THREE REVOLUTIONS IN THE KERNEL ARE WORSE THAN ONE

BENJAMIN JAYE AND FEDOR NAZAROV

Abstract. An example is constructed of a purely unrectifiable measure \( \mu \) for which the singular integral associated to the kernel \( K(z) = \frac{1}{z} \) is bounded in \( L^2(\mu) \). The singular integral fails to exist in the sense of principal value \( \mu \)-almost everywhere. This is in sharp contrast with the results known for the kernel \( \frac{1}{z} \) (the Cauchy transform).

1. Introduction

Let \( B(z, r) \) denote the closed disc in \( \mathbb{C} \) centred at \( z \) with radius \( r > 0 \). A finite Borel measure \( \mu \) is said to be 1-dimensional if \( \mathcal{H}^1(\text{supp}(\mu)) < \infty \), and there exists a constant \( C > 0 \) such that \( \mu(B(z, r)) \leq Cr \) for any \( z \in \mathbb{C} \) and \( r > 0 \).

For a kernel function \( K : \mathbb{C} \setminus \{0\} \to \mathbb{C} \), and a finite measure \( \mu \), we define the singular integral operator associated to \( K \) by

\[
T_\mu(f)(z) = \int_\mathbb{C} K(z - \xi)f(\xi)d\mu(\xi), \quad \text{for } z \notin \text{supp}(\mu).
\]

A well-known problem in harmonic analysis is to determine geometric properties of \( \mu \) from regularity properties of the operator \( T_\mu \), see for instance the monograph of David and Semmes [DS]. This paper concerns the question of characterizing those functions \( K \) with the following property:

\[\tag{*} \|T_\mu(1)\|_{L^\infty(\mathbb{C} \setminus \text{supp}(\mu))} < \infty \text{ implies that } \mu \text{ is rectifiable}.\]

The property that \( \|T_\mu(1)\|_{L^\infty(\mathbb{C} \setminus \text{supp}(\mu))} < \infty \) is equivalent to the boundedness of \( T_\mu \) as an operator in \( L^2(\mu) \), see for instance [NTV]. A measure \( \mu \) is rectifiable if \( \text{supp}(\mu) \) can be covered (up to an exceptional set of \( \mathcal{H}^1 \) measure zero) by a countable union of rectifiable curves. A measure \( \mu \) is purely unrectifiable if its support is purely unrectifiable, that is, \( \mathcal{H}^1(\Gamma \cap \text{supp}(\mu)) = 0 \) for any rectifiable curve \( \Gamma \).

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David and Léger [Leg] proved that the Cauchy kernel $\frac{1}{z}$ has property $(\ast)$. As is remarked in [CMPT], the proof in [Leg] extends to the case when the Cauchy kernel is replaced by either its real or imaginary part, i.e. $\frac{\Re(z)}{|z|^2}$ or $\frac{\Im(z)}{|z|^2}$. Recently in [CMPT], Chousionis, Mateu, Prat, and Tolsa extended the result of [Leg] and showed that kernels of the form $(\Re(z))^k \frac{1}{|z|^2}$ have property $(\ast)$ for any odd positive integer $k$. Both of these results use the Melnikov-Menger curvature method.

On the other hand, Huovinen [Huo2] has shown that there is a purely unrectifiable Ahlfors-David (AD)-regular set $E$ for which the singular integral associated to the kernel $\frac{\Re(z)}{|z|^2} - \frac{\Re(z)^3}{|z|^4}$ is bounded in $L^2(\mathcal{H}^1_E)$. In fact, an essentially stronger conclusion is proved that the principal values of the associated singular integral operator exist $\mathcal{H}^1$-a.e. on $E$. Huovinen takes advantage of several non-standard symmetries and cancellation properties in this kernel to construct his very nice example.

The result of this paper is that a weakened version of Huovinen’s theorem holds for a very simple kernel function. Indeed, it is perhaps the simplest example of a kernel for which the Menger curvature method fails to be directly applicable. From now on, we shall fix

$$K(z) = \frac{z}{z^2}, \quad z \in \mathbb{C} \setminus \{0\}.$$  

(1.1)

We prove the following result.

**Theorem 1.1.** There exists a 1-dimensional purely unrectifiable probability measure $\mu$ with the property that $\|T_\mu(1)\|_{L^\infty(\mathbb{C} \setminus \text{supp}(\mu))} < \infty$.

In other words, the kernel $K$ in (1.1) fails to satisfy property $(\ast)$. At this point, we would also like to mention Huovinen’s thesis work [Huo1], regarding the kernel function $K(z)$ from (1.1). It is proved that if $\liminf_{r \to 0} \frac{\mu(B(z,r))}{r} \in (0, \infty)$ $\mu$-a.e. (essentially the AD-regularity of $\mu$), then the $\mu$-almost everywhere existence of $T_\mu(1)$ in the sense of principal value implies that $\mu$ is rectifiable. This result was proved by building upon the theory of symmetric measures, developed by Mattila [Mat2], and Mattila and Preiss [MP]. Unfortunately the measure in Theorem 1.1 does not satisfy the AD-regularity condition. In view of Huovinen’s work it would be of interest to construct an AD-regular measure supported on an unrectifiable set for which the conclusion of Theorem 1.1 holds. We have not been able to construct such a measure (yet).

