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UNCONDITIONAL BASES OF EXPONENTIALS AND OF REPRODUCING KERNELS

CONTENTS

INTRODUCTION

The problem of expansion of a given function f defined on a finite interval I of real axis $\mathbb{R}$ in Dirichlet series with complex frequencies λ_n

$$f(x)=\sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n x}$$

is the nearest analog of the well-known Fourier analysis problem*). In general, the family of exponentials $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ is not orthogonal in the Hilbert space $L^2(I)$ of all square-summable functions on I and apart from that, it need not be complete on that interval. Leaving aside the difficult completeness problem (i.e. the problem of completeness of exponentials in $L^2(I)$), we shall focus our attention on a more narrow question: to describe families of frequencies $(\lambda_n)_{n\in\mathbb{Z}}$ producing "well-behaved" bases $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ in the space $L^2(I)$.

The convergence problem for o r t h o g o n a l expansions with respect to a general complete orthonormal system $(\varphi_n)_{n\in\mathbb{Z}}$ in $L^2(I)$ is solved by the famous V.A.Steklov theorem: such an expansion converges in L^2 to the function being expanded. Moreover, the system $(\varphi_n)_{n\in\mathbb{Z}}$ being orthogonal, the corresponding Fourier series converges unconditionally; that is it converges to the same sum after any permutation of its terms. This, surely, remains true for any system $(\psi_n)_{n\in\mathbb{Z}}$ (a so-called R i e s z b a s i s) which can be obtained from the system $(\varphi_n)_{n\in\mathbb{Z}}$ by an invertible bounded linear transformation of $L^2(I)$.

In what follows we shall use a slightly more general notion of unconditional basis to avoid the hypothesis $\|\psi_n\|\asymp 1$ (i.e.

*) To emphasize the relationship of a general problem to the classical one we shall use the set of all integers $\mathbb{Z}$ as an index set; this kind of numeration will be also highly convenient for the comparison of our results with the classical theory.

$\inf_n \|\psi_n\| > 0$, $\sup_n \|\psi_n\| < +\infty$) and to cover by the same token the case of exponentials with frequencies whose imaginary parts are bounded from one side.

According to the definition of *unconditional basis* (see, for example, [7], [18]) every element x of a given space can be uniquely decomposed in an unconditionally convergent series $x = \sum_{n\in\mathbb{Z}} a_n \psi_n$. In this paper we deal, aside from one exception, with a Hilbert space where by the classical G.Köthe - O.Toeplitz theorem a complete system $(\psi_n)_{n\in\mathbb{Z}}$ forms an unconditional basis iff the following "approximate Parseval identity" holds

$$\|\sum_{n\in\mathbb{Z}} a_n\psi_n\|_2^2 \asymp \sum_{n\in\mathbb{Z}} |a_n|^2 \|\psi_n\|_2^2 .$$

So we take the following definition as one suitable to work with.

DEFINITION. A family $(\psi_n)_{n\in\mathbb{Z}}$ of non-zero vectors in a Hilbert space H is called *an unconditional basis* in H if

1) the family $(\psi_n)_{n\in\mathbb{Z}}$ spans the space H ;

2) there are positive constants c, C such that for every finite sequence of complex numbers $(a_n)_{n\in\mathbb{Z}}$ the following inequalities hold

$$c\sum_n |a_n|^2 \|\psi_n\|^2 \leqslant \|\sum_n a_n\psi_n\|^2 \leqslant C\sum_n |a_n|^2 \|\psi_n\|^2 .$$

Thus every Riesz basis is unconditional and conversely every unconditional basis satisfying $\|\psi_n\| \asymp 1$ is a Riesz basis.

The purpose of this paper is to describe all subsets $\Lambda = \{\lambda_n : n\in\mathbb{Z}\}$ of a half-plane $\mathbb{C}_\gamma \overset{def}{=} \{\zeta\in\mathbb{C} : \operatorname{Im}\zeta > \gamma\}$, $\gamma\in\mathbb{R}$, (or of the half-plane $\{\zeta\in\mathbb{C} : \operatorname{Im}\zeta < \gamma\} = \mathbb{C}^\gamma$) such that the family $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ forms an unconditional basis in $L^2(I)$.

The first fundamental progress in the outlined area was attained by N.Wiener [58] and by N.Wiener and R.Paley [59] in 1934. They proved that the system $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ forms a Riesz basis in $L^2(0, 2\pi)$ if $\lambda_n \in \mathbb{R}$, $n\in\mathbb{Z}$, and if $\sup_n |n-\lambda_n| < \pi^{-2}$. This result has been repeatedly revised and generalized; see the history of the question in §7 of Part I. The most exquisite formulation of the achievements mentioned above can be obtained by comparison of the theorems due to A.Ingham [41] and M.I.Kadec [10].

THEOREM. Let $\delta > 0$. Every family $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ satisfying

$$\sup_{n\in\mathbb{Z}} |\lambda_n - n| = \delta, \ \delta > 0,$$

forms a Riesz basis in $L^2(0, 2\pi)$ if and only if $\delta < \frac{1}{4}$.

We obtain this theorem in Part I of the paper as a consequence of our main results.

In all papers, which have dealt with the subject discussed, it was assumed that $\sup_n |\operatorname{Im} \lambda_n| < \infty$ and the main tool of investigation was an idea stated in the remarkable book of N.Wiener and R.Paley [59]: to form a Riesz basis in $L^2(0, 2\pi)$ it is sufficient for the family $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ to be close enough to the usual trigonometrical system $(e^{inx})_{n\in\mathbb{Z}}$.

One can hardly expect that such an approach to the general problem will be successful, though a result of part III (see §4) exhibits some connection between the general and the classical case.

Another point of view, also originated in [59] has been advanced by B.Ja.Levin. In his method a central role is played by an entire function of exponential type with zeros λ_n, $n \in \mathbb{Z}$ and whose width of the indicator diagram coincides with the length of the interval where our basis is considered. We shall call this entire function "a generating function for the family $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$".

We are now going to state in terms of generating functions a condition sufficient for the exponentials to form a Riesz basis in $L^2(I)$.

DEFINITION. A countable subset $\Lambda = \{\lambda_n : n \in \mathbb{Z}\}$ of the complex plane $\mathbb{C}$ is named *separated* if

$$\inf_{n\neq m} |\lambda_n - \lambda_m| > 0. \qquad (S)$$

DEFINITION (B.Ja.Levin). An entire function f of exponential type is called a *sine-type function* (briefly STF) if its zero set is contained in a strip of a finite width, parallel to the real axis, and if

$$0 < \inf_{x\in\mathbb{R}} |f(x)| \leqslant \sup_{x\in\mathbb{R}} |f(x)| < +\infty.$$

The sine-type functions play an important role in the exponential bases problem and we will be returning from time to time to a discussion of their properties in the sequel.

THEOREM (B.Ja.Levin, V.D.Golovin [14], [6]). Let the generating function of a family $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ be a sine-type function with the width of the indicator diagram equal to a, $a>0$. Then $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ forms a Riesz basis in $L^2(I)$, $|I|=a$.

Some attempts have been made to unify the approaches mentioned above. The relevant result of V.È. Kacnelson [12] can be stated, broadly speaking, as follows. A transformation $\lambda_n \to \mu_n$, $n\in\mathbb{Z}$, of the zero set of a STF preserves the property to form a Riesz basis for the corresponding family of exponentials if the set $\{\mu_n : n\in\mathbb{Z}\}$ is separated and if $|\operatorname{Re}(\mu_n-\lambda_n)| < \frac{1}{4}\inf_{n\neq k}|\operatorname{Re}\lambda_n - \operatorname{Re}\lambda_k|$. The most subtle result has been proved by S.A.Avdonin [2], see §7 of the Part I below. The main tools of these papers are delicate estimates of canonical products.

The method of the present paper rests on completely different considerations. It comes from an explicit description of those families of exponentials $\mathcal{E}_\Lambda=(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ which form unconditional bases in their closed linear spans $\operatorname{span}\mathcal{E}_\Lambda$ in $L^2(\mathbb{R}_+)$, $\mathbb{R}_+ \overset{def}{=} (0,+\infty)$. This description is given by the famous C a r l e s o n c o n d i t i o n

$$\inf_n \prod_{k\neq n}\left|\frac{\lambda_k-\lambda_n}{\lambda_k-\overline{\lambda}_n}\right|>0; \tag{C}$$

see L.Carleson [28], H.Shapiro - A.Shields [56], V.È.Kacnelson [11], N.K.Nikol'skii - B.S.Pavlov [20]. If we deal with such a set of frequencies $\Lambda=\{\lambda_n : n\in\mathbb{Z}\}$ and if the transformation $f\to f\cdot\chi_{[0,a]}$, which, obviously, coincides with the orthogonal projection onto $L^2(0,a)$ *), is an isomorphism of $\operatorname{span}\mathcal{E}_\Lambda$ onto $L^2(0,a)$, then, clearly, the family $(e^{i\lambda_n x}\cdot\chi_{[0,a)})_{n\in\mathbb{Z}}$ will form an unconditional basis in $L^2(0,a)$.

This procedure is a chief ingredient of the proofs of all our results. It appeared for the first time in [22], and was used, in particular, in the proof of Levin - Golovin theorem. However, only five years later it became clear that these arguments lead not only to a full solution of the exponential Riesz bases problem [23], but also imply simple and transparent proofs of almost all known results in that area [25]. In the sequel, it turned out that the

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*) We assume that the space $L^2(0,a)$ is imbedded into $L^2(\mathbb{R}_+)$ in a natural way.

sphere of applications of the described method can be considerably extended to cover unconditional bases of exponentials as well as the bases formed by reproducing kernels [19].

Our method has several advantages in comparison with those of Wiener - Paley and Levin; requiring less in what concerns the non-perturbed basis, it allows one to redistribute the difficulties more uniformly between the investigation of non-perturbed bases in $L^2(\mathbb{R}_+)$ and perturbed ones in $L^2(0,a)$, $a>0$. Moreover, under the slight additional requirement that the projection does not distort the elements of our family too much, the above geometrical reasoning can be inverted.

It should be noted that the solution of a well-known problem, originated in the papers of L.Schwartz [55] and P.Koosis [43], is reduced to the application of the described method too. This is a problem of equivalence of norms $\left(\int_{\mathbb{R}_+}|f|^2\right)^{1/2}$, $\left(\int_I |f|^2\right)^{1/2}$ on the span of exponentials $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R}_+)$. Clearly, the norm equivalence together with Carleson's condition imply that the family $(e^{i\lambda_n x}\cdot\chi_I)_{n\in\mathbb{Z}}$ is an unconditional basis in its span.

The same procedure can be applied to the joint completeness problem of an operator and its adjoint (for dissipative operators and contractions). The application, outlined in [21] in a vague form, appears now more distinctly. The joint basis property is also discussed here. Both of them are important for the spectral theory of differential operators. They arise naturally, for example, in the investigation of the Sturm - Liouville problem containing a spactral parameter in the boundary condition:

$$-\frac{d^2u}{dx^2}=\lambda^2\rho^2(x)u; \quad u'(0)=0, \quad u'(a)-i\lambda u(a)=0.$$

A similar problem for the Schrödinger operator has been considered by T.Regge [52] in connection with a question of resonance scattering theory.

Aside from the systematic exposition of [19], [23], [25] and the applications to the theory of differential operators, our paper contains some new results too. The exposition is developed along the following plan.

The main purpose of Part I is to apply the above mentioned approach to the exponential bases problem; to formulate all our main results including the results for the reproducing kernels; and to discuss the connections between them. Apart from that there is a series of examples here illustrating the general theory. Part I is concluded with a short survey of the history of problem.

Part II deals with a bases problem for reproducing kernels. The connections of the bases problem with Hankel operators and with the B.Sz.-Nagy - C.Foiaş functional model are discussed. The bases close to orthogonal are considered here also. In conclusion we outline an interpretation of our results in terms of the interpolation theory and investigate the bases problem in L^p, $p \neq 2$.

The next part, Part III, is devoted to some applications of our approach in the classical domain. We prove here some results concerning the perturbation theory for exponential unconditional bases. In particular, a new proofs for the theorems of S.A.Avdonin and V.È.Kacnelson are given. In section 3 of Part III we state an example, which is due to S.A.Vinogradov and V.I.Vasjunin, of a generating function bounded on $\mathbb{R}$ together with its reciprocal and such that $\lim_{n\to\infty} \operatorname{Im} \lambda_n = +\infty$ for a sequence $(\lambda_n)_{n\in\mathbb{Z}}$ of its zeros. It is also proved (following to V.I.Vasjunin) that in many cases an unconditional basis $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ in its closed span in $L^2(0,a)$ can be extended to be an unconditional exponential basis in the whole space $L^2(0,a)$. The last section of the Part, § 4, deals with the problem of equiconvergence of Fourier series with respect to the general unconditional basis $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ in $L^2(0,2\pi)$ and of those with respect to $(e^{inx})_{n\in\mathbb{Z}}$. A theorem is proved generalizing the well-known Levinson theorem [48].

Part IV is devoted to the applications of our geometrical approach to the above mentioned Regge problem. The main purpose of this part of the paper is to indicate new possibilities of the method rather than prove accomplished results. So it is linked to the preceding Parts by the method of investigation.

Completing the discussion we mention that we have tried to make the bulk of the article intelligible to anyone with basic knowledge of functional analysis and function theory.

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PART I

BACKGROUND OF EXPONENTIAL BASES PROBLEM

1. Functional model

To translate our problem into the language used in B.Sz.-Nagy - C.Foias model some facts of common knowledge about the Hardy class H^2_+ in the upper half-plane $\mathbb{C}_+ \stackrel{def}{=} \{\zeta \in \mathbb{C} : \operatorname{Im} \zeta > 0\}$ are needed. The following sources [18], [33], [44], [54] contain the exhaustive information about the subject.

