

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David and Léger [Leg] proved that the Cauchy kernel $\frac{1}{z}$ has property $(*)$. As is remarked in [CMPT], the proof in [Leg] extends to the case when the Cauchy kernel is replaced by either its real or imaginary part, i.e. $\frac{\Re(z)}{|z|^2}$ or $\frac{\Im(z)}{|z|^2}$. Recently in [CMPT], Chousionis, Mateu, Prat, and Tolsa extended the result of [Leg] and showed that kernels of the form $\frac{(\Re(z))^k}{|z|^{k+1}}$ have property $(*)$ for any odd positive integer k. Both of these results use the Melnikov-Menger curvature method.

On the other hand, Huovinen [Huo2] has shown that there is a purely unrectifiable Ahlfors-David (AD)-regular set E for which the singular integral associcated to the kernel $\frac{\Re(z)}{|z|^2} - \frac{\Re(z)^3}{|z|^4}$ is bounded in $L^2(\mathcal{H}^1_{|E})$. In fact, an essentially stronger conclusion is proved that the principal values of the associated singular integral operator exist $\mathcal{H}^1$-a.e. on E. Huovinen takes advantage of several non-standard symmetries and cancellation properties in this kernel to construct his very nice example.

The result of this paper is that a weakened version of Huovinen's theorem holds for a very simple kernel function. Indeed, it is perhaps the simplest example of a kernel for which the Menger curvature method fails to be directly applicable. From now on, we shall fix

$$K(z) = \frac{\bar{z}}{z^2}, z \in \mathbb{C}\backslash\{0\}. \tag{1.1}$$

We prove the following result.

Theorem 1.1. *There exists a 1-dimensional purely unrectifiable probability measure μ with the property that $\|T_\mu(1)\|_{L^\infty(\mathbb{C}\setminus \operatorname{supp}(\mu))} < \infty$.*

In other words, the kernel K in (1.1) fails to satisfy property $(*)$. At this point, we would also like to mention Huovinen's thesis work [Huo1], regarding the kernel function $K(z)$ from (1.1). It is proved that if $\liminf_{r\to 0} \frac{\mu(B(z,r))}{r} \in (0,\infty)$ μ-a.e. (essentially the AD-regularity of μ), then the μ-almost everywhere existence of $T_\mu(1)$ in the sense of principal value implies that μ is rectifiable. This result was proved by building upon the theory of symmetric measures, developed by Mattila [Mat2], and Mattila and Preiss [MP]. Unfortunately the measure in Theorem 1.1 does not satisfy the AD-regularity condition. In view of Huovinen's work it would be of interest to construct an AD-regular measure supported on an unrectifiable set for which the conclusion of Theorem 1.1 holds. We have not been able to construct such a measure (yet).

For the measure μ constructed in Theorem 1.1, we show that $T_\mu(1)$ fails to exist in the sense of principal value μ-almost everywhere. Thus the two properties of $L^2(\mu)$ boundedness of the operator T_μ, and the existence of $T_\mu(1)$ in the sense of principal value, are quite distinct for this singular integral operator.

2. Notation

- Let m_2 denote the 2-dimensional Lebesgue measure normalized so that $m_2(B(0,1)) = 1$. We let m_1 denote the 1-dimensional Lebesgue measure.
- A collection of squares are essentially pairwise disjoint if the interiors of any two squares in the collection do not intersect. Throughout the paper, all squares are closed.
- We shall denote by C and c large and small absolute positive constants. The constant C should be thought of as large (at least 1), while c is to be thought of as small (smaller than 1).
- For $a > 1$, the disc aB denotes the concentric enlargement of a disc B by a factor of a.
- We define the $\mathcal{H}^1$-measure of a set E by $\mathcal{H}^1(E) = \sup_{\delta>0} \inf\bigl\{\sum_j r_j \,:\, E \subset \bigcup_j B(x_j, r_j) \text{ with } r_j \leq \delta\bigr\}$.
- For $z \in \mathbb{C}$ and $r > 0$, we define the annulus $A(z,r) = B(z,r)\backslash B(z, \frac{r}{2})$.
- The set $\text{supp}(\mu)$ denotes the closed support of μ.

3. A reflectionless measure

Let us make the key observation that allows us to prove Theorem 1.1.

Lemma 3.1. *Let $z \in \mathbb{C}$, $r > 0$. For any $\omega \in B(z,r)$,*

$$\int_{B(z,r)} K(\omega - \xi)\, dm_2(\xi) = 0.$$

Proof. Without loss of generality, we may set $z = 0$ and $r = 1$. If $|\omega| < |\xi|$, then

$$K(\omega - \xi) = \frac{\overline{\omega - \xi}}{\xi^2} \sum_{\ell=0}^{\infty} (\ell + 1)\Bigl(\frac{\omega}{\xi}\Bigr)^{\ell}.$$