For the measure $\mu$ constructed in Theorem 1.1, we show that $T_\mu(1)$ fails to exist in the sense of principal value $\mu$-almost everywhere. Thus the two properties of $L^2(\mu)$ boundedness of the operator $T_\mu$, and the
existence of $T_\mu(1)$ in the sense of principal value, are quite distinct for
this singular integral operator.

2. Notation

- Let $m_2$ denote the 2-dimensional Lebesgue measure normalized
  so that $m_2(B(0,1)) = 1$. We let $m_1$ denote the 1-dimensional
  Lebesgue measure.
- A collection of squares are essentially pairwise disjoint if the
  interiors of any two squares in the collection do not intersect.
  Throughout the paper, all squares are closed.
- We shall denote by $C$ and $c$ large and small absolute positive
  constants. The constant $C$ should be thought of as large (at
  least 1), while $c$ is to be thought of as small (smaller than 1).
- For $a > 1$, the disc $aB$ denotes the concentric enlargement of a
  disc $B$ by a factor of $a$.
- We define the $H^1$-measure of a set $E$ by
  \[ H^1(E) = \sup_{\delta > 0} \inf \left\{ \sum_j r_j : E \subset \bigcup_j B(x_j, r_j) \text{ with } r_j \leq \delta \right\}. \]
- For $z \in \mathbb{C}$ and $r > 0$, we define the annulus $A(z, r) = B(z, r) \setminus B(z, \frac{r}{2})$.
- The set $\text{supp}(\mu)$ denotes the closed support of $\mu$.

3. A REFLECTIONLESS MEASURE

Let us make the key observation that allows us to prove Theorem 1.1

Lemma 3.1. Let $z \in \mathbb{C}$, $r > 0$. For any $\omega \in B(z, r)$,

\[ \int_{B(z,r)} K(\omega - \xi) \, dm_2(\xi) = 0. \]

Proof. Without loss of generality, we may set $z = 0$ and $r = 1$. If
$|\omega| < |\xi|$, then

\[ K(\omega - \xi) = \frac{\omega - \xi}{\xi^2} \sum_{\ell=0}^{\infty} (\ell + 1) \left( \frac{\omega}{\xi} \right)^\ell. \]

So whenever $t > |\omega|$, we have $\int_{\partial B(0,t)} K(\omega - \xi) \, dm_1(\xi) = 0$. (This follows merely from the fact that $\int_{\partial B(0,t)} \xi^k \xi^k \, dm_1(\xi) = 0$ whenever $k, \ell \in \mathbb{Z}$ satisfy $k \neq \ell$.) On the other hand, if $|\xi| < |\omega|$, then

\[ K(\omega - \xi) = \frac{\omega - \xi}{\omega^2} \sum_{\ell=0}^{\infty} (\ell + 1) \left( \frac{\xi}{\omega} \right)^\ell. \]
Therefore, if \( t < |\omega| \), then

\[
\int_{\partial B(0,t)} K(\omega - \xi)dm_1(\xi) = 2\pi \left[ t\frac{\omega}{\omega^2} - \frac{t^3}{\omega^3} \right] = \frac{2\pi}{\omega^3} (t|\omega|^2 - 2t^3).
\]

Since \( \int_0^{|\omega|} (t|\omega|^2 - 2t^3)dt = 0 \), the desired conclusion follows. \( \square \)

The next lemma will form the basis of the proof of the non-existence of \( T_\mu (1) \) in the sense of principal value.

**Lemma 3.2.** There exists a constant \( \tilde{c} > 0 \) such that for any disc \( B(z, r) \), and \( \omega \in \partial B(z, r) \),

\[
\left| \int_{A(\omega,r) \cap B(z,r)} K(\omega - \xi) \frac{dm_2(\xi)}{r} \right| \geq \tilde{c}.
\]

**Proof.** By an appropriate translation and rescaling, we may assume that \( B(z, r) = B(i, 1) \), and \( \omega = 0 \). Making reference to Figure 1 above, we split the domain of integration into three regions, \( I = \{ \xi \in A(0,1) : \arg(\xi) \in \left[ \frac{\pi}{6}, \frac{5\pi}{6} \right] \} \), \( II = \{ \xi \in A(0,1) \cap B(i,1) : \arg(\xi) \in \left[ 0, \frac{\pi}{6} \right] \} \) and \( III = \{ \xi \in A(0,1) \cap B(i,1) : \arg(\xi) \in \left[ \frac{5\pi}{6}, \pi \right] \} \). The regions \( II \) and \( III \) are respectively the right and left grey shaded regions in Figure 1. Note that \( \Im K(-\xi) < 0 \) if \( \arg(\xi) \in \left[ \frac{\pi}{3}, \frac{2\pi}{3} \right] \), and \( \Im K(-\xi) > 0 \) if \( \arg(\xi) \in \left[ 0, \frac{\pi}{3} \right] \cup \left[ \frac{2\pi}{3}, \pi \right] \). Furthermore, note that

\[
\int_I \Im K(-\xi)dm_2(\xi) = \frac{1}{\pi} \int_{1/2}^1 \frac{1}{t} \int_{\pi/6}^{5\pi/6} -\Im(e^{-3\theta i})td\theta dt = 0.
\]