A function f which is analytic in $\mathbb{C}_+$ belongs to the Hardy class H^2_+ if

$$\|f\|^2 \stackrel{def}{=} \sup_{y>0} \frac{1}{2\pi} \int_{\mathbb{R}} |f(x+iy)|^2 \, dx < +\infty .$$

By Fatou's theorem the space H^2_+ may be considered as a closed subspace of $L^2(\mathbb{R})$. It is convenient to define an inner product in $L^2(\mathbb{R})$ by the formula

$$(f,g) \stackrel{def}{=} \frac{1}{2\pi} \int_{\mathbb{R}} f \bar{g} \, dx .$$

A nontrivial function f in H^2_+ can be factored uniquely as the product

$$f = c \cdot B \cdot S \cdot f_e ,$$

where c is a unimodular constant, $|c| = 1$; B is a Blaschke product; S is a singular inner function; and f_e is an outer function. A Blaschke product B with the zero sequence $(\lambda_n)_{n\in\mathbb{Z}}$ is an infinite product

$$B(z) = \prod_{n\in\mathbb{Z}} \varepsilon_n \frac{1 - z/\lambda_n}{1 - z/\bar{\lambda}_n} ,$$

where signs ε_n, $|\varepsilon_n| = 1$ make each factor in the product non-negative at the point $z = i$. A well-known Blaschke condition

$$\sum_{n\in\mathbb{Z}} \frac{\operatorname{Im} \lambda_n}{|\lambda_n + i|^2} < +\infty \tag{B}$$

is the necessary and sufficient one for the Blaschke product to converge. To describe the factor S one can consider a one-point compactification $\hat{\mathbb{R}} \overset{def}{=} \mathbb{R} \cup \{\infty\}$ of the real line $\mathbb{R}$. Then

$$S(z) = \exp\left\{-\frac{1}{\pi i}\int_{\hat{\mathbb{R}}} \frac{tz+1}{t-z}\, d\mu(t)\right\},$$

where μ is a non-negative finite measure on $\hat{\mathbb{R}}$ which, being restricted on $\mathbb{R}$, is singular with respect to the usual Lebesgue measure on $\mathbb{R}$. The measure μ of the full mass equal to $\pi \cdot a$, $a>0$, supported by the point ∞ corresponds, obviously, to the exponential e^{iaz}. The product $c\cdot B\cdot S$ is called an inner function. Inner functions can be described as elements of the algebra H^∞ of all uniformly bounded and holomorphic in $\mathbb{C}_+$ functions, whose boundary values are unimodular a.e. on $\mathbb{R}$. The outer part f_e of the function f is defined by

$$f_e(z) = \exp\left\{\frac{1}{\pi i}\int_{\mathbb{R}} \frac{tz+1}{t-z}\cdot\frac{\log|f(t)|}{t^2+1}\, dt\right\}.$$

It should be noted that the same factorization property holds for all Hardy classes H^p_+, $0<p\leqslant+\infty$ (H^p_+ consists of all functions f, analytic in $\mathbb{C}_+$ and satisfying

$$\|f\|_p^p \overset{def}{=} \sup_{y>0}\int_{\mathbb{R}} |f(x+iy)|^p dx < +\infty).$$

The well-known Paley - Wiener theorem asserts that the inverse Fourier transform

$$\mathcal{F}^* f(t) = \int_{\mathbb{R}} e^{it\gamma} f(\gamma)\, d\gamma$$

is a one to one norm-preserving mapping of $L^2(\mathbb{R}_+)$ onto H^2_+. By the inversion formula we have

$$f(\gamma) = \mathcal{F}\cdot\mathcal{F}^* f(\gamma) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{F}^* f(t)\, e^{-i\gamma t}\, dt.$$

Let, for the time being, $\Lambda = \{\lambda_n : n\in\mathbb{Z}\}$ be a fixed subset of $\mathbb{C}_+$ and let $a>0$. Clearly, the family $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ forms an unconditional basis in $L^2(0,a)$ iff the family $(e^{-i\bar{\lambda}_n x})_{n\in\mathbb{Z}}$ does. Let $\Lambda^* \overset{def}{=} \{-\bar{\lambda}_n : n\in\mathbb{Z}\}$. The Fourier transform $\mathcal{F}^*$ maps the closed span $\mathcal{E}_{\Lambda^*}$ of the family $(e^{i\lambda x}\cdot\chi_{[0,\infty)})_{\lambda\in\Lambda^*}$ onto the subspace

$$K_B \overset{def}{=} H^2_+ \ominus B H^2_+$$

in H^2_+, B being the Blaschke product for the sequence $(\lambda_n)_{n\in\mathbb{Z}}$ if it satisfies the Blaschke condition and the identically zero function otherwise. The proof of this fact rests on a simple calculation:

$$\mathcal{F}^*(e^{-i\bar{\lambda}\gamma}\cdot\chi_{[0,+\infty)})(z)=\int_0^\infty e^{i\gamma z}e^{-i\gamma\bar{\lambda}}\,d\gamma=\frac{i}{z-\bar{\lambda}}\ .$$

It remains only to observe that the span of the family $((z-\bar{\lambda})^{-1})_{\lambda\in\Lambda}$ is equal to K_B.

The space $\mathcal{F}^*\mathcal{E}_{\Lambda^*}$ being described, we have to do the same for the space $\mathcal{F}^*L^2(0,a)$. Let $\theta^a(z)\overset{def}{=\!=}e^{iaz}$, $a>0$. Clearly,

$$\mathcal{F}^*L^2(0,a)=\mathcal{F}^*L^2(\mathbb{R}_+)\ominus\mathcal{F}^*L^2(a,\infty)=H^2_+\ominus\theta^aH^2_+=K_{\theta^a}\,.$$

The program outlined in Introduction can be easily applied now. But it is natural to consider now a more general problem. Let θ be any inner function and let B be a Blaschke product with the sequence of zeros $(\lambda_n)_{n\in\mathbb{Z}}$. The function $k(z,\lambda)\overset{def}{=\!=}\frac{i}{z-\bar{\lambda}}$ is, obviously, the reproducing kernel for H^2_+:

$$(f,k(\cdot,\lambda))=\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{f(x)}{x-\lambda}\,dx=f(\lambda)\,,\quad \operatorname{Im}\lambda>0\,.$$

Let P_θ be an orthogonal projection onto the subspace K_θ. Then the function $k_\theta(\cdot,\lambda)\overset{def}{=\!=}P_\theta k(\cdot,\lambda)$ is the reproducing kernel for K_θ. Indeed, if $f\in K_\theta$ then

$$(f,k_\theta(\cdot,\lambda))=(f,P_\theta k(\cdot,\lambda))=(f,k(\cdot,\lambda))=f(\lambda)\,.$$

Simple computations show that

$$k_\theta(z,\lambda)=i\,\frac{1-\overline{\theta(\lambda)}\,\theta(z)}{z-\bar{\lambda}}\,.$$

Now we are in a position to formulate general problem of unconditional bases for reproducing kernels:

What is to be assumed about the pair (θ,Λ) for the family $(k_\theta(\cdot,\lambda))_{\lambda\in\Lambda}$ to be an unconditional basis in K_θ ?

2. Carleson condition

As it was already mentioned in Introduction, the test for the family $((z-\overline{\lambda}_n)^{-1})_{n\in\mathbb{Z}}$ to be an unconditional basis in its closed span in H^2_+ is given by the well-known Carleson condition:

$$\delta = \inf_n \prod_{k\neq n} \left|\frac{\lambda_n-\lambda_k}{\lambda_n-\overline{\lambda}_k}\right| > 0. \tag{C}$$

Clearly, (C) $\Longrightarrow$ (B) and therefore the Blaschke product

$$B=\prod_{n\in\mathbb{Z}} b_n\,, \qquad b_n(z) \overset{def}{=\!=} \varepsilon_n \frac{1-z/\lambda_n}{1-z/\overline{\lambda}_n}\,,$$

may be considered. Denoting $B_n \overset{def}{=\!=} B\cdot b_n^{-1}$, one may rewrite (C) in a more compact form

$$\inf_n |B_n(\lambda_n)|>0. \tag{C}$$

It is a matter of common knowledge, see for example [18], that the Carleson condition is equivalent to a purely geometrical one. Let $D(\zeta,r) \overset{def}{=\!=} \{\xi\in\mathbb{C} : |\xi-\zeta|<r\}$.

DEFINITION. A subset $\Lambda=\{\lambda_n : n\in\mathbb{Z}\}$ of $\mathbb{C}_+$ is called a *rare set* if there is a positive ε such that

$$D(\lambda_n,\varepsilon\,\mathrm{Im}\,\lambda_n)\cap D(\lambda_m,\varepsilon\,\mathrm{Im}\,\lambda_m)=\emptyset,\ m\neq n. \tag{R}$$

DEFINITION. A positive measure μ in $\mathbb{C}_+$ is called a C-*measure* if

$$\sup_{r>0,\,x\in\mathbb{R}} r^{-1}\mu(D(x,r))<+\infty. \tag{CM}$$

Then $\Lambda\in(C)$ iff $\Lambda\in(R)$ and the measure $\sum_{n\in\mathbb{Z}}\mathrm{Im}\lambda_n\cdot\delta_{\lambda_n}$ (δ_λ denotes the unit mass at λ) is a C-measure.

Here are two examples of sets satisfying (C): $\Lambda_1=\{2^n i : n\in\mathbb{Z}\}$, $\Lambda_2=\{i+n : n\in\mathbb{Z}\}$. In general, if $\Lambda\subset\{\xi\in\mathbb{C} : 0<c<\mathrm{Im}\,\xi<c^{-1}\}$ then the separation condition

$$\inf_{n\neq m}|\lambda_n-\lambda_m|>0 \tag{S}$$

and the Carleson condition are equivalent.

There is one more notion needed for the formulation of the main theorem on unconditional bases of rational fractions.

DEFINITION. A family of non-zero elements $(x_n)_{n\in\mathbb{Z}}$ of a Banach space X is called a **uniformly minimal family** if

$$\inf_n \operatorname{dist}\left(\frac{x_n}{\|x_n\|}, \operatorname{span}(x_k : k \neq n)\right) > 0.$$

Clearly, any basis, and, in particular, any unconditional basis, forms a uniformly minimal family. The converse assertion does not hold in general but it, nevertheless, holds for the families of rational fractions in H^2_+ ; see theorem A below. Apparently, the main reason of this phenomenon is rooted in simple formulae for the dual family

$$\psi_n = \frac{2\operatorname{Im}\lambda_n}{z-\overline{\lambda_n}} \cdot \frac{B_n(z)}{B_n(\lambda_n)}, \quad n \in \mathbb{Z},$$

of the family φ_n , $\varphi_n \overset{def}{=\!=} (z-\overline{\lambda}_n)^{-1}$ spanning the space

$$\operatorname{span}_{H^2_+}(\varphi_n : n \in \mathbb{Z}) = H^2_+ \ominus BH^2_+ = K_B.$$

It is an easy task to check that $\psi_n \in K_B$, $n \in \mathbb{Z}$, and that $\langle \varphi_n , \psi_k \rangle = \delta_{nk}$. The computation of the distance from $\|\varphi_n\|^{-1}\varphi_n$ to the $\operatorname{span}(\varphi_k : k \neq n)$ is now an elementary exercise: $\operatorname{dist}(\|\varphi_n\|^{-1}\varphi_n, \operatorname{span}(\varphi_k : k \neq n)) =$ (by the Hahn - Banach theorem) $= \|\varphi_n\|^{-1}_{H^2_+} \cdot \|\psi_n\|^{-1}_{H^2_+} = (2\operatorname{Im}\lambda_n)^{1/2}(2\operatorname{Im}\lambda_n)^{-1/2}|B_n(\lambda_n)| = |B_n(\lambda_n)|$.

COROLLARY. For a family $\{(z-\overline{\lambda}_n)^{-1}, n \in \mathbb{Z}\}$, $\operatorname{Im}\lambda_n > 0$, to be uniformly minimal it is necessary and sufficient that $(\lambda_n)_{n\in\mathbb{Z}} \in (C)$. ●

THEOREM A. Let $\Lambda = \{\lambda_n : n \in \mathbb{Z}\} \subset \mathbb{C}_+$. The following assertions are equivalent.

1. The family $((z-\overline{\lambda}_n)^{-1})_{n\in\mathbb{Z}}$ forms an unconditional basis in its own span in H^2_+.
2. The family $((z-\overline{\lambda}_n)^{-1})_{n\in\mathbb{Z}}$ is uniformly minimal in H^2_+.
3. The family $(e^{i\lambda_n x}\chi_{\mathbb{R}_+})_{n\in\mathbb{Z}}$ forms an unconditional basis in its $L^2(\mathbb{R}_+)$ -span.
4. $\Lambda \in (C)$.

In such form Theorem A has been obtained by N.K.Nikol'skii and B.S.Pavlov [63], [20] (see also [61], [62]) as a consequence of a more general theory. Their proof hinges on preceding results of L.Carleson [28] and of H.Shapiro - A.Shields [56], [64] from the interpolation theory.

There are many ways to reformulate the assertions 1-4 of Theorem A and, first of all, to link these assertions to the objects fundamental for our approach. We mean the expansions in Fourier series with respect to the eigen-functions of the so-called "model semigroup" and the well-known interpolation problem $f(\lambda_n)(\operatorname{Im}\lambda_n)^{1/2} = a_n$, $(a_n)_{n\in\mathbb{Z}} \in \ell^2$ in H^2_+. We leave the discussion of these links - for the time being - till §5, not to be led too far from exponential bases. Note, however, that it is just the operator-theoretical approach (connected with the model semigroup) the proof of Theorem A in [20] was based upon.