So whenever $t > |\omega|$, we have $\int_{\partial B(0,t)} K(\omega - \xi)\, dm_1(\xi) = 0$. (This follows merely from the fact that $\int_{\partial B(0,t)} \overline{\xi}^{\ell} \xi^k\, dm_1(\xi) = 0$ whenever $k, \ell \in \mathbb{Z}$ satisfy $k \neq \ell$.) On the other hand, if $|\xi| < |\omega|$, then

$$K(\omega - \xi) = \frac{\overline{\omega - \xi}}{\omega^2} \sum_{\ell=0}^{\infty} (\ell + 1)\Bigl(\frac{\xi}{\omega}\Bigr)^{\ell}.$$

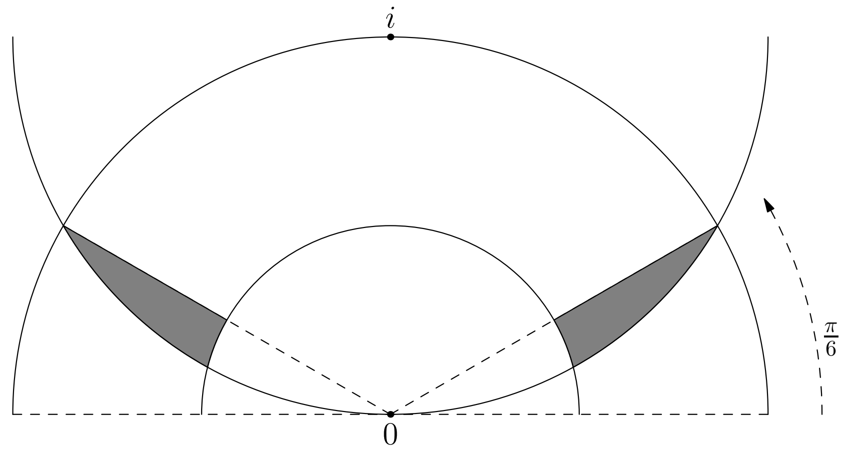

FIGURE 1. The set-up for the proof of Lemma 3.2.

Therefore, if $t < |\omega|$, then

$$\int_{\partial B(0,t)} K(\omega - \xi) dm_1(\xi) = 2\pi\Big[t\frac{\overline{\omega}}{\omega^2} - 2\frac{t^3}{\omega^3}\Big] = \frac{2\pi}{\omega^3}(t|\omega|^2 - 2t^3).$$

Since $\int_0^{|\omega|}(t|\omega|^2 - 2t^3)dt = 0$, the desired conclusion follows. □

The next lemma will form the basis of the proof of the non-existence of $T_\mu(1)$ in the sense of principal value.

Lemma 3.2. *There exists a constant $\tilde{c} > 0$ such that for any disc $B(z, r)$, and $\omega \in \partial B(z, r)$,*

$$\Big|\int_{A(\omega,r)\cap B(z,r)} K(\omega - \xi)\frac{dm_2(\xi)}{r}\Big| \geq \tilde{c}.$$

Proof. By an appropriate translation and rescaling, we may assume that $B(z, r) = B(i, 1)$, and $\omega = 0$. Making reference to Figure 1 above, we split the domain of integration into three regions, $I = \{\xi \in A(0, 1) : \arg(\xi) \in \big[\frac{\pi}{6}, \frac{5\pi}{6}\big]\}$, $II = \{\xi \in A(0, 1) \cap B(i, 1) : \arg(\xi) \in \big[0, \frac{\pi}{6}\big]\}$ and $III = \{\xi \in A(0, 1) \cap B(i, 1) : \arg(\xi) \in \big[\frac{5\pi}{6}, \pi\big]\}$. The regions II and III are respectively the right and left grey shaded regions in Figure 1. Note that $\Im K(-\xi) < 0$ if $\arg(\xi) \in \big[\frac{\pi}{3}, \frac{2\pi}{3}\big]$, and $\Im K(-\xi) > 0$ if $\arg(\xi) \in \big[0, \frac{\pi}{3}\big] \cup \big[\frac{2\pi}{3}, \pi\big]$. Furthermore, note that

$$\int_I \Im K(-\xi) dm_2(\xi) = \frac{1}{\pi}\int_{\frac{1}{2}}^{1} \frac{1}{t}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} -\Im\big(e^{-3\theta i}\big) t d\theta dt = 0.$$

But $\int_{II\cup III} \Im K(-\xi) dm_2(\xi) = 2\int_{II} \Im K(-\xi) dm_2(\xi) > 0$. Therefore, by setting $\tilde{c} = 2\int_{II} \Im K(-\xi)dm_2(\xi)$, the lemma follows. □

4. Packing squares in a disc

Fix $r, R \in (0,\infty)$ such that $r < \frac{R}{16}$ and $\frac{R}{r} \in \mathbb{N}$.