But \( \int_{II \cup III} \Im K(-\xi)dm_2(\xi) = 2 \int_{II} \Im K(-\xi)dm_2(\xi) > 0 \). Therefore, by setting \( \tilde{c} = 2 \int_{II} \Im K(-\xi)dm_2(\xi) \), the lemma follows. \( \square \)
4. Packing squares in a disc

Fix \( r, R \in (0, \infty) \) such that \( r < \frac{R}{10} \) and \( \frac{R}{r} \in \mathbb{N} \).

**Lemma 4.1.** One can pack \( \frac{R}{r} \) pairwise essentially disjoint squares of side length \( \sqrt{\pi r R} \) into a disc of radius \( R(1 + 4\sqrt{\frac{r}{R}}) \).

**Proof.** We may assume that the disc is centred at the origin. Consider the square lattice with mesh size \( \sqrt{\pi r R} \). Label those squares that intersect \( B(0, R) \) as \( Q_1, \ldots, Q_M \). These squares are contained in \( B(0, R(1 + 4\sqrt{\frac{r}{R}})) \). Since \( M \pi r = \sum_{j=1}^{M} m_2(Q_j) > m_2(B(0, R)) = R^2 \), we have that \( M > \frac{R}{r} \). By throwing away \( M - \frac{R}{r} \) of the least desirable squares, we arrive at the desired collection. \( \square \)

**Lemma 4.2.** Consider a disc \( B(z, R) \). Let \( Q_1, \ldots, Q_{R/r} \) be the collection of squares contained in \( B(z, R(1 + 4\sqrt{\frac{r}{R}})) \) found in Lemma 4.1. Then \( m_2(B(z, R) \triangle \bigcup_{j=1}^{R/r} Q_j) \leq C r^{1/2} R^{3/2} \).

**Proof.** Since \( m_2(B(z, R)) = m_2(\bigcup_{j=1}^{R/r} Q_j) = R^2 \), the property follows from the fact that both sets are contained in \( B(z, R(1 + 4\sqrt{\frac{r}{R}})) \). \( \square \)

5. The construction of the sparse Cantor set \( E \)

Let \( r_0 = 1 \), and choose \( r_j, j \in \mathbb{N} \), to be a sequence which tends to zero quickly. Assume that \( r_j < \frac{r_{j+1}}{100}, \frac{1}{r_j} \in \mathbb{N} \), and \( \frac{r_j}{r_{j+1}} \in \mathbb{N} \) for all \( j \geq 1 \).

Several additional requirements will be imposed on the decay of \( r_j \) over the course of the following analysis, and we make no attempt to optimize the conditions.

It will be convenient to let \( s_{n+1} = 4\sqrt{\frac{r_{n+1}}{r_n}} \) for \( n \in \mathbb{Z}_+ \).

First define \( \tilde{B}_1^{(0)} = B(0, 1) \). Given the \( n \)-th level collection of \( \frac{1}{r_n} \) discs \( \tilde{B}_j^{(n)} \) of radius \( r_n \), we construct the \( (n + 1) \)-st generation according to the following procedure:

- Fix a disc \( \tilde{B}_j^{(n)} \). Apply Lemma 4.1 with \( R = r_n \) and \( r = r_{n+1} \) to find \( \frac{r_n}{r_{n+1}} \) squares \( Q_{\ell}^{(n+1)} \) of side length \( \sqrt{\pi r_{n+1} r_n} \) that are pairwise essentially disjoint, and contained in \( (1 + s_{n+1}) \cdot \tilde{B}_j^{(n)} \). Let \( \tilde{z}_{\ell}^{(n+1)} \) be the centre of \( Q_{\ell}^{(n+1)} \), and set \( \tilde{B}_{\ell}^{(n+1)} = B(\tilde{z}_{\ell}^{(n+1)}, r_{n+1}) \). This procedure is carried out for each disc \( \tilde{B}_j^{(n)} \) from the \( n \)-th level collection. There are a total of \( \frac{1}{r_{n+1}} \) discs \( \tilde{B}_{\ell}^{(n+1)} \) in the \( (n + 1) \)-st level.

The above construction is executed for each \( n \in \mathbb{Z}_+ \).

Now, set \( B_j^{(n)} = (1 + s_{n+1}) \tilde{B}_j^{(n)} \). Define \( E^{(n)} = \bigcup_j B_j^{(n)} \). We shall repeatedly use the following properties of the construction:
Figure 2. The picture shows a single disc $B_j^{(n)}$ of radius $(1 + s_{n+1})r_n$. The grey shaded squares are the squares $Q_{\ell}^{(n+1)}$ of sidelength $\sqrt{\pi r_n r_{n+1}}$ formed by applying Lemma 4.1 to the disc $\tilde{B}_j^{(n)}$ of radius $r_n$. The boundary of the disc $\tilde{B}_j^{(n)}$ is the dashed circle. Deep inside each square $Q_{\ell}^{(n+1)}$ is the disc $B_{\ell}^{(n+1)}$ of radius $(1 + s_{n+2})r_{n+1}$.