Our last remark concerns the interplay between the unconditional bases property and the completeness problem for rational fractions in H^2_+. Obviously, (C) $\Longrightarrow$ (B), and therefore the uncompleteness is a necessary condition for the family $((z-\overline{\lambda}_n)^{-1})_{n\in\mathbb{Z}}$ to be an unconditional basis in its closed span in H^2_+.

Now we are in a position to make the first step towards the investigation of the basis property for exponentials. Namely, according to the plan stated in Introduction we are to prove that the Carleson condition (C) is necessary for exponentials to form an unconditional basis in $L^2(0,a)$. The next step will be to study the orthogonal projection P_{K_θ}, $\theta = \theta^a$. Because of the general nature of our geometrical reasoning, it is natural to deal with the general case of reproducing kernels at once; see the end of §1.

THEOREM 1. Let θ be an inner function and let $\Lambda = \{\lambda_n : n \in \mathbb{Z}\} \subset \mathbb{C}_+$.

1. If the family $(k_\theta(\cdot,\lambda_n))_{n\in\mathbb{Z}}$ is an unconditional basis in its span then $\Lambda \in (C)$.

2. If the family $(k_\theta(\cdot,\lambda_n))_{n\in\mathbb{Z}}$ is uniformly minimal and if

$$\sup_n |\theta(\lambda_n)| < 1 \tag{1}$$

then $\Lambda \in (C)$.

Leaving aside the proof of the assertion 1 till §1 of Part II, we shall give now a simple explanation of the assertion 2 of the theorem, which is sufficient for our analysis of exponential bases property. For $\theta = \theta^a$ the condition (1) implies, obviously, that $\Lambda \subset \mathbb{C}_\delta$, for some positive number δ . The role of the condition (1) in what follows becomes clear after we note that it is a necessary and sufficient condition for H^2_+-norms of the functions $(z-\overline{\lambda}_n)^{-1}$ and $P_\theta(z-\overline{\lambda}_n)^{-1} = k_\theta(\cdot,\lambda_n)$ to be comparable. If $\theta = \theta^a$ then it means

$$\Big(\int_{\mathbb{R}_+} |e^{i\lambda_n x}|^2\,dx\Big)^{1/2} \asymp \Big(\int_0^a |e^{i\lambda_n x}|^2\,dx\Big)^{1/2},\ n\in\mathbb{Z}. \qquad (1a)$$

The statement 2 of Theorem 1 is an immediate corollary of Theorem A and the following elementary Lemma.

LEMMA *). Let L be a bounded linear operator in a Banach space X and let $(x_n)_{n\in\mathbb{Z}}$ be a sequence of non-zero vectors in X satisfying $C \overset{\text{def}}{=} \sup_n \|x_n\|\,\|Lx_n\|^{-1} < \infty$. Then the family $(x_n)_{n\in\mathbb{Z}}$ is uniformly minimal if the same holds for the family $(Lx_n)_{n\in\mathbb{Z}}$.

PROOF. If $a_k \in \mathbb{C}$, $b_k = a_k\|Lx_k\|\,\|x_k\|^{-1}$, then

$$\|Lx_n\cdot\|Lx_n\|^{-1} - \sum_{k\neq n} a_k Lx_k\| = \|x_n\|\,\|Lx_n\|^{-1}\|Lx_n\cdot\|x_n\|^{-1} - \sum_{k\neq n} b_k Lx_k\| \leqslant$$

$$\leqslant C\|L\|\,\|x_n\cdot\|x_n\|^{-1} - \sum_{k\neq n} b_k x_k\|.$$

It follows that

$$\operatorname{dist}(x_n\|x_n\|^{-1}, \operatorname{span}(x_k, k\neq n)) \geqslant (C\|L\|)^{-1}\operatorname{dist}(Lx_n\|Lx_n\|^{-1}, \operatorname{span}(Lx_k: k\neq n)) \bullet$$

To prove the statement 2 of Theorem 1 let $x_n = (z-\bar\lambda_n)^{-1}$, $L = P_\theta$. Then it follows from the equalities $\|x_n\|^2 = (2\operatorname{Im}\lambda_n)^{-1}$, $\|Lx_n\|^2 = (1-|\theta(\lambda_n)|^2)(2\operatorname{Im}\lambda_n)^{-1}$ that $\sup_n \|x_n\|\cdot\|Lx_n\|^{-1} < +\infty$. The trivial part of Theorem A ($2 \Longrightarrow 4$) together with the Lemma imply $\Lambda = \{\lambda_n : n\in\mathbb{Z}\} \in (C)$. $\bullet$

A simple but, nevertheless, important remark is relevant now. Let $\theta = \theta^a$ for the time being. There are a few isomorphisms in $L^2(0,a)$ preserving the exponentials:

$$f(x) \longmapsto e^{i\alpha x} f(x),\quad \alpha\in\mathbb{C}$$

$$f(x) \longmapsto f(a-x)$$

$$f(x) \longmapsto \overline{f(x)}.$$

Any of these isomorphisms preserves, obviously, the property to be a uniformly minimal exponential family and the basis property as well. Using these isomorphisms we always can move a frequency

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*) An analogous lemma may be found in [22].

set $\Lambda = \{\lambda_n : n \in \mathbb{Z}\}$ from any half-plane $\mathbb{C}_\gamma$ (or $\mathbb{C}^\gamma$), $\gamma \in \mathbb{R}$, to the half-plane $\mathbb{C}_\delta$, $\delta > 0$. So the assumption (1) does not restrict the generality if we deal with the sets Λ contained in a half-plane $\mathbb{C}_\gamma$ (or $\mathbb{C}^\gamma$), $\gamma \in \mathbb{R}$.

The second step in splitting up our problem into two independent ones is made by theorem 2 below. We again not only formulate the theorem in its natural generality, but also give a special formulation (Theorem 2$'$) for the important case of exponentials.

THEOREM 2. Let θ be an inner function, $\Lambda = \{\lambda_n : n \in \mathbb{Z}\} \subset \subset \mathbb{C}$, and let $\Lambda \in$ (1). Then the following statements are equivalent.

1. The family $(k_\theta(\cdot, \lambda_n))_{n \in \mathbb{Z}}$ forms an unconditional basis in K_θ.

2. a) $\Lambda \in (C)$; b) the operator $P_\theta | K_B$ maps isomorphically the space K_B onto K_θ , B being the Blaschke product for the sequence $(\lambda_n)_{n \in \mathbb{Z}}$.

THEOREM 2$'$. Let $\Lambda = \{\lambda_n : n \in \mathbb{Z}\} \subset \mathbb{C}_\delta$, $\delta > 0$, and let a be a positive number. The following statements are equivalent.

1. The family $(e^{i\lambda_n x} \cdot \chi_{[0,a)})_{n \in \mathbb{Z}}$ is an unconditional basis in $L^2(0,a)$.

2. a) $\Lambda \in (C)$; b) the restriction of the orthogonal projection $f \mapsto \chi_{[0,a)} \cdot f$ onto $\operatorname{span}_{L^2(\mathbb{R}_+)}(e^{i\lambda_n x} : n \in \mathbb{Z})$ is an isomorphism of the span onto $L^2(0,a)$.

It is clear from §1 that Theorem 2$'$ is covered by Theorem 2.

THE PROOF OF THEOREM 2. 1 $\Longrightarrow$ 2. From Theorem 1 it follows that $\Lambda \in (C)$ and therefore the family $((z-\bar\lambda_n)^{-1})_{n \in \mathbb{Z}}$ is an unconditional basis in its closed span $K_B = H^2 \ominus BH^2$ by Theorem A . Using the condition $\|(z-\bar\lambda_n)^{-1}\|_{H^2_+} \asymp \asymp \|P_\theta (z-\bar\lambda_n)^{-1}\|_{H^2_+}$ implied by (1), we see that

$$\|P_\theta \sum_n a_n (z-\bar\lambda_n)^{-1}\|^2_{H^2_+} = \|\sum_n a_n k_\theta(\cdot,\lambda_n)\|^2_{H^2_+} \asymp$$

$$\asymp \sum_n |a_n|^2 \cdot \|k_\theta(\cdot,\lambda_n)\|^2_{H^2_+} \asymp \sum_n |a_n|^2 \|(z_n-\bar\lambda_n)^{-1}\|^2_{H^2_+} \asymp$$

$$\asymp \|\sum_n a_n (z-\bar\lambda_n)^{-1}\|^2_{H^2_+}.$$

This, clearly, implies that the map $P_\theta : K_B \longrightarrow K_\theta$ is an isomorphism.

2 $\Longrightarrow$ 1. The set Λ satisfying the Carleson condition, it follows by Theorem A that the family $((z-\bar\lambda_n)^{-1})_{n \in \mathbb{Z}}$

forms an unconditional basis in K_B. The family $k_\theta(\cdot,\lambda_n)= = P_\theta(z-\bar\lambda_n)^{-1}$ is now an unconditional basis in K_θ because it is assumed in the conditions of the theorem that the operator $P_\theta \mid K_B$ is an isomorphism. ●

Thus the unconditional basis problem for exponentials defined on a finite interval, as well as the more general problem for reproducing kernels in K_θ, is reduced to the study of the conditions of invertibility of the operator $P_\theta : K_B \to K_\theta$.
We shall describe later, see §§ 3,5, all pairs of inner functions (θ_1, θ_2) such that $P_{\theta_1}: K_{\theta_2} \longrightarrow K_{\theta_1}$ is an isomorphism, and shall be especially detailed in the leading case $\theta_1 = \exp iaz$, $\theta_2 = B \overset{def}{=\!=} \prod_{n\in\mathbb{Z}} b_{\lambda_n}$. Such a description, see § 4, may be given directly in terms of the distribution of numbers $(\lambda_n)_{n\in\mathbb{Z}}$, and all known results on exponential bases in $L^2(0,a)$ can be easily derived after that.

To end this section we note that Theorems 2, 2′ can be given a form covering the case of unconditional bases in their closed linear span (i.e. not assuming the family under consideration to be complete in the whole space). Let us do this, e.g., for Theorem 2.

THEOREM 2 bis. Let θ be an inner function, let $\Lambda = \{\lambda_n : n\in\mathbb{Z}\} \subset \mathbb{C}$, and let $\Lambda \in (1)$. Then the following statements are equivalent.

1. The family $\{k_\theta(\cdot,\lambda_n) : n\in\mathbb{Z}\}$ forms an unconditional basis in its closed linear span.

2. a) $\Lambda \in (C)$, b) the operator $P_\theta : K_B \longrightarrow K_\theta$ is left-invertible.

3. The invertibility tests for $P_\theta \mid K_B$: geometrical and analytical aspects

Let M and N be closed subspaces of a Hilbert space H. The invertibility of the operator $P_M \mid N$ means clearly that the subspaces are "close" (in a sense). Geometrically speaking this "closeness" can be expressed as the positivity of the angle $\langle N, M^\perp\rangle$ formed by subspaces N and $M^\perp \overset{def}{=\!=} H \ominus M$; a precise definition of the angle $\langle X, Y\rangle$ will be given later (§ 2, Part II; now it will be used only nominally). Note for the time being, that $\cos\langle X, Y\rangle = \| P_X \mid Y \|$ (see § 2, II).

The following Lemma gives simple geometrical conditions for the operator $P_\theta \mid K_B$ to be invertible.

LEMMA. Let M and N denote closed subspaces of a Hilbert space H. The following statements are equivalent: 1. $\mathrm{Ker}(P_M|N)=\{\mathbb{O}\}$; 2. $M^\perp \cap N=\{\mathbb{O}\}$; 3. $\mathrm{clos}(M+N^\perp)=H$, 4. $\mathrm{clos}\, P_N M=N$.

The following statements are equivalent: 1. $P_M|N$ is left-invertible; 2. $\|P_N|M^\perp\|<1$; 3. $0<\langle N, M^\perp\rangle$; 4. $H=M+N^\perp$.

There is no sense to burden our text with the highly standard proof of the Lemma; see however Lemma 2.1, § 2, Part II. Considering $P_M|N$ as a mapping of N into M we see that

$$(P_M|N)^*=P_N|M,$$

so that the Lemma yields the following useful conclusion.

COROLLARY. The following statements are equivalent.

1. The projection P_M maps the subspace N isomorphically onto M.
2. $\max(\|P_N|M^\perp\|,\ \|P_M|N^\perp\|)<1$.
3. $0<\langle N, M^\perp\rangle$ and $N+M^\perp=H$.
4. $\|P_N|M^\perp\|<1$, $M\cap N^\perp=\{\mathbb{O}\}$.

We may now return to the problem of the invertibility of the operator $P_\theta|K_B$ arisen at the end of § 2. Let $\mathbb{P}_+$ be the orthogonal projection of $L^2(\mathbb{R})$ onto H^2_+, and let $\mathbb{P}_-=I-\mathbb{P}_+$.

LEMMA. $P_\theta=\theta\mathbb{P}_-\bar\theta\,|H^2_+$.

PROOF. It is clear that $\theta\mathbb{P}_-\bar\theta x=\mathbb{O}$ if $x\in\theta H^2_+$. If $x\perp\theta H^2_+$, then, obviously, $\bar\theta x\perp H^2_+$ and therefore $\theta\mathbb{P}_-\bar\theta x=x$. ●

THEOREM 3. Let θ_j be an inner function, $P_{\theta_j}=P_{K_{\theta_j}}$, $K_{\theta_j}=H^2_+\ominus\theta_j H^2_+$, $j=1,2$. The following statements are equivalent.