Lemma 4.1. *One can pack $\frac{R}{r}$ pairwise essentially disjoint squares of side length $\sqrt{\pi r R}$ into a disc of radius $R(1+4\sqrt{\frac{r}{R}})$.*

Proof. We may assume that the disc is centred at the origin. Consider the square lattice with mesh size $\sqrt{\pi r R}$. Label those squares that intersect $B(0,R)$ as $Q_1,\dots,Q_M$. These squares are contained in $B(0,R(1+4\sqrt{\frac{r}{R}}))$. Since $MrR = \sum_{j=1}^M m_2(Q_j) > m_2(B(0,R)) = R^2$, we have that $M > \frac{R}{r}$. By throwing away $M - \frac{R}{r}$ of the least desirable squares, we arrive at the desired collection. □

Lemma 4.2. *Consider a disc $B(z,R)$. Let $Q_1,\dots,Q_{R/r}$ be the collection of squares contained in $B(z,R(1+4\sqrt{\frac{r}{R}}))$ found in Lemma 4.1. Then $m_2(B(z,R)\triangle\bigcup_{j=1}^{R/r} Q_j) \le Cr^{1/2}R^{3/2}$.*

Proof. Since $m_2(B(z,R)) = m_2\big(\bigcup_{j=1}^{R/r} Q_j\big) = R^2$, the property follows from the fact that both sets are contained in $B(z,R(1+4\sqrt{\frac{r}{R}}))$. □

5. The construction of the sparse Cantor set E

Let $r_0 = 1$, and choose r_j, $j \in \mathbb{N}$, to be a sequence which tends to zero quickly. Assume that $r_j < \frac{r_{j-1}}{100}$, $\frac{1}{r_j} \in \mathbb{N}$, and $\frac{r_j}{r_{j+1}} \in \mathbb{N}$ for all $j \ge 1$.

Several additional requirements will be imposed on the decay of r_j over the course of the following analysis, and we make no attempt to optimize the conditions.

It will be convenient to let $s_{n+1} = 4\sqrt{\frac{r_{n+1}}{r_n}}$ for $n \in \mathbb{Z}_+$.

First define $\widetilde{B}_1^{(0)} = B(0,1)$. Given the n-th level collection of $\frac{1}{r_n}$ discs $\widetilde{B}_j^{(n)}$ of radius r_n, we construct the $(n+1)$-st generation according to the following procedure:

Fix a disc $\widetilde{B}_j^{(n)}$. Apply Lemma 4.1 with $R = r_n$ and $r = r_{n+1}$ to find $\frac{r_n}{r_{n+1}}$ squares $Q_\ell^{(n+1)}$ of side length $\sqrt{\pi r_{n+1} r_n}$ that are pairwise essentially disjoint, and contained in $(1+s_{n+1})\cdot\widetilde{B}_j^{(n)}$. Let $z_\ell^{(n+1)}$ be the centre of $Q_\ell^{(n+1)}$, and set $\widetilde{B}_\ell^{(n+1)} = B(z_\ell^{(n+1)}, r_{n+1})$. This procedure is carried out for each disc $\widetilde{B}_j^{(n)}$ from the n-th level collection. There are a total of $\frac{1}{r_{n+1}}$ discs $\widetilde{B}_\ell^{(n+1)}$ in the $(n+1)$-st level.

The above construction is executed for each $n \in \mathbb{Z}_+$.

Now, set $B_j^{(n)} = (1+s_{n+1})\widetilde{B}_j^{(n)}$. Define $E^{(n)} = \bigcup_j B_j^{(n)}$. We shall repeatedly use the following properties of the construction:

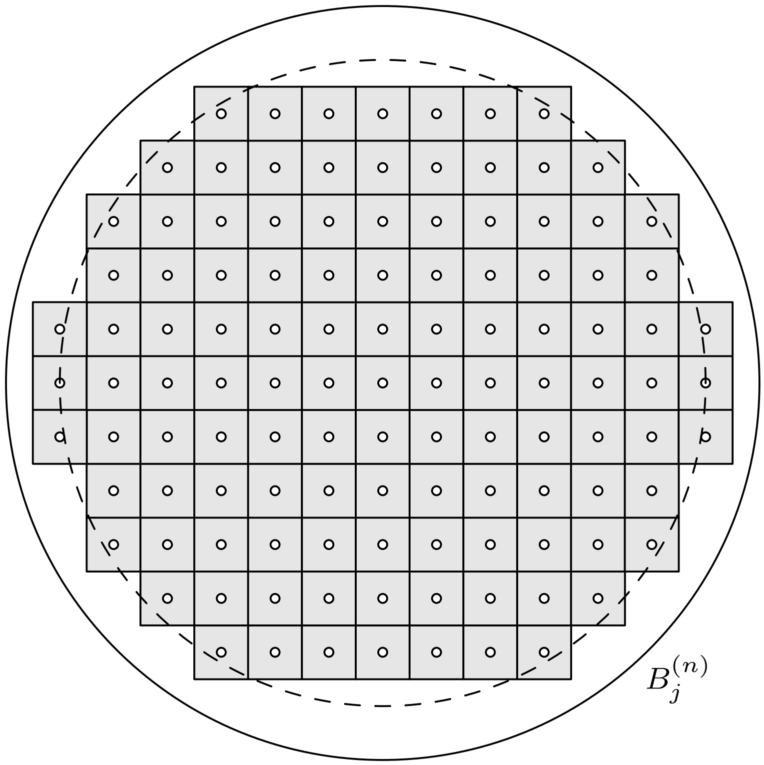

FIGURE 2. The picture shows a single disc $B_j^{(n)}$ of radius $(1+s_{n+1})r_n$. The grey shaded squares are the squares $Q_\ell^{(n+1)}$ of sidelength $\sqrt{\pi r_n r_{n+1}}$ formed by applying Lemma 4.1 to the disc $\widetilde{B}_j^{(n)}$ of radius r_n. The boundary of the disc $\widetilde{B}_j^{(n)}$ is the dashed circle. Deep inside each square $Q_\ell^{(n+1)}$ is the disc $B_\ell^{(n+1)}$ of radius $(1+s_{n+2})r_{n+1}$.