(a) $\bigcup_{\ell} Q_{\ell}^{(n+1)} \subset E^{(n)}$, for all $n \geq 0$.
(b) $B_j^{(n)} \subset Q_j^{(n)}$ for each $n \geq 1$. Moreover, $\text{dist}(B_j^{(n)}, \partial Q_j^{(n)}) \geq \frac{1}{2} \sqrt{r_n - 1} r_n$.
(c) $\text{dist}(B_j^{(n)}, B_k^{(n)}) \geq \frac{1}{2} \sqrt{r_n - 1} r_n$ whenever $j \neq k$, $n \geq 0$.

Property (a) is immediate. To see property (b), merely note that $\text{dist}(B_j^{(n)}, \partial Q_j^{(n)}) = \frac{\sqrt{r_n - 1} r_n}{2} - (1 + s_{n+1})r_n \geq \frac{1}{2} \sqrt{r_n - 1} r_n$. For property (c), we shall use induction. If $n = 0$, then the claim is trivial. Using (b), the claimed estimate is clear if $Q_j^{(n)}$ and $Q_k^{(n)}$ have been created by an application of Lemma 4.1 in a common disc $\tilde{B}_{\ell}^{(n-1)}$. Otherwise, the squares are born out of applying Lemma 4.1 to different discs at the $(n - 1)$-st level, and those parent discs are already separated by $\frac{1}{2} \sqrt{r_{n-1} - 1} r_{n-1}$. 
Courts of properties (a) and (b), we see that \( E^{(n+1)} \) \( \subseteq \) \( E^{(n)} \) for each \( n \geq 0 \). Set \( E = \bigcap_{n \geq 0} E^{(n)} \). Each \( z \in E^{(n)} \) is contained in a unique disc \( B^{(n)}_z \) (or square \( Q^{(n)}_z \)) which we shall denote by \( B^{(n)}(z) \) (respectively \( Q^{(n)}(z) \)).

If \( m \geq n \geq 0 \), then \( E \cap B^{(n)}_z \) is covered by the \( \frac{\ell}{r_m} \) discs \( B^{(m)}_z \) that are contained in \( B^{(n)}_z \), each of which has radius \( (1 + s_{m+1})r_m \leq 2r_m \). Therefore \( \mathcal{H}^1(E \cap B^{(n)}_z) \leq 2r_n \). Taking \( n = 0 \) yields \( \mathcal{H}^1(E) \leq 2 \).

6. The measure \( \mu \)

Define \( \mu^{(n)}_j = \frac{1}{r_n} \chi_{B^{(n)}_j} m_2 \). Set \( \mu^{(n)} = \sum_j \mu^{(n)}_j \). Then \( \operatorname{supp}(\mu^{(n)}) \subseteq E^{(n)} \), and \( \mu^{(n)}(C) = 1 \) for all \( n \). Therefore, there exists a subsequence of the sequence of measures \( \mu^{(n)} \) that converges weakly to a measure \( \mu \), with \( \mu(C) = 1 \) and \( \operatorname{supp}(\mu) \subseteq E \).

The following three properties hold:

(i) \( \operatorname{supp}(\mu^{(m)}) \subseteq \bigcup_j B^{(n)}_j \) whenever \( m \geq n \),

(ii) \( \mu^{(m)}(B^{(n)}_j) = r_n \) for \( m \geq n \), and

(iii) there exists \( C_0 > 0 \) such that \( \mu^{(n)}(B(z, r)) \leq C_0 r \) for any \( z \in C \), \( r > 0 \) and \( n \geq 0 \).

Properties (i) and (ii) follow immediately from the construction of \( E^{(n)} \). To see the third property, note that since \( \mu^{(n)} \) is a probability measure, the property is clear if \( r \geq 1 \). If \( r < 1 \), then \( r \in (r_{m+1}, r_m) \) for some \( m \in \mathbb{Z}_+ \). If \( m \geq n \), then \( B(z, r) \) intersects at most one disc \( B^{(n)}_j \). Then \( \mu^{(n)}(B(z, r)) = \frac{1}{r_n} m_2(B(z, r) \cap \tilde{B}^{(n)}_j) \leq \frac{r^2}{r_n} \leq r \). Otherwise \( m < n \). In this case, note that since the discs \( B^{(m+1)}_j \) are \( 1 - \frac{r}{\sqrt{r_m r_{m+1}}} \) separated, \( B(z, r) \) intersects at most \( 1 + C(1 \sqrt{r_m r_{m+1}}) \) discs \( B^{(m+1)}_j \).

Hence, by property (ii), we see that

\[
\mu^{(n)}(B(z, r)) = \sum_j \mu^{(n)}(B(z, r) \cap B^{(m+1)}_j) \leq \left[ 1 + C \left( \frac{r}{\sqrt{r_m r_{m+1}}} \right)^2 \right] r_m,
\]

which is at most \( Cr \).