1. The operator $P_{\theta_1}: K_{\theta_2}\longrightarrow K_{\theta_1}$ is invertible.
2. $\mathrm{dist}(\theta_1\bar\theta_2, H^\infty)<1$, $\mathrm{dist}(\theta_2\bar\theta_1, H^\infty)<1$.
3. $\mathrm{dist}(\theta_1\bar\theta_2, H^\infty)<1$, $\bar\theta_2\theta_1 H^2_-\cap H^2_+=\{\mathbb{O}\}$.
4. $\mathrm{dist}(\theta_2\bar\theta_1, H^\infty)<1$, $\theta_2\bar\theta_1 H^2_-\cap H^2_+=\{\mathbb{O}\}$.
5. $0<\langle\theta_2 H^2_-, \theta_1 H^2_+\rangle$, $\theta_2 H^2_-+\theta_1 H^2_+=L^2(\mathbb{R})$.
6. $0<\langle\theta_1 H^2_-, \theta_2 H^2_+\rangle$, $\theta_1 H^2_-+\theta_2 H^2_+=L^2(\mathbb{R})$.

PROOF. To use the obtained tests of invertibility of $P_M|N$, where $M=K_{\theta_1}$, $N=K_{\theta_2}$, we are to calculate the norm $\|P_{\theta_2}|K^\perp_{\theta_1}\|$:

$$\|P_{\theta_2}|K^\perp_{\theta_1}\|=\|P_{\theta_2}|\theta_1 H^2_+\|=\|\theta_2\mathbb{P}_-\bar\theta_2|\theta_1 H^2_+\|=$$

$$=\sup\Big\{\Big|\int_{\mathbb{R}}\bar\theta_2\theta_1 h_1\bar h_2\Big|: h_1\in H^2_+,\ h_2\in H^2_-,\ \|h_i\|\leqslant 1\Big\}=$$

(we use well-known properties of spaces $H^p_+ : H^2_- = \{\bar f : f \in H^2_+\}$; the unit ball of H^1_+ coincides with the set $\{fg : \|f\|_{H^2_+} \leqslant 1,$ $\|g\|_{H^2_+} \leqslant 1\}$; see the sources indicated at the begining of § 1)

$$= \sup\{|\int_{\mathbb{R}} \bar\theta_2 \theta_1 h| : h \in H^1_+,\ \|h_1\| \leqslant 1\} =$$

(the Hahn - Banach theorem) $= \operatorname{dist}(\bar\theta_2\theta_1, H^\infty)$.

So 1 $\Longleftrightarrow$ 2, as was to be proved. The remaining assertions can be obtained by a formal application of the corollary stated above. It is useful to note that $\theta H^2_- = H^2_- \oplus K_\theta$ for any inner function θ. ●

The same arguments lead to the following tests.

THEOREM 3 bis. Let the conditions of Theorem 3 be satisfied. Then the following assertions are equivalent.

1. The operator $P_{\theta_1} : K_{\theta_2} \longrightarrow K_{\theta_1}$ is left-invertible.
2. $\operatorname{dist}(\theta_1\bar\theta_2, H^\infty) < 1$.
3. $0 < \langle \theta_2 H^2_-, \theta_1 H^2_+ \rangle$.
4. $L^2(\mathbb{R}) = \theta_1 H^2_- + \theta_2 H^2_+$. ●

Any reader familiar with the Hankel operators may descry the Hankel operator $H_{\bar\theta_1\theta_2}$ at the right-hand side of the formula

$$P_{\theta_1} | K^\perp_{\theta_2} = \theta_1 \mathbb{P}_- \bar\theta_1 | \theta_2 H^2_+ .$$

This connection of the bases problem with the Hankel (and Toeplitz) operators and with their spectral theory will be very useful. Remind necessary definitions.

Let $L^\infty(\mathbb{R})$ be the space of all bounded measurable functions φ on $\mathbb{R}$ with the natural norm

$$\|\varphi\|_\infty = \operatorname{ess\,sup}_{\mathbb{R}} |\varphi| .$$

DEFINITION. Let $\varphi \in L^\infty(\mathbb{R})$. The T o e p l i t z o p e r a t o r with the symbol φ is the operator T_φ on H^2_+ defined by

$$T_\varphi f = \mathbb{P}_+ \varphi f, \quad f \in H^2_+ .$$

The H a n k e l o p e r a t o r H_φ with the same symbol is defined by the formula

$$H_\varphi f = \mathbb{P}_- \varphi f, \quad f \in H^2_+ .$$

The operators T_φ and H_φ are different parts of the multiplication operator

$$\varphi f = H_\varphi f + T_\varphi f\ ,\ f \in H^2_+ . \tag{2}$$

Now we see that

$$P_{\theta_2} | K^\perp_{\theta_1} = \theta_2 P_- \bar\theta_2 | \theta_1 H^2_+ = \theta_2 H_{\bar\theta_2 \theta_1} \cdot \bar\theta_1 | K^\perp_{\theta_1} \tag{3}$$

and therefore

$$\| P_{\theta_2} | K^\perp_{\theta_1} \| = \| H_{\bar\theta_2\theta_1} \|.$$

Returning to the Theorem 3, one can immediately note that it is reduced to the well-known Nehari theorem.

THEOREM (Z.Nehari [50], [54]). If $\varphi \in L^\infty(\mathbb{R})$, then

$$\| H_\varphi \| = dist\ (\varphi, H^\infty).$$

On the other hand the Hankel operators appearing in Theorems 3 and 3bis have unimodular symbols $\varphi = \theta_1 \bar\theta_2$, $\varphi = \theta_2 \bar\theta_1$. Then it follows from (2) that

$\| H_\varphi \| < 1$ iff T_φ is left-invertible;

$\| H_\varphi \| < 1$, $\| H_{\bar\varphi} \| < 1$ iff T_φ is an invertible operator.

Putting these remarks together with Theorems 3 and 3bis, we obtain the following result.

THEOREM 4. Let θ_j be an inner function for $j = 1,2$. Then the operator $P_{\theta_1} : K_{\theta_2} \longrightarrow K_{\theta_1}$ is an isomorphism (respectively left-invertible) if and only if the Toeplitz operator $T_{\theta_1 \bar\theta_2}$ is invertible (respectively left-invertible). ●

In order to translate now the invertibility of $P_{\theta_1} | K_{\theta_2}$ into "the language of inner functions" θ_1, θ_2 (or returning to exponential and reproducing kernel bases - into the language of the Blachske product B with the zero set $\Lambda = \{\lambda_n : n \in \mathbb{Z}\}$) we can apply the invertibility criteria of the Toeplitz operator theory, and in particular A.Devinatz's - H.Widom's theorem [31], [57], [54]. For its formulation a new portion of definitions is needed.

The first deals with the Hilbert transform in $L^\infty(\mathbb{R})$. The space $L^\infty(\mathbb{R})$ being not contained in $L^2(\mathbb{R})$ it is impossible to extend the Hilbert transform (from $L^2(\mathbb{R})$) by means of the usual Cauchy integral. We shall use the conformally-invariant form to remove the singularity at infinity. Namely, we define the Hilbert transform $\widetilde{v}$ of a function v, $v \in L^\infty(\mathbb{R})$ by

$$\widetilde{v}(x) = \frac{1}{\pi} \,(\text{v. p.}) \int_{\mathbb{R}} \left\{ \frac{1}{x-t} + \frac{t}{1+t^2} \right\} v(t)\,dt .$$

The Schwarz formula

$$V(z) = \frac{1}{\pi i} \int_{\mathbb{R}} \left\{ \frac{1}{t-z} - \frac{t}{1+t^2} \right\} v(t)\,dt$$

recovers the function V by its real part v only, provided $V \in H^\infty$ and $\operatorname{Im} V(i) = 0$.

DEFINITION. A non-negative function w is called a function satisfying Helson - Szegö condition (briefly $w \in (HS)$) if there are functions u, v in $L^\infty(\mathbb{R})$ such that

$$\|v\|_\infty < \pi/2 \quad \text{and} \quad w = \exp\{u + \widetilde{v}\}. \tag{HS}$$

Another form of the Helson -Szegö condition has been obtained in a remarkable paper of B.Muckenhoupt, R.Hunt and R.Wheeden [40]. Let $\mathcal{J}$ be the family of all intervals on $\mathbb{R}$.

THEOREM (R.A.Hunt, B.Muckenhoupt, R.L.Wheeden [40]). The (HS)-condition is equivalent to (A_2)-condition of Muckenhoupt:

$$\sup_{I \in \mathcal{J}} \frac{1}{|I|} \int_I w\,dx \cdot \frac{1}{|I|} \int_I w^{-1}\,dx < \infty. \tag{A_2}$$

THEOREM (A.Devinatz, H.Widom [31], [57]). A Toeplitz operator T_φ with a unimodular symbol φ ($|\varphi| = 1$ a.e.) is invertible if and only if

$$\varphi = e^{i(\widetilde{u} + v + c)}, \text{ where } c \in \mathbb{R};\ u, v \in L^\infty(\mathbb{R}),\ \|v\|_\infty < \pi/2.$$

The next theorem combined with Theorem 4 will be a **key** tool for the proofs of many efficient basis tests.

THEOREM 5. Let φ be a unimodular function. The following conditions are equivalent.

1. The Toeplitz operator is invertible.
2. $\operatorname{dist}_{L^\infty}(\varphi, H^\infty) < 1$, $\operatorname{dist}_{L^\infty}(\overline{\varphi}, H^\infty) < 1$.
3. There is an outer function f, $f \in H^\infty$, satisfying

$$\|\varphi - f\|_\infty < 1.$$

4. There is a branch of the argument α of the unimodular function φ, $\varphi(x) = e^{i\alpha(x)}$, such that

$$\inf\{\|\alpha - \tilde{v} - c\|_\infty : v \in L^\infty(\mathbb{R}),\ c \in \mathbb{R}\} < \pi/2.$$

5. There are a unimodular constant λ and an outer function h such that

$$\varphi = \lambda \cdot \frac{\bar{h}}{h}, \quad |h|^2 \in (HS) \qquad (\text{or } |h|^2 \in (A_2)\ ).$$

To obtain a list of invertibility tests for $P_\theta | K_B$ it remains only to put $\varphi = \bar{B}\theta$ in the condition of the theorem.

Referring the reader to § 2, Part II for the proof of Theorem 5, we mention that the equivalence 1 $\Longleftrightarrow$ 2 has been already proved and the equivalence 1 $\Longleftrightarrow$ 5 is a simple consequence of the A.Devinatz - H.Widom theorem.

4. Basis property of exponentials on an interval

Comparing Theorems 2,2$'$ and 2bis with Theorems 3,3bis, 4 and 5 one can easily obtain a series of tests for the basis property mentioned in the title of the section. Nevertheless, for the convenience of the reader we formulate one of them.

Let $\widetilde{L^\infty} \overset{def}{=} \{\tilde{v} : v \in L^\infty(\mathbb{R})\}$ and let $\widetilde{L^\infty} + \mathbb{C} = \{u + c : u \in \widetilde{L^\infty},\ c \in \mathbb{C}\}$. It is useful to note that non-zero constants can not coincide with $\tilde{v}$, the harmonic continuation of $\tilde{v}$ is vanishing at the point i. For any function f defined on $\mathbb{R}$ let

$$\operatorname{dist}_{L^\infty}(f, \widetilde{L^\infty} + \mathbb{C}) \overset{def}{=} \inf\{\|f - g\|_\infty : g \in \widetilde{L^\infty} + \mathbb{C}\},$$

assuming that $\|f - g\|_\infty = +\infty$ if $f - g \notin L^\infty(\mathbb{R})$.

Let $\Lambda \subset \mathbb{C}_\delta$, $\delta > 0$, and let B be a Blaschke product with the zero set Λ. It is easy to see that the function α_Λ defined by

$$\alpha_\Lambda(x) = 2\int_0^x \sum_{\lambda \in \Lambda} \frac{\operatorname{Im}\lambda}{|\lambda - t|^2}\, dt - ax, \quad x \in \mathbb{R},$$

is a continuous branch of argument, up to an additive constant, of the unimodular function $B\bar{\theta}^a$ on $\mathbb{R}$.

THEOREM 6. Let $\Lambda=\{\lambda_n : n\in\mathbb{Z}\}\subset\mathbb{C}_\delta$, $\delta>0$. Then the family $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ forms an unconditional basis in $L^2(0,a)$ if and only if

$$\Lambda\in(C),\quad \mathrm{dist}_{L^\infty}(\alpha_\Lambda,\widetilde{L^\infty}+C)<\pi/2.$$

The sufficiency part of Theorem 6 is a simple consequence of Theorems 2′, 4 and 5. We put aside the proof of the necessity till §1 of Part III where it will be proved that the function α, arising in Theorem 5 (see assertion 4 of that theorem), is automatically continuous under the conditions of Theorem 6. This will imply, obviously, $\alpha-\alpha_\Lambda\equiv const$.

The M.I.Kadec theorem can be easily obtained as a corollary of Theorem 6. The same reasonings fit in for the proof S.A.Avdonin and V.È.Kacnelson theorems as well; see §2 of Part III.

COROLLARY. Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers and let $\sup_{n\in\mathbb{Z}}|n-\lambda_n|<1/4$. Then the family $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ is a Riesz basis in $L^2(0,2\pi)$.