(a) $\bigcup_\ell Q_\ell^{(n+1)} \subset E^{(n)}$, for all $n \geq 0$.

(b) $B_j^{(n)} \subset Q_j^{(n)}$ for each $n \geq 1$. Moreover, $\mathrm{dist}(B_j^{(n)}, \partial Q_j^{(n)}) \geq \frac{1}{2}\sqrt{r_{n-1}r_n}$.

(c) $\mathrm{dist}(B_j^{(n)}, B_k^{(n)}) \geq \frac{1}{2}\sqrt{r_{n-1}r_n}$ whenever $j \neq k$, $n \geq 0$.

Property (a) is immediate. To see property (b), merely note that $\mathrm{dist}(B_j^{(n)}, \partial Q_j^{(n)}) = \frac{\sqrt{\pi r_{n-1}r_n}}{2} - (1+s_{n+1})r_n \geq \frac{1}{2}\sqrt{r_{n-1}r_n}$. For property (c), we shall use induction. If $n=0$, then the claim is trivial. Using (b), the claimed estimate is clear if $Q_j^{(n)}$ and $Q_k^{(n)}$ have been created by an application of Lemma 4.1 in a common disc $\widetilde{B}_\ell^{(n-1)}$. Otherwise, the squares are born out of applying Lemma 4.1 to different discs at the $(n-1)$-st level, and those parent discs are already separated by $\frac{1}{2}\sqrt{r_{n-2}r_{n-1}}$.

Courtesy of properties (a) and (b), we see that $E^{(n+1)} \subset E^{(n)}$ for each $n \geq 0$. Set $E = \bigcap_{n\geq 0} E^{(n)}$. Each $z \in E^{(n)}$ is contained in a unique disc $B_j^{(n)}$ (or square $Q_j^{(n)}$) which we shall denote by $B^{(n)}(z)$ (respectively $Q^{(n)}(z)$).

If $m \geq n \geq 0$, then $E \cap B_j^{(n)}$ is covered by the $\frac{r_n}{r_m}$ discs $B_\ell^{(m)}$ that are contained in $B_j^{(n)}$, each of which has radius $(1 + s_{m+1})r_m \leq 2r_m$. Therefore $\mathcal{H}^1(E \cap B_j^{(n)}) \leq 2r_n$. Taking $n = 0$ yields $\mathcal{H}^1(E) \leq 2$.

6. The measure μ

Define $\mu_j^{(n)} = \frac{1}{r_n}\chi_{\widetilde{B}_j^{(n)}} m_2$. Set $\mu^{(n)} = \sum_j \mu_j^{(n)}$. Then $\text{supp}(\mu^{(n)}) \subset E^{(n)}$, and $\mu^{(n)}(\mathbb{C}) = 1$ for all n. Therefore, there exists a subsequence of the sequence of measures $\mu^{(n)}$ that converges weakly to a measure μ, with $\mu(\mathbb{C}) = 1$ and $\text{supp}(\mu) \subset E$.

The following three properties hold:

(i) $\text{supp}(\mu^{(m)}) \subset \bigcup_j B_j^{(n)}$ whenever $m \geq n$,

(ii) $\mu^{(m)}(B_j^{(n)}) = r_n$ for $m \geq n$, and

(iii) there exists $C_0 > 0$ such that $\mu^{(n)}(B(z, r)) \leq C_0 r$ for any $z \in \mathbb{C}$, $r > 0$ and $n \geq 0$.

Properties (i) and (ii) follow immediately from the construction of $E^{(n)}$. To see the third property, note that since $\mu^{(n)}$ is a probability measure, the property is clear if $r \geq 1$. If $r < 1$, then $r \in (r_{m+1}, r_m)$ for some $m \in \mathbb{Z}_+$. If $m \geq n$, then $B(z, r)$ intersects at most one disc $B_j^{(n)}$. Then $\mu^{(n)}(B(z, r)) = \frac{1}{r_n} m_2(B(z, r) \cap \widetilde{B}_j^{(n)}) \leq \frac{r^2}{r_n} \leq r$. Otherwise $m < n$. In this case, note that since the discs $B_j^{(m+1)}$ are $\frac{1}{2}\sqrt{r_m r_{m+1}}$ separated, $B(z, r)$ intersects at most $1 + C\big(\frac{r}{\sqrt{r_m r_{m+1}}}\big)^2$ discs $B_j^{(m+1)}$. Hence, by property (ii), we see that

$$\mu^{(n)}(B(z, r)) = \sum_j \mu^{(n)}(B(z, r) \cap B_j^{(m+1)}) \leq \Big[1 + C\Big(\frac{r}{\sqrt{r_m r_{m+1}}}\Big)^2\Big] r_{m+1},$$

which is at most Cr.