The weak convergence of a subsequence of \( \mu^{(n)} \) to the measure \( \mu \), along with property (iii), yields that \( \mu(B(z, r)) \leq C_0 r \) for any disc \( B(z, r) \). We shall henceforth refer to this property by saying that \( \mu \) is \( C_0 \)-nice. We have now shown that \( \mu \) is 1-dimensional.

Notice that we also have \( \mathcal{H}^1(E) \geq \frac{1}{C_0} \mu(E) > 0 \).
7. The boundedness of $T_\mu(1)$ off the support of $\mu$

As a simple consequence of the weak convergence of $\mu^{(n)}$ to $\mu$, the property that $\|T_\mu(1)\|_{L^\infty(\mathbb{C}\setminus\text{supp}(\mu))} < \infty$ will follow from the following proposition.

**Proposition 7.1.** Provided that $\sum_{n \geq 1} \sqrt{s_n} < \infty$, there exists a constant $C > 0$ so that the following holds:

Suppose that $\text{dist}(z, \text{supp}(\mu)) = \varepsilon > 0$. Then for any $m \in \mathbb{Z}_+$ with $r_m < \frac{\varepsilon}{4}$,

$$\left| \int_{\mathbb{C}} K(z - \xi) d\mu^{(m)}(\xi) \right| \leq C.$$

To begin the proof, fix $r_m$ with $r_m < \frac{\varepsilon}{4}$. Let $z^* \in \text{supp}(\mu)$ with $\text{dist}(z, z^*) = \varepsilon$. For any $\xi \in \text{supp}(\mu)$, $B^{(m)}(\xi) \cap \text{supp}(\mu^{(m)}) \neq \emptyset$, so $\text{dist}(z, \text{supp}(\mu^{(m)})) \geq \varepsilon - (1 + s_{m+1})r_m \geq \frac{\varepsilon}{2}$.

Now, let $q$ be the least integer with $r_q \leq \varepsilon$ (so $m \geq q$). Then by property (ii) of the previous section,

$$(7.1) \quad \int_{B^{(q)}(z^*)} |K(z, \xi)| d\mu^{(m)}(\xi) \leq \frac{2}{\varepsilon} \mu^{(m)}(B^{(q)}(z^*)) = \frac{2r_q}{\varepsilon} \leq 2.$$

The crux of the matter is the following lemma.

**Lemma 7.2.** There exists $C > 0$ such that for any $n \in \mathbb{Z}_+$ with $1 \leq n \leq q$,

$$\left| \int_{B^{(n-1)}(z^*) \setminus B^{(n)}(z^*)} K(z - \xi) d\mu^{(m)}(\xi) \right| \leq C \sqrt{s_n} + C \sqrt{\frac{\varepsilon}{r_{n-1}}}.$$

For the proof of Lemma 7.2 we shall require the following simple comparison estimate.

**Lemma 7.3.** Let $z_0 \in \mathbb{C}$, and $\lambda > 0$. Fix $r, R \in (0,1]$ with $100r \leq R$. Suppose that $\nu_1$ and $\nu_2$ are Borel measures, such that $\text{supp}(\nu_1) \subset Q(z_0, \sqrt{\pi Rr}) = Q$, $\text{supp}(\nu_2) \subset B(z_0, 2r) = B$, and $\nu_1(\mathbb{C}) = \nu_2(\mathbb{C})$. Then, for any $z \in \mathbb{C}$ with $\text{dist}(z, Q) \geq \lambda \sqrt{rR}$, we have

$$\left| \int_{\mathbb{C}} K(z - \xi) d\nu_1(\xi) - \int_{B} K(z - \xi) d\nu_2(\xi) \right| \leq \frac{1}{\lambda^2} \int_{\mathbb{C}} \frac{C \sqrt{Rr}}{|z - \xi|^2} d\nu_1(\xi) + \frac{1}{\lambda^2} \int_{B} \frac{Cr}{|z - \xi|^2} d\nu_2(\xi).$$

**Proof.** Note that the left hand side of the inequality can be written as

$$\left| \int_{\mathbb{C}} [K(z - \xi) - K(z - z_0)] d(\nu_1 - \nu_2)(\xi) \right|.$$
But, under the hypothesis on \( z \), we have that 
\[
|K(z - \xi) - K(z - z_0)| \leq C|z - z_0| \sqrt[n]{|z - \xi|^2},
\]
for any \( \xi \in Q \). Plugging this estimate into the integral and taking into account the supports of \( \nu_1 \) and \( \nu_2 \), the inequality follows. \( \square \)

**Proof of Lemma 7.2.** Write

\[
\mathcal{A} = \{ j : B_j^{(n)} \neq B^{(n)}(z^*) \text{ and } B_j^{(n)} \subset B^{(n-1)}(z^*) \}.
\]