PROOF. According to our remark on p. 229, we may without loss of generality consider a family of frequencies $(\lambda_n+iy)_{n\in\mathbb{Z}}$, $y>0$. It is clear that the family $(e^{i(n+iy)x})_{n\in\mathbb{Z}}$ is a Riesz basis in $L^2(0,2\pi)$. This example is a good illustration for Theorem 5. Let $\varepsilon=\exp(-2\pi y)$, then

$$\frac{\theta^{2\pi}(z)-\varepsilon}{1-\varepsilon\theta^{2\pi}(z)}\stackrel{\mathrm{def}}{=\!=}B_0(z)=\prod_{n\in\mathbb{Z}}\frac{1-\frac{z}{n+iy}}{1-\frac{z}{n-iy}}\,.$$

We may conclude therefore that

$$B_0\bar\theta^{2\pi}=\frac{\overline{1-\varepsilon\theta^{2\pi}}}{1-\varepsilon\theta^{2\pi}}\,.$$

The function $h=1-\varepsilon\theta^{2\pi}$ is outer and

$$1-\varepsilon\leqslant\inf_{x\in\mathbb{R}}|h(x)|\leqslant\sup_{x\in\mathbb{R}}|h(x)|\leqslant 1+\varepsilon.$$

Therefore statement 5 of Theorem 5 holds and the Toeplitz operator $T_{B\bar\theta^{2\pi}}$ is invertible by that theorem. Obviously, $\mathbb{Z}+iy\in\in(C)$. So the combination of Theorems 2′ and 4 implies among other things the Riesz basis property for the family $(e^{inx}\cdot e^{-y})_{n\in\mathbb{Z}}$ in $L^2(0,2\pi)$. The function $\alpha_{\mathbb{Z}+iy}$, up to an additive constant, is an argument of the unimodular function $B_0\bar\theta^{2\pi}$

This implies

$$\alpha_{\mathbb{Z}+iy}(x) = c + \widetilde{\log|h^2|}(x), \quad c \in \mathbb{R},$$

and $\alpha_{\mathbb{Z}+iy} \in \tilde{L}^\infty + \mathbb{C}$. Moreover $\alpha_{\mathbb{Z}+iy} \in \mathrm{Re}\, H^\infty$ as $\log h^2 \in H^\infty$.

Now we may compare the functions $\alpha_{\mathbb{Z}+iy}$ and $\alpha_{\Lambda+iy}$. Let $\lambda_n = n + \delta_n$, $n \in \mathbb{Z}$. Then

$$\alpha_{\mathbb{Z}+iy}(x) - \alpha_{\Lambda+iy}(x) = 2\sum_{n\in\mathbb{Z}} \left\{ \int_0^x \frac{y}{(t-n)^2+y^2}\,dt - \int_0^x \frac{y}{(t-\lambda_n)^2+y^2}\,dt \right\} =$$

$$= 2\sum_{n\in\mathbb{Z}} \int_{x-\delta_n}^{x} \frac{y}{(t-n)^2+y^2}\,dt - 2\sum_{n\in\mathbb{Z}} \int_{-\delta_n}^{0} \frac{y}{(t-n)^2+y^2}\,dt .$$

It is time to remember that $\delta \overset{\mathrm{def}}{=} \sup_n |\delta_n| < 1/4$. An obvious estimate shows

$$\left|\sum_{n\in\mathbb{Z}} \int_{x-\delta_n}^{x} \frac{y}{(t-n)^2+y^2}\,dt \right| \leq \sum_{n\in\mathbb{Z}} \int_{x-\delta}^{x} \frac{y}{(t-n)^2+y^2}\,dt = \int_{x-\delta}^{x} \sum_{n\in\mathbb{Z}} \frac{y}{(t-n)^2+y^2}\,dt .$$

It remains to show that the right-hand side of the equality is bounded by $\pi/4$ uniformly on $\mathbb{R}$. Very simple reasonings lead to this conclusion. The periodic function

$$t \longmapsto \sum_{n\in\mathbb{Z}} \frac{y}{(t-n)^2+y^2}$$

tends to a constant uniformly in t as $y \to +\infty$. Its integral along the interval $[0,1]$ is π. So the integral along any interval with length smaller than $1/4$ will be smaller than $\pi/4$ if y is sufficiently large.

We may, certainly, use a more formal calculation. By the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} \frac{y}{y^2+(t-n)^2} = \pi \cdot \frac{1-\varepsilon^2}{1-2\varepsilon\cos 2\pi t + \varepsilon^2} ,$$

$\varepsilon = \exp(-2\pi y)$. It is clear that

$$\sup_{x\in\mathbb{R}} \int_{x-\delta}^{x} \sum_{n\in\mathbb{Z}} \frac{y}{y^2+(t-n)^2}\,dt = \pi\cdot\int_{-\delta/2}^{\delta/2} \frac{1-\varepsilon^2}{1-2\varepsilon\cos 2\pi t+\varepsilon^2}\,dt$$

and the right-hand side tends to $\pi\delta < \pi/4$ as $y\to+\infty$. ●

REMARK. See another proof of the Corollary in [19], [18] p.342.

Our next topic concerns the relationship between the bases problem and the theory of entire functions. Entire functions arise in the unconditional bases problem in a natural way. Assuming the family $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ is an unconditional basis in $L^2(0,a)$ we see that the co-dimension of $\operatorname{span}\{e^{i\lambda_n x}\cdot\chi_{[0,a)} : n\in\mathbb{Z}\setminus\{0\}\}$ in $L^2(0,a)$ is equal to 1. By the Hahn - Banach theorem

$$\dim\{f\in L^2(0,a): \int_0^a e^{i\lambda_n x} f(x)\,dx = 0,\ n\in\mathbb{Z}\setminus\{0\}\} = 1. \tag{4}$$

It follows that the Fourier - Laplace transform

$$\hat{f} = \int_0^a e^{izt} f(t)\,dt$$

vanishes exactly on the set $\{\lambda_n : n\in\mathbb{Z}\setminus\{0\}\}$ if the function f, $f\not\equiv 0$ belongs to the one-dimensional subspace considered in (4). Indeed, every zero μ ($\hat{f}(\mu)=0$) not belonging to the set gives rise to a function g belonging to the subspace defined by (4) and not a scalar multiple of f. Indeed, let $\hat{f}(z)\cdot(z-\mu)^{-1} = \hat{g}(z)$, where $g = -ie^{-i\mu x}\int_0^x e^{i\mu s} f(s)\,ds$, $g\in L^2(0,a)$. So the function F_Λ,

$$F_\Lambda = \left(1-\frac{z}{\lambda_0}\right)\int_0^a e^{izt} f(t)\,dt \tag{5}$$

is an entire function of the exponential type a with the zero set $\{\lambda_n : n\in\mathbb{Z}\}$. It follows from (5) that the conjugate diagram of F_Λ is the segment $[0, ia]$ *). Let $\mathcal{E}_a$ denote the set of all entire functions of exponential type

*) The exhausting information about diagrams, and in general about the growth theory, may be found in [13], [27]. In our case a is the length of the interval on which the basis problem is considered.

with the conjugate diagram $[0, ia]$. An entire function of exponential type without zeros coincides with one exponential $\exp \lambda z$, $\lambda \in \mathbb{C}$. Therefore the functions in $\mathcal{E}_a$ are defined by their zero-set up to a multiplicative constant.

DEFINITION. Let $\Lambda \subset \mathbb{C}_+$, $a > 0$. An entire function F_Λ in $\mathcal{E}_a$ is called a g e n e r a t i n g f u n c t i o n for the pair (Λ, a) if its zero set is Λ and if $F_\Lambda(0) = 1$.

THEOREM 7. Let $\Lambda = \{\lambda_n : n \in \mathbb{Z}\} \subset \mathbb{C}_\delta$, $\delta > 0$ and let $a > 0$. The following conditions are equivalent.

1. The family $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ is a Riesz basis in $L^2(0,a)$
2. $\Lambda \in (C)$ and there is a generating function F_Λ for the pair (Λ, a) satisfying $|F_\Lambda|^2 | \mathbb{R} \in (HS)$ (or equivalently $|F_\Lambda|^2 | \mathbb{R} \in (A_2)$).

We shall give now only an idea of the proof, the details may be found in Part III. What we are to prove is the equivalence of the inclusion $|F_\Lambda|^2 | \mathbb{R} \in (HS)$ and of the invertibility of the Toeplitz operator $T_{\bar\theta^a B}$. By Theorem 5 (see the statements 1 and 5) the operator $T_{\bar\theta^a B}$ is invertible if and only if the unimodular function $\bar\theta^a B$ can be factored in a form $\bar\theta^a B = c\bar h h^{-1}$, $|c| = 1$, $c \in \mathbb{C}$, $|h^2| \in (HS)$. This implies the equality

$$Bh = c\theta^a \bar h \tag{6}$$

holds a.e. on $\mathbb{R}$ for the outer function h . It follows from $|h^2| \in (HS)$ by V.I.Smirnov theorem that $h(z+i)^{-1} \in H^2_+$. The equality (6) means that the boundary values of the function Bh analytic in the upper half-plane coincide with the ones of $z \to c\,\theta^a(z)\,\overline{h(\bar z)}$, which is, obviously, analytic in the lower half-plane. Using the inclusion $h(z+i)^{-1} \in H^2_+$ one can easily deduce that the function Bh is a restriction of an entire function F onto $\mathbb{C}_+$. Standard estimates show that $F \in \mathcal{E}_a$. The zero set of F is Λ . We see also that $|F|^2 = |h|^2$ on $\mathbb{R}$. These arguments can be easily converted.

REMARK. The Levin - Golovin theorem (see Introduction for the formulation) is an obvious corollary of Theorem 7.

Let now $\Lambda = \{\lambda_n : n \in \mathbb{Z}\} \subset \mathbb{R}$. It would be pleasant to have a test for the unconditional bases property in terms of this set only. To do this let

$$N_\Lambda(x) = \begin{cases} \operatorname{card} \Lambda \cap [0,x], & x \geqslant 0. \\ -\operatorname{card} \Lambda \cap [x,0), & x < 0. \end{cases}$$

The function N_Λ is non-decreasing on $\mathbb{R}$. An asymptotic property of N_Λ equivalent to the unconditional bases property for the family $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ in $L^2(0,a)$ will be given in terms of the well-known class $BMO(\mathbb{R})$. The space $BMO(\mathbb{R})$ consists of locally integrable functions f on $\mathbb{R}$ satisfying

$$\|f\|_* = \sup_{I\in\mathcal{J}} \frac{1}{|I|}\int_I |f-f_I|\,dx < \infty, \quad f_I \overset{\text{def}}{=} \frac{1}{|I|}\int_I f\,dx.$$

Here $\mathcal{J}$ stands for the family of all intervals on $\mathbb{R}$. An important property of $BMO(\mathbb{R})$ is that this class as well as the class of function satisfying (A_2)-condition, has a completely different description. A function f belongs to BMO iff there are bounded measurable functions u, v such that $f = u + \tilde{v}$. This and other properties of BMO may be found in [44], [54]. If $f \in BMO$ then it follows that

$$\int_{\mathbb{R}} \frac{|f(x)|}{1+x^2}\,dx < +\infty$$

and so every function f in BMO has a harmonic continuation into $\mathbb{C}_+$:

$$\mathcal{U}_f(z) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\operatorname{Im} z}{|t-z|^2} f(t)\,dt.$$

Let symbol $\mathcal{P}_\gamma$ denote the set of all f in BMO satisfying the following condition. There are a positive number y, a real number c and bounded measurable functions u, v such that

$$\mathcal{U}_f(x+iy) = c + \tilde{u}(x) + v(x); \quad x\in\mathbb{R}, \quad \|v\|_\infty < \gamma.$$

THEOREM 8. Let $\lambda_n \in \mathbb{R}$, $n\in\mathbb{Z}$. Then the family $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ forms a Riesz basis in $L^2(0,a)$, $a>0$, iff

1. $\inf_{n\neq m} |\lambda_n - \lambda_m| > 0$;
2. $N_\Lambda - \frac{a}{2\pi}x \in \mathcal{P}_{1/4}$.

The condition 2 of Theorem 8 defines a number a uniquely because the linear function $x \mapsto x$ does not belong to BMO (indeed, $\int_{\mathbb{R}} \frac{|x|}{1+x^2}\,dx = +\infty$).

It is interesting to compare Theorem 8 with known theorems concerning the completeness problem. It follows from the condition

$$\int_{\mathbb{R}} \frac{|N_\Lambda(x) - ax|}{1+x^2}\,dx < +\infty$$

by the Beurling - Malliavin theorem that the family $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ is complete on any interval I , $|I| < a$; see theorem 71 in [51]. We see therefore that the conditions implying the unconditional basis property for a family of exponentials on I are considerably more restrictive than those for the completeness property.