The weak convergence of a subsequence of $\mu^{(n)}$ to the measure μ, along with property (iii), yields that $\mu(B(z, r)) \leq C_0 r$ for any disc $B(z, r)$. We shall henceforth refer to this property by saying that μ *is C_0-nice*. We have now shown that μ is 1-dimensional.

Notice that we also have $\mathcal{H}^1(E) \geq \frac{1}{C_0}\mu(E) > 0$.

7. The boundedness of $T_\mu(1)$ off the support of μ

As a simple consequence of the weak convergence of $\mu^{(n)}$ to μ, the property that $\|T_\mu(1)\|_{L^\infty(\mathbb{C}\setminus\operatorname{supp}(\mu))} < \infty$ will follow from the following proposition.

Proposition 7.1. *Provided that $\sum_{n\geq 1}\sqrt{s_n} < \infty$, there exists a constant $C > 0$ so that the following holds:*

Suppose that $\operatorname{dist}(z, \operatorname{supp}(\mu)) = \varepsilon > 0$. *Then for any $m \in \mathbb{Z}_+$ with $r_m < \frac{\varepsilon}{4}$,*

$$\Big|\int_{\mathbb{C}} K(z-\xi)d\mu^{(m)}(\xi)\Big| \leq C.$$

To begin the proof, fix r_m with $r_m < \frac{\varepsilon}{4}$. Let $z^* \in \operatorname{supp}(\mu)$ with $\operatorname{dist}(z, z^*) = \varepsilon$. For any $\xi \in \operatorname{supp}(\mu)$, $B^{(m)}(\xi) \cap \operatorname{supp}(\mu^{(m)}) \neq \varnothing$, so $\operatorname{dist}(z, \operatorname{supp}(\mu^{(m)})) \geq \varepsilon - (1 + s_{m+1})r_m \geq \frac{\varepsilon}{2}$.

Now, let q be the least integer with $r_q \leq \varepsilon$ (so $m \geq q$). Then by property (ii) of the previous section,

$$\int_{B^{(q)}(z^*)} |K(z,\xi)|d\mu^{(m)}(\xi) \leq \frac{2}{\varepsilon}\mu^{(m)}(B^{(q)}(z^*)) = \frac{2r_q}{\varepsilon} \leq 2. \tag{7.1}$$

The crux of the matter is the following lemma.

Lemma 7.2. *There exists $C > 0$ such that for any $n \in \mathbb{Z}_+$ with $1 \leq n \leq q$,*

$$\Big|\int_{B^{(n-1)}(z^*)\setminus B^{(n)}(z^*)} K(z-\xi)d\mu^{(m)}(\xi)\Big| \leq C\sqrt{s_n} + C\sqrt{\tfrac{\varepsilon}{r_{n-1}}}.$$

For the proof of Lemma 7.2, we shall require the following simple comparison estimate.

Lemma 7.3. *Let $z_0 \in \mathbb{C}$, and $\lambda > 0$. Fix $r, R \in (0,1]$ with $100r \leq R$. Suppose that ν_1 and ν_2 are Borel measures, such that* $\operatorname{supp}(\nu_1) \subset Q(z_0, \sqrt{\pi Rr}) = Q$, $\operatorname{supp}(\nu_2) \subset B(z_0, 2r) = B$, *and* $\nu_1(\mathbb{C}) = \nu_2(\mathbb{C})$. *Then, for any $z \in \mathbb{C}$ with dist$(z, Q) \geq \lambda\sqrt{rR}$, we have*

$$\Big|\int_Q K(z-\xi)d\nu_1(\xi) - \int_B K(z-\xi)d\nu_2(\xi)\Big|$$
$$\leq \frac{1}{\lambda^2}\int_Q \frac{C\sqrt{Rr}}{|z-\xi|^2}d\nu_1(\xi) + \frac{1}{\lambda^2}\int_B \frac{Cr}{|z-\xi|^2}d\nu_2(\xi).$$

Proof. Note that the left hand side of the inequality can be written as

$$\Big|\int_Q [K(z-\xi) - K(z-z_0)]d(\nu_1-\nu_2)(\xi)\Big|.$$

But, under the hypothesis on z, we have that $|K(z-\xi)-K(z-z_0)| \leq \frac{C|\xi-z_0|}{\lambda^2|z-\xi|^2}$ for any $\xi \in Q$. Plugging this estimate into the integral and taking into account the supports of ν_1 and ν_2, the inequality follows. □

Proof of Lemma 7.2. Write

$$\mathcal{A} = \{j : B_j^{(n)} \neq B^{(n)}(z^*) \text{ and } B_j^{(n)} \subset B^{(n-1)}(z^*)\}.$$