First suppose that \( \text{dist}(z, Q_j^{(n)}) \geq \frac{1}{4} \sqrt[n]{r_{n-1}r_n} \) for \( j \in \mathcal{A} \). Then the hypothesis of Lemma 7.3 are satisfied with \( \nu_1 = \chi_{Q_j^{(n)}} \frac{m_2}{r_{n-1}}, \nu_2 = \chi_{B_j^{(n)}} \mu^{(m)}, \) \( R = r_{n-1}, r = r_n \), and \( z_0 = z_{Q_j^{(n)}} \). Thus

\[
\left| \int_{Q_j^{(n)}} K(z - \xi) \frac{dm_2(\xi)}{r_{n-1}} - \int_{B_j^{(n)}} K(z - \xi) d\mu^{(m)}(\xi) \right| 
\leq \frac{C}{r_{n-1}} \sqrt[n]{r_{n-1}r_n} \frac{dm_2(\xi)}{r_{n-1}} + \int_{B_j^{(n)}} \frac{Cr_n d\mu^{(m)}(\xi)}{|z - \xi|^2}.
\]

Now suppose that \( j \in \mathcal{A} \) and \( \text{dist}(z, Q_j^{(n)}) \leq \frac{1}{4} \sqrt[n]{r_{n-1}r_n} \). Since \( \text{dist}(z, Q_j^{(n)}) \geq \text{dist}(z^*, Q_j^{(n)}) - \text{dist}(z, z^*) \geq \frac{1}{2} \sqrt[n]{r_{n-1}r_n} - \epsilon \), we must have that \( \epsilon \geq \frac{1}{2} \sqrt[n]{r_{n-1}r_n} \). But as \( \text{dist}(z, \text{supp}(\mu^{(m)})) \geq \frac{\epsilon}{2} \), and \( \mu^{(m)}(B_j^{(n)}) = r_n \), we have the following crude bound

\[
\left| \int_{Q_j^{(n)}} K(z - \xi) \frac{dm_2(\xi)}{r_{n-1}} - \int_{B_j^{(n)}} K(z - \xi) d\mu^{(m)}(\xi) \right| 
\leq \frac{C}{r_{n-1}} \sqrt[n]{m_2(Q_j^{(n)})} + \frac{2}{\epsilon} \mu^{(m)}(B_j^{(n)}) \leq C_\nu n.
\]

(Here it is used that \( \int_{A} |K(\xi)| dm_2(\xi) \leq C \sqrt[n]{m_2(A)} \) for any Borel measurable set \( A \subset \mathbb{C} \) of finite \( m_2 \)-measure.)

At most 4 of the essentially pairwise disjoint squares \( Q_j^{(n)}, j \in \mathcal{A} \), can satisfy \( \text{dist}(z, Q_j^{(m)}) \leq \frac{1}{4} \sqrt[n]{r_{n-1}r_n} \) (and it can only happen at all if \( n = q \)). Therefore by summing (7.2) and (7.3) over \( j \in \mathcal{A} \) in the cases when \( \text{dist}(z, Q_j^{(n)}) \geq \frac{1}{4} \sqrt[n]{r_{n-1}r_n} \) and \( \text{dist}(z, Q_j^{(n)}) \leq \frac{1}{4} \sqrt[n]{r_{n-1}r_n} \) respectively, we see that the quantity

\[
\left| \int_{\bigcup_{j \in \mathcal{A}} Q_j^{(n)}} K(z - \xi) \frac{dm_2(\xi)}{r_{n-1}} - \int_{B^{(n-1)}(z^*) \setminus B^{(n)}(z^*)} K(z - \xi) d\mu^{(m)}(\xi) \right|
\]

is no greater than a constant multiple of

\[
\int_{B(z, 2r_{n-1}) \setminus B(z, \frac{1}{4} \sqrt[n]{r_{n-1}r_n})} \sqrt[n]{r_{n-1}r_n} \frac{dm_2(\xi)}{r_{n-1} |z - \xi|^2} + \int_{C \setminus B(z, \frac{1}{4} \sqrt[n]{r_{n-1}r_n})} r_n d\mu^{(m)}(\xi) + s_n.
\]
The first term here is bounded by $C_\sqrt{r_n \log \left( \frac{r_n-1}{r_n} \right)} \leq Cs_n \log \left( \frac{1}{s_n} \right) \leq C\sqrt{s_n}$. Since $\mu(m)$ is $C_0$-nice, we bound the second term by

$$Cr_n \int_{\frac{1}{4} \sqrt{r_n r_{n-1}}}^{\infty} \frac{dr}{r^2} \leq Cr_n \frac{1}{\sqrt{r_n r_{n-1}}} \leq Cs_n.$$  

We now wish to estimate $\int_{\cup_{j \in A} Q_j^{(n)}} K(z - \xi) \frac{dm_2(\xi)}{r_{n-1}}$. With a slight abuse of notation, write $\tilde{B}^{(n-1)}(z^*) = \tilde{B}_j^{(n-1)}$ if $z^* \in \tilde{B}_j^{(n-1)}$. Then

$$\left|\int_{\tilde{B}^{(n-1)}(z^*)} K(z - \xi) \frac{dm_2(\xi)}{r_{n-1}} - \int_{\cup_{j \in A} Q_j^{(n)}} K(z - \xi) \frac{dm_2(\xi)}{r_{n-1}}\right|$$

is bounded by $\frac{C}{r_{n-1}} (m_2(\tilde{B}^{(n-1)}(z^*) \triangle \cup_{j \in A} Q_j^{(n)})) \frac{1}{2}$. By Lemma 4.2, this quantity is no greater than $\frac{C}{r_{n-1}} \sqrt{r_n^{1/2} r_{n-1}^{3/2} + r_n r_{n-1}} \leq C\sqrt{s_n}$.