The Kadec theorem may be also proved with the help of Theorem 8. Here is a sketch of the proof. Let $f(x) = N_{\mathbb{Z}}(x) - x$, $x \in \mathbb{R}$. Then the function $x \mapsto \mathcal{U}_f(x+iy)$, $y>0$ belongs to $\widetilde{L}^\infty + \mathbb{C}$. If $\lambda_n = n + \delta_n$ and if $\sup_n |\delta_n| = \delta < 1/4$, then

$$N_{\mathbb{Z}}(x) - N_\Lambda(x) = \sum_{n\in\mathbb{Z}} \operatorname{sign}\delta_n \cdot \chi_{[n,n+\delta_n)}(x).$$

Therefore the Poisson integral of $N_{\mathbb{Z}} - N_\Lambda$ is equal to

$$\sum_{n\in\mathbb{Z}} \int_{x-\delta_n}^{x} \frac{y}{(t-n)^2+y^2}\,dt.$$

The proof is finished as on p. 238 ●

The next result demonstrates the close relationship existing between general unconditional exponential bases and the classical orthogonal system $(e^{inx})_{n\in\mathbb{Z}}$ in $L^2(-\pi,\pi)$. Let $\Lambda = \{\lambda_n : n \in \mathbb{Z}\} \subset \mathbb{C}_+$ and let the family $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ be an unconditional basis in $L^2(-\pi,\pi)$. Let $(h_n)_{n\in\mathbb{Z}}$ be the dual family for $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ in $L^2(-\pi;\pi)$:

$$(e^{i\lambda_k x}, h_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\lambda_k x}\cdot\overline{h_n(x)}\,dx = \begin{cases} 1, & n=k,\\ 0, & n\neq k.\end{cases}$$

Then it is possible to associate to every function f in $L^2(-\pi,\pi)$ the non-harmonic Fourier series

$$f \sim \sum_{n\in\mathbb{Z}} (f,h_n)\,e^{i\lambda_n x}$$

which, in accordance with our assumption, converges unconditionally in L^2 to the function f. However, the question of the pointwise convergence of such a non-harmonic Fourier series is interesting too. It were again R.Paley and N.Wiener who have studied the problem for the first time [59]. After that N.Levinson in his well-known book [48] has proved, assuming $\Lambda \subset \mathbb{R}$, $\sup_n |\lambda_n - n| < 1/4$, that for every function f in $L^2(-\pi,\pi)$

$$\lim_{N\to+\infty}\Big\{\sum_{|n|\le N}\hat{f}(n)e^{inx}-\sum_{|n|\le N}(f,h_n)e^{i\lambda_n x}\Big\}=0$$

uniformly on every compact subset of the interval $(-\pi,\pi)$. Here $\hat{f}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-inx}f(x)\,dx$ stands for usual Fourier coefficients of f. In §4 of Part III this theorem is extended on each family $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$, with $\Lambda\subset\mathbb{C}_+$, which forms an unconditional basis in $L^2(-\pi,\pi)$.

5. Hilbert space geometry of exponentials and reproducing kernels, and the spectral expansion of the model semigroup

Let us return once more to the Carleson condition (C) for the set $\Lambda=\{\lambda_n : n\in\mathbb{Z}\}$, $\lambda_n\in\mathbb{C}_+$. As we have already noted, this condition appeared originally in the papers of L.Carleson [28], W.K.Hayman [66], D.J.Newman [65] as a condition for the solvability of the interpolation problem in H^∞. H.Shapiro and A.Shields proved later that (C) is a necessary and sufficient condition for the following interpolation problem in H^2_+ to be solvable for any given sequence $(a_n)_{n\in\mathbb{Z}}$, $(a_n)_{n\in\mathbb{Z}}\in\ell^2(\mathbb{Z})$:

$$f\in H^2_+;\quad f(\lambda_n)\cdot\sqrt{2\,\mathrm{Im}\,\lambda_n}=a_n .$$

A formal solution of the problem is given by the formula

$$f(z)=\sum_{n\in\mathbb{Z}}\frac{2i\,\mathrm{Im}\,\lambda_n}{z-\bar{\lambda}_n}\cdot\frac{B_n(z)}{B_n(\lambda_n)}\,a_n+Bg,\quad g\in H^2_+ .$$

The series under the condition $\Lambda\in(C)$ turns out to be unconditionally convergent for every $(a_n)_{n\in\mathbb{Z}}\in\ell^2$. The solution f corresponding to $g=0$ has the minimal norm among other solutions and belongs to $K_B=H^2_+\ominus BH^2_+$.

In the paper [20] it was observed that the considered series is the Fourier series expansion with respect to the eigen-functions of the so-called model contractive semigroup. The model semigroup has been thoroughly studied in papers of B.Sz-Nagy, C.Foiaş, V.M.Adamjan, D.Z.Arov, M.G.Krein and others. The semigroup we want to deal with is defined in K_B by the formula

$$Z_t f = P_B \cdot U_t f, \quad f \in K_B, \quad t > 0,$$

where $U_t f(z) = \exp(izt) \cdot f(z)$. The inner function B is named the characteristic function of the semigroup $(Z_t)_{t>0}$.

Spectral properties of $(Z_t)_{t>0}$ are now well-studied, see for example [18]. We mention only that the generator A of a model semigroup, $Z_t = \exp(iAt)$, $t > 0$ is a simple dissipative operator and its spectrum σ coincides with the spectrum of the characteristic function B.

In particular, every simple zero λ_n of B is a simple eigen-value for $A = A_B$ and the corresponding eigen-function is defined by

$$\psi_n(z) = \frac{(2 \operatorname{Im} \lambda_n)^{-1/2}}{z - \overline{\lambda}_n} \cdot \frac{B_{\lambda_n}(z)}{B_{\lambda_n}(\lambda_n)}.$$

If B is a Blaschke product then the family $(\psi_n)_{n \in \mathbb{Z}}$ of eigen-functions of A_B is complete in K_B. The dual system, being the family of eigen-functions for the conjugate operator A_B^*, is defined by

$$\varphi_n(z) = \frac{(2 \operatorname{Im} \lambda_n)^{1/2}}{z - \overline{\lambda}_n},$$

and $A_B^* \varphi_{\lambda_n} = \overline{\lambda}_n \varphi_{\lambda_n}$, $n \in \mathbb{Z}$.

By Theorem A the Carleson condition is a necessary and sufficient condition for $(\varphi_n)_{n \in \mathbb{Z}}$, as well as for $(\psi_n)_{n \in \mathbb{Z}}$ to form an unconditional basis in K_B.

Let now θ denote a singular inner function and let B denote a Blaschke product. The invertibility problem for the operator $P_\theta : K_B \to K_\theta$, which is central for the unconditional basis problem, can be reformulated in terms of model operators. To do this consider the subspace $\operatorname{clos}(K_B + K_\theta)$ in H^2_+.

LEMMA. $\operatorname{clos}(K_B + K_\theta) = K_{B\theta}$.

THE PROOF is an elementary calculation: if $f \perp K_B + K_\theta$ then by definition $f \in BH^2_+ \cap \theta H^2_+ = B\theta H^2_+$. ●

Let A be a model dissipative operator in $K_{B\theta}$ with a characteristic function $B\theta$ and let $(Z_t)_{t\geqslant 0}$:

$$Z_t f = P_K e^{izt} f = e^{iAt} f,$$

be a corresponding semigroup of contractions.

The spaces K_B and K_θ have a well-defined spectral sense.

LEMMA A. The space K_B is the subspace of discrete spectrum for A^* and the space K_θ is the subspace of singular continuous spectrum for A^*. Their orthogonal complements in $K \stackrel{\text{def}}{=} K_{B\theta}$

$$K \ominus K_B = BK_\theta, \quad K \ominus K_\theta = \theta K_B$$

are the spaces of singular continuous spectrum and the space of discrete spectrum for the operator A respectively.

The point discrete spectrum $\sigma_d(A)$ of A coincides with $\Lambda = \{\lambda_n : n \in \mathbb{Z}\}$ and $\bar\Lambda = \{\bar\lambda_n : n \in \mathbb{Z}\} = \sigma_d(A^*)$.

If $\theta = \theta^a$ then the point ∞ belongs to the singular continuous spectrum of the both operators A and A^*.

The interested reader can find the proof of a proposition analogous to the Lemma in [18].

The spectral interpretation of the completeness problem and the unconditional bases problem requires to remind the reader one definition more.

DEFINITION (see [7], p.382). A family of vectors $(\varphi_n)_{n\in\mathbb{Z}}$, $\|\varphi_n\| \asymp 1$ in a Hilbert space is named ω-linearly independent if the conditions

$$\lim_{N\to\infty} \sum_{|n|\leqslant N} a_n \varphi_n = 0, \quad (a_n)_{n\in\mathbb{Z}} \in \ell^2$$

imply $a_n = 0$, $n \in \mathbb{Z}$.

To emphasize the spectral sense of the subspaces K_B and $\theta K_B = K \ominus K_\theta$ we shall use the following notation $E_d^* \stackrel{\text{def}}{=} K_B$, $E_d \stackrel{\text{def}}{=} K \ominus K_\theta$.

Let now $\varphi_n = \frac{(2\operatorname{Im}\lambda_n)^{1/2}}{z-\bar\lambda_n}$ and let $\sup_{n\in\mathbb{Z}} |\theta(\lambda_n)| < 1$. Then it follows from § 2 that $\|P_\theta \varphi_n\| \asymp 1$.

LEMMA. The following statements are equivalent:

1. the family $(P_\theta \varphi_n)_{n\in\mathbb{Z}}$ is complete in K_θ;
2. $K_\theta \cap K_B^\perp = \{\mathbb{O}\}$.

If $\Lambda \in (C)$ then the following statements are equivalent:

3. the family $(P_\theta \varphi_n)_{n\in\mathbb{Z}}$ is a ω-linearly independent;

4. $K_\theta^\perp \cap K_B = \{\mathbb{O}\}$.

PROOF. 1 ⇔ 2. Let $f \in K_\theta \ominus \operatorname{span}(P_\theta \varphi_n : n \in \mathbb{Z})$. Then

$$0 = (P_\theta \varphi_n, f) = (\varphi_n, P_\theta f) = (\varphi_n, f)$$

and therefore $f \perp K_B$.

3 ⟺ 4. The family $(\varphi_n)_{n\in\mathbb{Z}}$ is a Riesz basis in K_B by Theorem A . Therefore for every f in $P_\theta K_B$ one may find a sequence $(a_n)_{n\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$ such that

$$f = \sum_{n\in\mathbb{Z}} a_n P_\theta \varphi_n .$$

On the other hand each sum of such a form is the orthogonal projection of a function in K_B . Therefore the condition $P_\theta f = 0$, $f \in K_B$ appears to be equivalent to $\sum a_n P_\theta \varphi_n = 0$, $(a_n)_{n\in\mathbb{Z}} \in \ell^2$. But the kernel of the operator $P_\theta | K_B$ is $K_B \cap K_\theta^\perp$. ●

LEMMA. The following statements are equivalent:

1. the family $(P_\theta \varphi_n)_{n\in\mathbb{Z}}$ is complete in K_θ;
2. $K = \operatorname{clos}(E_d + E_d^*)$.

If the family of eigen-functions of A (or A^*) forms an unconditional basis in its own span, then the following statements are equivalent:

3. the family $(P_\theta \varphi_n)_{n\in\mathbb{Z}}$ is ω-linearly independent;
4. $E_d \cap E_d^* = \{\mathbb{O}\}$.

PROOF. Apply Lemma A . ●

It is easy to obtain the spectral test for the invertibility of $P_\theta : K_B \to K_\theta$.

LEMMA. The operator $P_\theta : K_B \to K_\theta$ is invertible if and only if

a) $K = \operatorname{clos}(E_d + E_d^*)$;

b) $0 < \langle E_d, E_d^* \rangle$.

The following theorem finds its application in Part IV for the case $\theta = \theta^a$.

THEOREM 9. Let θ be an inner function, B be a Blaschke product. Suppose that the point spectrum $\sigma_p(A)$ of the model operator A defined in $K \overset{\text{def}}{=\!=} K_{B\theta}$ satisfies $\sup_{\lambda\in\sigma_p} |\theta(\lambda)| < 1$ and let eigen-vectors $\left\{ \frac{\theta}{z-\lambda} : \lambda \in \sigma_p(A) \right\}$ of A form an unconditional basis in their span. Then the following conditions are equivalent.

1. The operator $P_\theta : K_B \to K_\theta$ is invertible.

2. The family of reproducing kernels $\{(1-\overline{\theta(\lambda)}\theta)(z-\bar\lambda)^{-1} : \lambda\in\sigma_p(A)\}$ forms an unconditional basis in K_θ.

3. The joint family of eigen-functions for A and A^* forms an unconditional basis in K.

PROOF. The implications $1 \Longleftrightarrow 2$ are a simple corollary of Theorem 2. The statement $1 \Longleftrightarrow 3$ is implied by the spectral test of the invertibility of $P_\theta \mid K_B$. ●

REMARK. Clearly

$$P_B \mid K_\theta = (P_\theta \mid K_B)^*.$$

It follows that the operator $P_\theta \mid K_B$ has a bounded inverse operator if and only if the subspaces of continuous singular spectrum for A, A^* span the space $K = K_{B\theta}$ and form a positive angle.

6. Bases problem in the disc and in the half-plane

In § 1 it was shown that the unconditional exponential bases problem leads to a more general one. By some reasons it is convenient to deal with the general case of reproducing kernels in the setting of Hardy classes in the unit disc $\mathbb{D} = \{\zeta\in\mathbb{C} : |\zeta| < 1\}$. The main purpose of the section is to establish the connection between the Hardy classes theory in the half-plane and that in the disc.