First suppose that $\operatorname{dist}(z, Q_j^{(n)}) \geq \frac{1}{4}\sqrt{r_{n-1}r_n}$ for $j \in \mathcal{A}$. Then the hypothesis of Lemma 7.3 are satisfied with $\nu_1 = \chi_{Q_j^{(n)}}\frac{m_2}{r_{n-1}}$, $\nu_2 = \chi_{B_j^{(n)}}\mu^{(m)}$, $R = r_{n-1}$, $r = r_n$, and $z_0 = z_{Q_j^{(n)}}$. Thus

$$\begin{aligned}(7.2)\qquad &\Big|\int_{Q_j^{(n)}} K(z-\xi)\frac{dm_2(\xi)}{r_{n-1}} - \int_{B_j^{(n)}} K(z-\xi)d\mu^{(m)}(\xi)\Big| \\ &\qquad\leq \int_{Q_j^{(n)}} \frac{C\sqrt{r_{n-1}r_n}}{|z-\xi|^2}\frac{dm_2(\xi)}{r_{n-1}} + \int_{B_j^{(n)}} \frac{Cr_n d\mu^{(m)}(\xi)}{|z-\xi|^2}.\end{aligned}$$

Now suppose that $j \in \mathcal{A}$ and $\operatorname{dist}(z, Q_j^{(n)}) \leq \frac{1}{4}\sqrt{r_{n-1}r_n}$. Since $\operatorname{dist}(z, Q_j^{(n)}) \geq \operatorname{dist}(z^*, Q_j^{(n)}) - \operatorname{dist}(z, z^*) \geq \frac{1}{2}\sqrt{r_{n-1}r_n} - \varepsilon$, we must have that $\varepsilon \geq \frac{1}{4}\sqrt{r_{n-1}r_n}$. But as $\operatorname{dist}(z, \operatorname{supp}(\mu^{(m)})) \geq \frac{\varepsilon}{2}$, and $\mu^{(m)}(B_j^{(n)}) = r_n$, we have the following crude bound

$$\begin{aligned}(7.3)\qquad &\Big|\int_{Q_j^{(n)}} K(z-\xi)\frac{dm_2(\xi)}{r_{n-1}} - \int_{B_j^{(n)}} K(z-\xi)d\mu^{(m)}(\xi)\Big| \\ &\qquad\leq \frac{C}{r_{n-1}}\sqrt{m_2(Q_j^{(n)})} + \frac{2}{\varepsilon}\mu^{(m)}(B_j^{(n)}) \leq Cs_n.\end{aligned}$$

(Here it is used that $\int_A |K(\xi)|dm_2(\xi) \leq C\sqrt{m_2(A)}$ for any Borel measurable set $A \subset \mathbb{C}$ of finite m_2-measure.)

At most 4 of the essentially pairwise disjoint squares $Q_j^{(n)}$, $j \in \mathcal{A}$, can satisfy $\operatorname{dist}(z, Q_j^{(m)}) \leq \frac{1}{4}\sqrt{r_{n-1}r_n}$ (and it can only happen at all if $n = q$). Therefore by summing (7.2) and (7.3) over $j \in \mathcal{A}$ in the cases when $\operatorname{dist}(z, Q_j^{(n)}) \geq \frac{1}{4}\sqrt{r_{n-1}r_n}$ and $\operatorname{dist}(z, Q_j^{(n)}) \leq \frac{1}{4}\sqrt{r_{n-1}r_n}$ respectively, we see that the quantity

$$\Big|\int_{\bigcup_{j\in\mathcal{A}} Q_j^{(n)}} K(z-\xi)\frac{dm_2(\xi)}{r_{n-1}} - \int_{B^{(n-1)}(z^*)\setminus B^{(n)}(z^*)} K(z-\xi)d\mu^{(m)}(\xi)\Big|,$$

is no greater than a constant multiple of

$$\int_{B(z,2r_{n-1})\setminus B(z,\frac{1}{4}\sqrt{r_n r_{n-1}})} \sqrt{\frac{r_n}{r_{n-1}}}\frac{dm_2(\xi)}{|z-\xi|^2} + \int_{\mathbb{C}\setminus B(z,\frac{1}{4}\sqrt{r_n r_{n-1}})} \frac{r_n d\mu^{(m)}(\xi)}{|z-\xi|^2} + s_n.$$

The first term here is bounded by $C\sqrt{\frac{r_n}{r_{n-1}}}\log\big(\frac{r_{n-1}}{r_n}\big)\leq Cs_n\log(\frac{1}{s_n})\leq C\sqrt{s_n}$. Since $\mu^{(m)}$ is C_0-nice, we bound the second term by

$$Cr_n\int_{\frac{1}{4}\sqrt{r_nr_{n-1}}}^{\infty}\frac{dr}{r^2}\leq Cr_n\frac{1}{\sqrt{r_nr_{n-1}}}\leq Cs_n.$$

We now wish to estimate $\int_{\bigcup_{j\in\mathcal{A}}Q_j^{(n)}}K(z-\xi)\frac{dm_2(\xi)}{r_{n-1}}$. With a slight abuse of notation, write $\widetilde{B}^{(n-1)}(z^*)=\widetilde{B}_j^{(n-1)}$ if $z^*\in B_j^{(n-1)}$. Then

$$\Big|\int_{\widetilde{B}^{(n-1)}(z^*)}K(z-\xi)\frac{dm_2(\xi)}{r_{n-1}}-\int_{\bigcup_{j\in\mathcal{A}}Q_j^{(n)}}K(z-\xi)\frac{dm_2(\xi)}{r_{n-1}}\Big|$$

is bounded by $\frac{C}{r_{n-1}}\big(m_2\big(\widetilde{B}^{(n-1)}(z^*)\triangle\bigcup_{j\in\mathcal{A}}Q_j^{(n)}\big)\big)^{\frac{1}{2}}$. By Lemma 4.2, this quantity is no greater than $\frac{C}{r_{n-1}}\sqrt{r_n^{1/2}r_{n-1}^{3/2}+r_nr_{n-1}}\leq C\sqrt{s_n}$.