It remains to employ the reflectionless property (Lemma 3.1). Since $z \in (1 + \frac{\varepsilon}{r_{n-1}}) B^{(n-1)}(z^*)$, we use Lemma 3.1 to infer that

$$\left|\int_{\tilde{B}^{(n-1)}(z^*)} K(z - \xi) \frac{dm_2(\xi)}{r_{n-1}} \right| = \left|\int_{(1 + \frac{\varepsilon}{r_{n-1}}) B^{(n-1)}(z^*) \setminus \tilde{B}^{(n-1)}(z^*)} K(z - \xi) \frac{dm_2(\xi)}{r_{n-1}} \right|.$$  

This quantity is bounded by $\frac{C}{r_{n-1}} (m_2((1 + \frac{\varepsilon}{r_{n-1}}) B^{(n-1)}(z^*) \setminus \tilde{B}^{(n-1)}(z^*))) \frac{1}{2} \leq C\sqrt{s_n + \frac{s_n}{r_{n-1}}}$. The lemma follows. \hfill \Box

With Lemma 7.2 in hand, we may complete the proof of Proposition 7.1. First write

$$\int_{\mathbb{C}} K(z - \xi) d\mu^{(m)}(\xi) = \int_{B^{(n)}(z^*)} K(z - \xi) d\mu^{(m)}(\xi)$$

$$+ \sum_{n=1}^{q} \int_{B^{(n-1)}(z^*) \setminus B^{(n)}(z^*)} K(z - \xi) d\mu^{(m)}(\xi). \tag{7.4}$$

Next note that $\frac{\varepsilon}{r_{n-1}} \leq 1$ if $n = q$, and $\sqrt{\frac{\varepsilon}{r_{n-1}}} \leq s_n$ for $1 \leq n < q$. As $\sum_{n \geq 1} \sqrt{s_n} < \infty$, it follows from Lemma 7.2 that the sum appearing in the right hand side of (7.4) is bounded in absolute value independently of $q$, $m$ and $\varepsilon$. The remaining term on the right hand side of (7.4) has already been shown to be bounded in absolute value, see (7.1).
8. \( T_\mu(1) \) fails to exist in the sense of principal value \( \mu \)-almost everywhere

We now turn to consider the operator in the sense of principal value. The primary part of the argument will be the following lemma.

**Lemma 8.1.** Provided that \( n \) is sufficiently large, there exists a constant \( c_0 > 0 \) such that for any disc \( B_j^{(n)} \), and \( z \in \mathbb{C} \) satisfying \( \text{dist}(z, \partial B_j^{(n)}) \leq c_0 r_n \),

\[
\left| \int_{A(z, r_n)} K(z - \xi) d\mu(\xi) \right| \geq c_0.
\]

Before proving the lemma, we deduce from it that \( T_\mu(1) \) fails to exist in the sense of principal value for \( \mu \)-almost every \( z \in \mathbb{C} \). To this end, we set \( F = \{ z \in E : z \in (1 - c_0)B^{(n)}(z) \text{ for all but finitely many } n \} \). It suffices to show that \( \mu(F) = 0 \).

First note that, with \( F_n = \{ z \in E : z \in (1 - c_0)B^{(m)}(z) \text{ for all } m \geq n \} \), we have \( F \subset \bigcup_{n \geq 0} F_n \), so it suffices to show that \( \mu(F_n) = 0 \) for all \( n \).

To do this, note that for each \( m \geq 0 \), at most \( (1 - c_0) r_m + C \sqrt{r_m r_{m+1}} \) squares \( Q_{\ell}^{(m+1)} \) can intersect \( (1 - c_0)B_j^{(m)} \). Thus

\[
\mu\left( \bigcup_{\ell} \left\{ B_{\ell}^{(m+1)} : B_{\ell}^{(m+1)} \cap (1 - c_0)B_j^{(m)} \neq \emptyset \right\} \right) \leq (1 - c_0) r_m + C \sqrt{r_m r_{m+1}}
\]

\[
= (1 - c_0) \mu(B_j^{(m)}) + C s_{m+1} r_m \leq \left( 1 - \frac{c_0}{2} \right) \mu(B_j^{(m)}),
\]

where the last inequality holds provided that \( m \) is sufficiently large. But then, as long as \( n \) is large enough, this inequality may be iterated to yield

\[
\mu\left( \{ z \in E : z \in (1 - c_0)B^{(n+k)}(z) \text{ for } k = 1, \ldots, m \} \right) \leq (1 - \frac{c_0}{2})^m.
\]

Hence \( \mu(F_n) = 0 \).

In preparation for proving Lemma 8.1, we make the following claim.