Let $\mathbb{T} \overset{def}{=\!=} \{\zeta\in\mathbb{C} : |\zeta| = 1\}$ denote the unit circle of the complex plane and let $L^2(\mathbb{T})$ be the Hilbert space of all square-summable functions on $\mathbb{T}$ with respect to the normalized Lebesgue measure m on $\mathbb{T}$. The Hardy class $H^2(\mathbb{D})$ is defined as the space of all holomorphic functions g in $\mathbb{D}$ satisfying

$$\sup_{0<r<1} \int_{\mathbb{T}} |g(r\zeta)|^2\, dm(\zeta) < +\infty.$$

By Fatou's theorem the space $H^2(\mathbb{D})$ may be considered as a closed subspace of $L^2(\mathbb{T})$. Let θ be an inner function in $\mathbb{D}$ and let $K_\theta = H^2(\mathbb{D}) \ominus \theta H^2(\mathbb{D})$. The reproducing kernel for $H^2(\mathbb{D})$ being defined by $k(z,\lambda) = (1-\bar\lambda z)^{-1}$, the reproducing kernel for K_θ is equal to

$$k_\theta(z,\lambda)=\frac{1-\overline{\theta(\lambda)}\,\theta(z)}{1-\bar{\lambda}z}\ .$$

Let Λ be a subset of $\mathbb{D}$ satisfying the Blaschke condition

$$\sum_{\lambda\in\Lambda}(1-|\lambda|)<+\infty \tag{B}$$

and let B denote the Blaschke product

$$B=\prod_{\lambda\in\Lambda}\frac{\bar{\lambda}}{|\lambda|}\,\frac{\lambda-z}{1-\bar{\lambda}z}\ .$$

We remind that the Carleson condition for $\mathbb{D}$ has the same form as for $\mathbb{C}_+$. Namely, $\Lambda\in(C)$ if

$$\inf_{\lambda\in\Lambda}|B_\lambda(\lambda)|>0\,,\quad B_\lambda=B\cdot\frac{1-\bar{\lambda}z}{\lambda-z}\ .$$

It also may be split up into two parts; see [18].

THEOREM 10. Let $\Lambda\in(B)$ and let B be the Blaschke product with the zero set Λ. Let θ be an inner function in $\mathbb{D}$ satisfying $\sup\limits_{\lambda\in\Lambda}|\theta(\lambda)|<1$. The following statements are equivalent.

1. The family $\left\{\frac{1-\overline{\theta(\lambda)}\,\theta}{1-\bar{\lambda}z}:\lambda\in\Lambda\right\}$ forms an unconditional basis in $K_\theta=H^2(\mathbb{D})\ominus\theta H^2(\mathbb{D})$.

2. $\Lambda\in(C)$ and the operator P_θ maps K_B isomorphically onto K_θ.

The operator $P_\theta\,|\,K_B$ is invertible iff the Toeplitz operator $T_{\bar{B}\theta}$ does. The tests for the last are given by an analog of Theorem 5; see §3.

In conclusion, some words about the relationship between the Hardy classes in the disc and in the half-plane. Clearly, the operator

$$Uf(x)\overset{\mathrm{def}}{=\!=}\frac{1}{\sqrt{\pi}}\,\frac{1}{x+i}\,f\left(\frac{x-i}{x+i}\right),\quad x\in\mathbb{R},$$

is an isometry of $L^2(\mathbb{T})$ onto $L^2(\mathbb{R})$. Let $\gamma(x)\overset{\mathrm{def}}{=\!=}\frac{x-i}{x+i}$, $x\in\mathbb{R}$. Then it is aesy to check that

$$U\cdot M_\varphi=M_{\varphi\circ\gamma}U,$$

where M_φ stands for the multiplication operator in L^2 and $\varphi \in L^\infty$. It follows from the equality $UH^2(\mathbb{D}) = H^2_+$ that an analogous formula holds for the Hankel and Toeplitz operators. It should be also noted that $UK_\theta = K_{\theta\circ\gamma}$ and that the operator U establishes a one-to-one correspondence between the reproducing kernels of K_θ and those of $K_{\theta\circ\gamma}$. So the unitary operator U allows one to move from the disc into the half-plane and vice versa.

The special condition $\sup_{\lambda\in\Lambda} |\theta(\lambda)| < 1$ imposed onto the pair (Λ,θ) plays the same role as in § 1-4: simplifying the problem it leads to the more elegant formulations. When θ is a function "with a single charged point" this condition does not constitute a real restriction, a linear fractional transformation (linear $z \mapsto z + iy$, $y>0$, when $\theta(z) = e^{iaz}$) of Λ gives a set with the required property. We give also a general criterion for the family to form an unconditional basis. But the criterion being somewhat cumbersome, we prefer not to quote it here (see § 4, Part II).

7. Some remarks concerning the history of the problem

As we already pointed out in Introduction the problem we have discussed goes back to the fundamental book of R.Paley and N.Wiener [59]. It was also mentioned that the problem of Riesz bases of exponentials, as it was posed by R.Paley and N.Wiener, has been solved by M.I.Kadec in [10]. The intermediate result with $\delta < \pi^{-1}\cdot \log 2$ was proved in [34]. The elegant proof of R.Duffin and J.Eachus may be found in the book [16], p.227. For the sake of completeness we represent here, essentially following the N. Levinson's book [48], an example of A.Ingham which shows that the constant $1/4$ in the Kadec theorem can not be increased.

EXAMPLE (A.Ingham). Let $\lambda_0 = 0$, let $\lambda_n = n - 1/4$ if $n>0$, $n\in\mathbb{Z}$, and let $\lambda_n = -\lambda_{-n}$ if $n<0$, $n\in\mathbb{Z}$. Then

$$L^2(0,2\pi) = \operatorname{span}(e^{i\lambda_n x}\cdot\chi_{[0,2\pi)} : n\in\mathbb{Z}\setminus\{0\}).$$

In particular, the family $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ is not minimal in $L^2(0,2\pi)$.

It is sufficient to prove that the generating function F_Λ (which d o e s exist in this case) satisfies

$$\int_{\mathbb{R}} \frac{|F_\Lambda(x)|}{1+x^2}\,dx = +\infty .$$

The last assertion as well as the existence of F_Λ is a consequence of the formula

$$z^{-1}\cdot e^{-i\pi z}\cdot F_\Lambda(z) = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{\lambda_n^2}\right) = c^{-1}\int_{-\pi}^{\pi} e^{izt}\left(\cos\frac{t}{2}\right)^{-1/2} dt,$$

$C \overset{def}{=} \int_{-\pi}^{\pi}\left(\cos\frac{t}{2}\right)^{-1/2} dt$, because the function $t \to \left(\cos\frac{t}{2}\right)^{-1/2}$ does not belong to $L^2(-\pi,\pi)$, although it belongs, obviously, to $\bigcap_{p<2} L^p(-\pi,\pi)$. To prove the formula we are only to check that the zero set of $I(z) = \int_{-\pi}^{\pi} e^{izt}(\cos t/2)^{-1/2}dt$ coincides with $\{\lambda_n : n\in\mathbb{Z}\setminus\{0\}\}$. We have for $n\in\mathbb{Z}$, $n\geqslant 1$:

$$\int_{-\pi}^{\pi} e^{i\lambda_n x}\left(\cos\frac{x}{2}\right)^{-1/2} dx = \int_{-\pi}^{\pi} e^{i\lambda_n x}\cdot\left(\frac{e^{ix/2}+e^{-ix/2}}{2}\right)^{-1/2} dx =$$

$$= \sqrt{2}\int_{-\pi}^{\pi} e^{inx}\left(1+e^{ix}\right)^{-1/2} dx = 0$$

since $(1+z)^{-1/2} \in H^1(\mathbb{D})$. Now we are going to prove that if $I(z)=0$ and if $\operatorname{Re} z \geqslant 0$ then $z=\lambda_n$ for some n in $\mathbb{Z}$. The function $\left(\cos\frac{t}{2}\right)^{-1/2}$ being even this would imply the desired conclusion. By the Taylor formula

$$(1+\zeta)^{-1/2} = \sum_{k=0}^{\infty} (-1)^k \frac{(2k-1)!!}{(2k)!!}\zeta^k , \quad |\zeta|\leqslant 1.$$

Let now $\operatorname{Re} w \geqslant 0$ and let $\lambda \overset{def}{=} w + 1/4$. Then

$$\int_{-\pi}^{\pi} e^{iwt}\left(\cos\frac{t}{2}\right)^{-1/2} dt = \sqrt{2}\int_{-\pi}^{\pi} e^{i\lambda t}\left(1+e^{it}\right)^{-1/2} dt =$$

$$= 2\sqrt{2}\,\sin(\lambda\pi)\cdot\sum_{k=0}^{\infty}\frac{(2k-1)!!}{(2k)!!}\cdot\frac{1}{\lambda+k} .$$

But obviously,

$$\operatorname{Re}\sum_{k=0}^{\infty}\frac{(2k-1)!!}{(2k)!!}\cdot\frac{1}{\lambda+k}=\sum_{k=0}^{\infty}\frac{(2k-1)!!}{(2k)!!}\;\frac{k+\operatorname{Re}\lambda}{|\lambda+k|^2}>0$$

if $\operatorname{Re} w \geqslant 0$. ●

As R.M.Young noted in [60], the condition

$$|\lambda_n - n| < 1/4, \quad n\in\mathbb{Z}$$

is also insufficient for the family $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ to form a Riesz basis in $L^2(0,2\pi)$. This observation is based on the following theorem.

THEOREM (R.Duffin, A.Schaeffer [35]). Let $(\mu_n)_{n\in\mathbb{Z}}$ be a real sequence such that the family $(e^{i\mu_n x})_{n\in\mathbb{Z}}$ forms a Riesz basis in $L^2(0,a)$. Then there exists a positive number δ, $\delta>0$, such that any family $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$, satisfying $\sup_n|\lambda_n-\mu_n|<\delta$ is also a Riesz basis in $L^2(0,a)$.

We obtain in Part III a generalization of this result.

It was B.Ja.Levin who showed the significance of the notion of generating function. Generalizing his definition of a sine type function, see the definition in Introduction, we give the following one.

DEFINITION. An entire function S of exponential type is called a generalized sine type function (briefly $S\in$ GSTF) if all its zeros are in $\mathbb{C}_\delta$ for some δ, $\delta>0$ and if above that

$$0<\inf_{x\in\mathbb{R}}|S(x)|\leqslant\sup_{x\in\mathbb{R}}|S(x)|<+\infty .$$

It is not a difficult task to give an example of GST function whose zeros $\Lambda=\{\lambda_n : n\in\mathbb{Z}\}$ satisfy the condition $\varliminf_{|n|\to\infty}\operatorname{Im}\lambda_n=+\infty$. It appears nevertheless, and this is a subtle result due to S.A. Vinogradov, see §3 of Part III, that there is such an example satisfying in addition $\Lambda\in(C)$. In [14] B.Ja.Levin has proved that a family $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ is a basis in $L^2(0,a)$ if the set $\{\lambda_n : n\in\mathbb{Z}\}$ is separated and if it coincides with the zero set of a STF having the width of the indicator diagram equal to a. V.D.Golovin remarked later that in fact these families are Riesz bases in $L^2(0,a)$, see [5], [6]. Now the Levin - Golovin theorem is a simple concequence of Theorem 7 of the present paper, but at that time it was a fundamental step forward. V.È. Kacnelson has generalized the Levin-Golovin theorem as well as that of Kadec.

THEOREM (V.È. Kacnelson [12]). Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a zero sequence of STF with the width of the indicator diagram equal to a, $a>0$. Let $(\mu_n)_{n\in\mathbb{Z}}$ be a sequence of points in $\mathbb{C}_+$ satisfying

$$\sup_n \operatorname{Im}\mu_n<+\infty, \quad |\operatorname{Re}\lambda_n-\operatorname{Re}\mu_n|\leqslant d\rho_n,$$

where $d<1/4$ and $\rho_n=\inf_{k,k\neq n}|\operatorname{Re}\lambda_k-\operatorname{Re}\lambda_n|$. Let at last $\inf_{n\neq m}|\mu_n-\mu_m|>0$. Then the family $(e^{i\mu_n t})_{n\in\mathbb{Z}}$ is a Riesz basis in $L^2(0,a)$.

This theorem has been strengthened by S.A.Avdonin in [2] and [3]. To formulate his results the next definition is needed.

DEFINITION. Let $\Lambda=\{\lambda_n : n\in\mathbb{Z}\}$ be a separated subset of a strip of a finite width, parallel to the real axis. A partitioning $\Lambda=\bigcup_{k\in\mathbb{Z}}\Lambda_k$ of Λ by some vertical lines into disjoint subsets Λ_k is named an Λ -p a r t i t i o n i n g if the distances ℓ_k between the lines bounding each group Λ_k are uniformly bounded.

THEOREM (S.A.Avdonin [2]). Let Λ be a zero set of STF with the width of the indicator diagram equal to a, $a>0$. Let $(\delta_\lambda)_{\lambda\in\Lambda}$ be a bounded family of complex numbers satisfying

$$\Big|\sum_{\lambda\in\Lambda_j}\operatorname{Re}\delta_\lambda\Big|\leqslant d\ell_j$$

for some Λ-partitioning, where $d<1/4$. Suppose, that the set $\{\lambda+\delta_\lambda\}_{\lambda\in\Lambda}$ is separated. Then the family $(e^{i(\lambda+\delta_\lambda)x})_{\lambda\in\Lambda}$ forms a Riesz basis in $L^2(0,a)$.

A new proof of Kacnelson and Avdonin theorems will be given in §2 of Part III. The paper of Avdonin [2] contains also a theorem very similar to one of the corollaries of our Theorem 7. Let, for the time being, $\mathcal{M}$ denote the set of all positive functions φ defined on $[0,+\infty)$ and such that the function $\gamma(x)=x\cdot\frac{\varphi'(x)}{\varphi(x)}$ satisfies the following conditions

$$|\gamma(x)|\leqslant a<1/2, \quad \gamma'(x)=O\left(\frac{1}{x}\right), \quad x\to+\infty.$$

THEOREM (S.A.Avdonin [2]). Let Λ be a zero set of the entire function F with the width of the indicator diagram equal to a. Suppose that $0<\inf_{\lambda\in\Lambda}\operatorname{Im}\lambda\leqslant\sup_{\lambda\in\Lambda}\operatorname{Im}\lambda<+\infty$ and suppose there is a function φ in $\mathcal{M}$ satisfying

$$0 < \inf_{x\in\mathbb{R}} \frac{|F(x)|}{\varphi(x)} \leqslant \sup_{x\in\mathbb{R}} \frac{|F(x)|}{\varphi(x)} < +\infty .$$

Then the family $(e^{i\lambda t})_{\lambda\in\Lambda}$ is a Riesz basis in $L^2(0,a)$.