It remains to employ the reflectionless property (Lemma 3.1). Since $z\in(1+\frac{\varepsilon}{r_{n-1}})B^{(n-1)}(z^*)$, we use Lemma 3.1 to infer that

$$\Big|\int_{\widetilde{B}^{(n-1)}(z^*)}K(z-\xi)\frac{dm_2(\xi)}{r_{n-1}}\Big|=\Big|\int_{(1+\frac{\varepsilon}{r_{n-1}})B^{(n-1)}(z^*)\setminus\widetilde{B}^{(n-1)}(z^*)}K(z-\xi)\frac{dm_2(\xi)}{r_{n-1}}\Big|.$$

This quantity is bounded by $\frac{C}{r_{n-1}}\big(m_2((1+\frac{\varepsilon}{r_{n-1}})B^{(n-1)}(z^*)\setminus\widetilde{B}^{(n-1)}(z^*))\big)^{\frac{1}{2}}\leq C\sqrt{s_n+\frac{\varepsilon}{r_{n-1}}}$. The lemma follows. □

With Lemma 7.2 in hand, we may complete the proof of Proposition 7.1. First write

$$\begin{aligned}\int_{\mathbb{C}}K(z-\xi)d\mu^{(m)}(\xi)&=\int_{B^{(q)}(z^*)}K(z-\xi)d\mu^{(m)}(\xi)\\&\quad+\sum_{n=1}^{q}\int_{B^{(n-1)}(z^*)\setminus B^{(n)}(z^*)}K(z-\xi)d\mu^{(m)}(\xi).\end{aligned}\tag{7.4}$$

Next note that that $\frac{\varepsilon}{r_{n-1}}\leq 1$ if $n=q$, and $\sqrt{\frac{\varepsilon}{r_{n-1}}}\leq s_n$ for $1\leq n<q$. As $\sum_{n\geq1}\sqrt{s_n}<\infty$, it follows from Lemma 7.2 that the sum appearing in the right hand side of (7.4) is bounded in absolute value independently of q, m and ε. The remaining term on the right hand side of (7.4) has already been shown to be bounded in absolute value, see (7.1).

8. $T_\mu(1)$ FAILS TO EXIST IN THE SENSE OF PRINCIPAL VALUE μ-ALMOST EVERYWHERE

We now turn to consider the operator in the sense of principal value. The primary part of the argument will be the following lemma.

Lemma 8.1. *Provided that n is sufficiently large, there exists a constant $c_0 > 0$ such that for any disc $B_j^{(n)}$, and $z \in \mathbb{C}$ satisfying* $\operatorname{dist}(z, \partial B_j^{(n)}) \leq c_0 r_n$,

$$\Big|\int_{A(z,r_n)} K(z-\xi)d\mu(\xi)\Big| \geq c_0.$$

Before proving the lemma, we deduce from it that $T_\mu(1)$ fails to exist in the sense of principal value for μ-almost every $z \in \mathbb{C}$. To this end, we set $F = \{z \in E : z \in (1-c_0)B^{(n)}(z)$ for all but finitely many $n\}$. It suffices to show that $\mu(F) = 0$.

First note that, with $F_n = \{z \in E : z \in (1-c_0)B^{(m)}(z)$ for all $m \geq n\}$, we have $F \subset \bigcup_{n\geq 0} F_n$, so it suffices to show that $\mu(F_n) = 0$ for all n.

To do this, note that for each $m \geq 0$, at most $(1-c_0)\frac{r_m}{r_{m+1}} + C\sqrt{\frac{r_m}{r_{m+1}}}$ squares $Q_\ell^{(m+1)}$ can intersect $(1-c_0)B_j^{(m)}$. Thus

$$\mu\Big(\bigcup_\ell \Big\{B_\ell^{(m+1)} : B_\ell^{(m+1)} \cap (1-c_0)B_j^{(m)} \neq \varnothing\Big\}\Big) \leq (1-c_0)r_m + C\sqrt{\frac{r_m}{r_{m+1}}}r_{m+1}$$

$$= (1-c_0)\mu(B_j^{(m)}) + Cs_{m+1}r_m \leq \Big(1 - \frac{c_0}{2}\Big)\mu(B_j^{(m)}),$$

where the last inequality holds provided that m is sufficiently large. But then, as long as n is large enough, this inequality may be iterated to yield

$$\mu\big(\big\{z \in E : z \in (1-c_0)B^{(n+k)}(z) \text{ for } k = 1, \dots, m\big\}\big) \leq (1 - \tfrac{c_0}{2})^m.$$

Hence $\mu(F_n) = 0$.

In preparation for proving Lemma 8.1, we make the following claim.