**Claim 8.2.** Let \( n \in \mathbb{Z}_+ \). For any disc \( B_j^{(n)} \), and \( z \in \mathbb{C} \), we have

\[
\left| \int_{A(z, r_n) \cap B_j^{(n)}} K(z - \xi) d\left( \mu - \frac{m_2}{r_n} \right)(\xi) \right| \leq C s_{n+1}.
\]

**Proof.** To derive this claim, first suppose that a square \( Q_{\ell}^{(n+1)} \subset A(z, r_n) \). Then from a crude application of Lemma 7.3 (see (7.2)), we infer that

\[
\left| \int_{Q_{\ell}^{(n+1)}} K(z - \xi) d(\mu - \frac{m_2}{r_n})(\xi) \right| \leq C \frac{r_n r_{n+1}}{r_n^2} \leq C \left( \frac{r_{n+1}}{r_n} \right)^{\frac{3}{2}}.
\]
If it instead holds that $Q_{\ell}^{(n+1)} \cap \partial A(z, r_n) \neq \emptyset$, then we have the blunt estimate
\[
\left| \int_{Q_{\ell}^{(n+1)} \cap A(z, r_n)} K(z - \xi) d(\mu - \frac{m_2}{r_n})(\xi) \right| \leq \frac{2}{r_n} \left[ \mu(Q_{\ell}^{(n+1)}) + \frac{m_2(Q_{\ell}^{(n+1)})}{r_n} \right],
\]
which is bounded by $\frac{Cr_{n+1}}{r_n}$. There are most $\frac{r_n}{r_{n+1}}$ squares $Q_{\ell}^{(n+1)}$ contained in $A(z, r_n)$, and no more than $C \sqrt{\frac{r_n}{r_{n+1}}}$ squares $Q_{\ell}^{(n+1)}$ can intersect the boundary of $A(z, r_n)$.

On the other hand, the set $\tilde{A}$ consisting of the points in $A(z, r_n) \cap B^n_j$ not covered by any square $Q_{\ell}^{(n+1)}$ has $m_2$-measure no greater than $\frac{Cr_1}{2} \frac{1}{n+1} \frac{r_3}{2} \frac{r_n}{r_{n+1}}$ (see Lemma 4.2). Thus $\int_{\tilde{A}} |K(z - \xi)| \frac{dm_2(\xi)}{r_n} \leq \frac{2m_2(\tilde{A})}{r_n} \leq C r_{n+1}$.

Bringing these estimates together establishes Claim 8.2. □

Let us now complete the proof of Lemma 8.1

**Proof of Lemma 8.1** Note that $\int_{A(z, r_n) \cap B^n_j} K(z - \xi) \frac{dm_2(\xi)}{r_n}$ is a Lipschitz continuous function in $C$, with Lipschitz norm at most $\frac{C}{r_n}$. Thus, we infer from Lemma 3.2 that there is a constant $c_0 > 0$ such that
\[
\left| \int_{A(z, r_n) \cap B^n_j} K(z - \xi) \frac{dm_2(\xi)}{r_n} \right| \geq \frac{\tilde{c}}{2},
\]
whenever $\text{dist}(z, \partial B^n_j) \leq c_0 r_n$. But now we apply Claim 8.2 to deduce that for all such $z$, $\left| \int_{A(z, r_n)} K(z - \xi) \frac{dm_2(\xi)}{r_n} \right| \geq \frac{\tilde{c}}{2} - C s_{n+1}$ (the only part of the support of $\mu$ that $A(z, r_n)$ intersects is contained in $B^n_j$). The right hand side here is at least $\frac{\tilde{c}}{4}$ for all sufficiently large $n$. □

9. The set $E$ is purely unrectifiable

We now show that $E$ is purely unrectifiable, that is, $\mathcal{H}^1(E \cap \Gamma) = 0$ for any rectifiable curve $\Gamma$. The proof that follows is a simple special case of the well known fact that any set with zero lower $\mathcal{H}^1$-density is unrectifiable (one can in fact say much more, see for instance [Mat1]).

First notice that for each $z \in C$ and $n \geq 1$, $B(z, \frac{1}{4} \sqrt{r_n r_{n-1}})$ can intersect at most one of the discs $B^n_j$. Hence
\[
\mathcal{H}^1(E \cap B(z, \frac{1}{4} \sqrt{r_n r_{n-1}})) \leq 2r_n.
\]
A rectifiable curve $\Gamma$ can be covered by discs $B(z_j, \frac{1}{4} \sqrt{r_n r_{n-1}})$, $j = 1, \ldots, N$, the sum of whose radii is at most $\ell(\Gamma)$. 
Thus $\mathcal{H}^1(E \cap \Gamma) \leq \sum_{j=1}^N \mathcal{H}^1(E \cap B(z_j, \frac{1}{4}\sqrt{r_n r_{n-1}})) \leq 2 \sum_{j=1}^N r_n$. But $\sum_{j=1}^N \frac{1}{4}\sqrt{r_n r_{n-1}} \leq \ell(\Gamma)$, and so $\mathcal{H}^1(\Gamma \cap E) \leq 8 \frac{r_n}{r_{n-1}} \ell(\Gamma)$, which tends to zero as $n \to \infty$ (the sequence $\sqrt{s_n}$ is summable).

REFERENCES