Claim 8.2. Let $n \in \mathbb{Z}_+$. For any disc $B_j^{(n)}$, and $z \in \mathbb{C}$, we have

$$\Big|\int_{A(z,r_n)\cap B_j^{(n)}} K(z-\xi)d\big(\mu - \frac{m_2}{r_n}\big)(\xi)\Big| \leq Cs_{n+1}.$$

Proof. To derive this claim, first suppose that a square $Q_\ell^{(n+1)} \subset A(z, r_n)$. Then from a crude application of Lemma 7.3 (see (7.2)), we infer that

$$\Big|\int_{Q_\ell^{(n+1)}} K(z-\xi)d(\mu - \frac{m_2}{r_n})(\xi)\Big| \leq C\frac{\sqrt{r_n r_{n+1}}}{r_n^2}r_{n+1} \leq C\Big(\frac{r_{n+1}}{r_n}\Big)^{\frac{3}{2}}.$$

If it instead holds that $Q_\ell^{(n+1)} \cap \partial A(z, r_n) \neq \varnothing$, then we have the blunt estimate

$$\Big|\int_{Q_\ell^{(n+1)}\cap A(z,r_n)} K(z-\xi)d(\mu - \frac{m_2}{r_n})(\xi)\Big| \leq \frac{2}{r_n}\Big[\mu(Q_\ell^{(n+1)}) + \frac{m_2(Q_\ell^{(n+1)})}{r_n}\Big],$$

which is bounded by $\frac{Cr_{n+1}}{r_n}$. There are most $\frac{r_n}{r_{n+1}}$ squares $Q_\ell^{(n+1)}$ contained in $A(z, r_n)$, and no more than $C\sqrt{\frac{r_n}{r_{n+1}}}$ squares $Q_\ell^{(n+1)}$ can intersect the boundary of $A(z, r_n)$.

On the other hand, the set $\widetilde{A}$ consisting of the points in $A(z, r_n) \cap B_j^{(n)}$ not covered by any square $Q_\ell^{(n+1)}$ has m_2-measure no greater than $Cr_{n+1}^{1/2} r_n^{3/2}$ (see Lemma 4.2). Thus $\int_{\widetilde{A}} |K(z-\xi)| \frac{dm_2(\xi)}{r_n} \leq \frac{2m_2(\widetilde{A})}{r_n^2} \leq Cs_{n+1}$.

Bringing these estimates together establishes Claim 8.2. □

Let us now complete the proof of Lemma 8.1

Proof of Lemma 8.1. Note that $\int_{A(z,r_n)\cap B_j^{(n)}} K(z-\xi)\frac{dm_2(\xi)}{r_n}$ is a Lipschitz continuous function in $\mathbb{C}$, with Lipschitz norm at most $\frac{C}{r_n}$. Thus, we infer from Lemma 3.2 that there is a constant $c_0 > 0$ such that

$$\Big|\int_{A(z,r_n)\cap B_j^{(n)}} K(z-\xi)\frac{dm_2(\xi)}{r_n}\Big| \geq \frac{\tilde{c}}{2},$$

whenever $\mathrm{dist}(z, \partial B_j^{(n)}) \leq c_0 r_n$. But now we apply Claim 8.2 to deduce that for all such z, $\big|\int_{A(z,r_n)} K(z-\xi)d\mu(\xi)\big| \geq \frac{\tilde{c}}{2} - Cs_{n+1}$ (the only part of the support of μ that $A(z, r_n)$ intersects is contained in $B_j^{(n)}$). The right hand side here is at least $\frac{\tilde{c}}{4}$ for all sufficiently large n. □

9. The set E is purely unrectifiable

We now show that E is purely unrectifiable, that is, $\mathcal{H}^1(E \cap \Gamma) = 0$ for any rectifiable curve Γ. The proof that follows is a simple special case of the well known fact that any set with zero lower $\mathcal{H}^1$-density is unrectifiable (one can in fact say much more, see for instance [Mat1]).

First notice that for each $z \in \mathbb{C}$ and $n \geq 1$, $B(z, \frac{1}{4}\sqrt{r_n r_{n-1}})$ can intersect at most one of the discs $B_j^{(n)}$. Hence

$$\mathcal{H}^1(E \cap B(z, \tfrac{1}{4}\sqrt{r_n r_{n-1}})) \leq 2r_n.$$

A rectifiable curve Γ can be covered by discs $B(z_j, \frac{1}{4}\sqrt{r_n r_{n-1}})$, $j = 1, \ldots, N$, the sum of whose radii is at most $\ell(\Gamma)$.

Thus $\mathcal{H}^1(E\cap\Gamma)\le\sum_{j=1}^N\mathcal{H}^1(E\cap B(z_j,\frac14\sqrt{r_nr_{n-1}}))\le 2\sum_{j=1}^N r_n$. But $\sum_{j=1}^N\frac14\sqrt{r_nr_{n-1}}\le\ell(\Gamma)$, and so $\mathcal{H}^1(\Gamma\cap E)\le 8\sqrt{\frac{r_n}{r_{n-1}}}\ell(\Gamma)$, which tends to zero as $n\to\infty$ (the sequence $\sqrt{s_n}$ is summable).

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