The paper [2] contains also some examples which show that the modulus $|F_\Lambda|^2 \mid \mathbb{R}$ satisfying (A_2) can, nevertheless, bahave irregularly.

The problem of unconditional exponential bases is closely connected with the completeness problem and with the spectral theory of Toeplitz operators. It is interesting to note that all machinery needed for the solution of the problem of exponential Riesz bases (as is given by Theorem 6) was ready in the early 60-ies. The papers [43], [45], were especially close to the solution. The paper [43], containing really a characterization of Blaschke products (for the upper half-plane) generating compact Hankel operators $H_{\bar{B}\theta^a}$ for every a, $a>0$, contains also various combinations of all attributes of our description of bases. The same can be said on the paper [32] by R.Douglas and D. Sarason containing sufficient conditions of the completeness of exponentials involving invertibility of the Toeplitz operators $T_{\bar{B}\theta^a}$. Let us mention the paper [49] (indicated to one of us by P.Koosis), where one can find the trick employed in our proof of Kadec's theorem on $1/4$.

On the other hand, the idea of preservation of Riesz bases under some orthogonal projections was formulated (and used for a proof of the Levin-Golovin theorem) by one of us as early as in 1973 in the paper [22].

And in conclusion we indicate the paper [29] where bases of reproducing kernels of spaces K_θ are studied. But these bases are very close to orthogonal (à la Wiener - Paley theorem). This causes strong restrictions imposed on the inner function θ (see also § 5 Part II below). Riesz bases (of exponentials or of reproducing kernels) are connected with the problem of free interpolation by analytic functions (at corresponding knots). Almost every work devoted to exponential bases, beginning from the book by N.Wiener and R.Paley, contains some interpolatory corollaries. One can also find such corollaries in § 7 Part II.

PART II
BASES OF REPRODUCING KERNELS

1. Carleson condition

In §1 Part I we have formulated the general problem concerning unconditional bases composed of reproducing kernels. Now we recall it:

Given a pair (θ, Λ) with θ an inner function in the disc $\mathbb{D}$ and $\Lambda \subset \mathbb{D}$, find necessary and sufficient conditions for the family

$$k_\theta(z,\lambda) = \frac{1-\overline{\theta(\lambda)}\,\theta(z)}{1-\bar{\lambda} z}, \quad \lambda \in \Lambda$$

to be an unconditional basis of K_θ (or of the subspace of K_θ it generates).

This problem generalizes the problem concerning bases of rational fractions (and coincides with it when $\theta = B = \prod_{\lambda\in\Lambda} b_\lambda$), described in §2 Part I.

To link together the problems discussed we need a part of the well-known N.K.Bari theorem on Riesz bases (a proof may be found e.g. in [18], p.172).

THEOREM (N.K.Bari). Let $(\varphi_n)_{n\in\mathbb{Z}}$ be a family of nonzero vectors in a Hilbert space H and set $\psi_n = \varphi_n \|\varphi_n\|^{-1}$, $n \in \mathbb{Z}$. The following assertions are equivalent.

1. The family $(\varphi_n)_{n\in\mathbb{Z}}$ is an unconditional basis of H.
2. The Gram matrix $\{(\psi_n, \psi_m)\}_{n,m\in\mathbb{Z}}$ generates a continuous and invertible operator in the space $\ell^2(\mathbb{Z})$ and $H = \operatorname{span}(\varphi_n)_{n\in\mathbb{Z}}$.

We state now the main result of this section.

THEOREM 1.1. Suppose that the family $\{k_\theta(\cdot,\lambda) : \lambda \in \Lambda\}$ is an unconditional basis in its closed linear span. Then $\Lambda \in (C)$.

PROOF. We shall extract all information we need from the Gram matrix $\Gamma = \{(\psi_n, \psi_m)\}_{n,m\in\mathbb{Z}}$ corresponding in the same way as in N.K.Bari Theorem to the family of functions

$$\varphi_n = \frac{1-\overline{\theta(\lambda_n)}\,\theta(z)}{1-\bar{\lambda}_n z}, \quad n \in \mathbb{Z},$$

$\{\lambda_n : n \in \mathbb{Z}\}$ being an enumeration of Λ. Using the definition of the reproducing kernel, we obtain

$$(\varphi_n,\varphi_m)=\frac{1-\overline{\theta(\lambda_n)}\theta(\lambda_m)}{1-\overline{\lambda_n}\lambda_m}$$

and, in particular, $\|\varphi_n\|_2^2=(1-|\theta(\lambda_n)|^2)(1-|\lambda_n|^2)^{-1}$.
Hence

$$(\psi_n,\psi_m)=\frac{(1-|\lambda_n|^2)^{1/2}(1-|\lambda_m|^2)^{1/2}}{1-\overline{\lambda_n}\lambda_m}:\frac{(1-|\theta(\lambda_n)|^2)^{1/2}(1-|\theta(\lambda_m)|^2)^{1/2}}{1-\overline{\theta(\lambda_n)}\theta(\lambda_m)}.$$

Note that the absolute value of the divisor in the right-hand side of the last formula is less than 1:

$$\frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{w}z|^2}=1-\left|\frac{w-z}{1-\overline{w}z}\right|^2.$$

Let $\{e_n\}_{n\in\mathbb{Z}}$ be the standard unit vector basis in $\ell^2(\mathbb{Z})$: $e_n(k)=0$ for $k\neq n$, $e_n(n)=1$. The fact that the Gram matrix defines a bounded operator in $\ell^2(\mathbb{Z})$ implies the inequality

$$\sum_{n\in\mathbb{Z}}|(\psi_n,\psi_m)|^2=\|\Gamma e_m\|^2\leqslant\|\Gamma\|^2<\infty,$$

from which it follows in view of the preceding remarks that

$$\sup_{m\in\mathbb{Z}}\sum_n\frac{(1-|\lambda_n|^2)(1-|\lambda_m|^2)}{|1-\overline{\lambda_n}\lambda_m|^2}\leqslant\|\Gamma\|^2<\infty.$$

But the last condition is necessary and sufficient for the measure $\sum_n(1-|\lambda_n|)\delta_{\lambda_n}$ to be a Carleson one (for the proof see [18] or [44])*).

Let us check now the rarity condition. If $(\varphi_n)_{n\in\mathbb{Z}}$ is an unconditional basis in H then the normed family $(\psi_n)_{n\in\mathbb{Z}}$ is uniformly disjoint (i.e. $\inf\{\|\psi_n-\psi_m\|:n\neq m\}>0$), and, consequently, $\sup_{n\neq m}|(\psi_n,\psi_m)|^2=\gamma<1$. In the

*) It should be noted that the Carleson condition (C), as well as the rarity condition (R) and the condition that the corresponding measure is a Carleson one may be transferred from the half-plane $\mathbb{C}_+$ to the disc $\mathbb{D}$ by means of conformal mapping. The equivalence (C) $\Longleftrightarrow$ (CM) & (R) still holds in $\mathbb{D}$, cf. §2.6 of Part I for the details.

case we examine this inequality may be rewriten as follows:

$$\sup_{n\neq m}\left(1-\left|\frac{\lambda_n-\lambda_m}{1-\overline{\lambda_n}\lambda_m}\right|^2\right)\left(1-\left|\frac{\theta(\lambda_n)-\theta(\lambda_m)}{1-\overline{\theta(\lambda_n)}\,\theta(\lambda_m)}\right|^2\right)^{-1}=\gamma<1 .$$

This implies that $\inf_{n\neq m}\left|\frac{\lambda_n-\lambda_m}{1-\overline{\lambda_n}\lambda_m}\right|\geqslant 1-\gamma$ and hence $(\lambda_n)_{n\in\mathbb{Z}}$ satisfies the rarity condition ($\mathcal{R}$). ●

Let P_θ be the orthogonal projection onto the space K_θ. Theorem 1.1 shows that each unconditional basis of the form $\{k_\theta(\cdot,\lambda):\lambda\in\Lambda\}$ in K_θ is necessarily the image under P_θ of some unconditional basis consisting of rational fractions (namely, the basis $\{(1-\overline{\lambda}z)^{-1}:\lambda\in\Lambda\}$ in K_B).

Let us assume now that P_θ does not distort very much the norms of the rational fractions:

$$\sup_{\lambda\in\Lambda}\|(1-\overline{\lambda}z)^{-1}\|_{H^2}\cdot\|P_\theta(1-\overline{\lambda}z)^{-1}\|_{H^2}^{-1}<\infty . \text{*)}$$

Since $\|(1-\overline{\lambda}z)^{-1}\|^2_{H^2}=(k(\cdot,\lambda),k(\cdot,\lambda))=(1-|\lambda|^2)^{-1}$ and $\|P_\theta(1-\overline{\lambda}z)^{-1}\|^2_{H^2}=k_\theta(\lambda,\lambda)=(1-|\theta(\lambda)|^2)(1-|\lambda|^2)^{-1}$, the last condition is equivalent to the following inequality:

$$\sup_{\lambda\in\Lambda}|\theta(\lambda)|<1 . \tag{1}$$

This inequality means that (a) the poles of the rational fractions $(1-\overline{\lambda}z)^{-1}$, $\lambda\in\Lambda$ can accumulate only to the spectrum of θ on $\mathbb{T}$ (i.e. to the set $\{\zeta\in\mathbb{T}:\lim_{\xi\to\zeta}|\theta(\xi)|=0\}$); and, moreover, (b) this accumulation must be in a sense nontangential with respect to the unit circle. We shall see later that the condition (a) is implied by the fact that the functions $\{k_\theta(\cdot,\lambda):\lambda\in\Lambda\}$ form an unconditional basis of the space they generate (see corollary 4.2 and its comments, page 268 and §6 p. 276).

THEOREM 1.2. Suppose that the pair (θ,Λ) satisfies condition (1). Then the following assertions are equivalent.

1. The family $\{k_\theta(\cdot,\lambda):\lambda\in\Lambda\}$ is an unconditional basis in K_θ (resp., in the subspace of K_θ it generates).

2. $\Lambda\in(C)$ and $P_\theta\,|\,K_B$ is an isomorphism of K_B onto K_θ (resp., of K_B onto $P_\theta(K_B)$).

PROOF follows the same lines as the proof of Theorem 2 (Part 1, §2). Here is its shortened version.

1$\Rightarrow$2. Theorem 1.1 implies that $\Lambda\in(C)$. In view of Theorem A (cf. Part 1, §2) the fractions $k(\cdot,\lambda)$,$\lambda\in\Lambda$ form an

*) From now on $H^2\overset{\mathrm{def}}{=}H^2(\mathbb{D})$.

unconditional basis in K_B . Combining this with (1) we obtain that $P_\theta \mid K_B$ is an isomorphism.

Implication 2 $\Longrightarrow$ 1 is a consequence of Theorem A and inequality (1). ●

2. Projecting onto K_θ and Toeplitz operators

The condition " $P_\theta | K_B$ is an isomorphism onto its image" may be restated in geometric terms. To do this we need some notations and definitions.

Given a closed subspace M of a Hilbert space H we denote by $M^\perp$ the orthogonal complement to M and by P_M the orthogonal projection of H onto M.

By the angle between two subspaces M and N we mean a number (denoted $\langle N, M\rangle$) uniquely determined by $\langle N, M\rangle \in [0, \frac{\pi}{2}]$ and

$$\cos\langle N, M\rangle = \sup\{|(n,m)| : \|n\| = \|m\| = 1,\ n\in N,\ m\in M\}.$$

Clearly $\cos\langle N,M\rangle = \sup\{\|P_M x\| : \|x\| = 1,\ x\in N\} =$
$= \|P_M \mid N\| = \|P_N \mid M\| = \|P_N P_M\|$ and

$$\inf\{\|P_M n\|^2 : n\in N, \|n\|=1\} = 1 - \sup\{\|P_{M^\perp} n\|^2 : n\in N, \|n\|=1\} = \sin^2\langle N, M^\perp\rangle. \quad (*)$$

Let M, N be two subspaces of H with $M\cap N = \{0\}$. Define a (possibly discontinuous) projection $\mathcal{P}_{M\|N}$ on $M+N$ by

$$\mathcal{P}_{M\|N}(m+n) = m \qquad (m\in M,\ n\in N).$$

We call it the projection onto M along N ($\mathcal{P}^2 = \mathcal{P};\ \mathcal{P}|M = I;\ \mathcal{P}|N = \mathbb{O}$) . It follows from the closed graph theorem that this projection is continuous if and only if $M+N$ is closed. Also we have

$$\sin\langle M, N\rangle = \inf_{x\in M} \frac{\|(I-P_M)x\|}{\|x\|} = \|\mathcal{P}_{M\|N}\|^{-1}.$$

LEMMA 2.1. Let M and N be closed subspaces of a Hilbert space H . The following assertions are equivalent.

1. $Ker(P_M|N)=\{\mathbb{O}\}$.
2. $M^{\perp}\cap N=\{\mathbb{O}\}$.
3. $clos(M+N^{\perp})=H$.
4. $clos\, P_N M=N$.

The following assertions are also equivalent.

1a. $P_M|N$ is an isomorphism (onto its image).
2a. $\cos\langle N,M^{\perp}\rangle<1$.
3a. $<N,M^{\perp}>\;>0$.

Finally, $P_M|N$ is an isomorphism of N o n t o M if and only if any of the following (equivalent) conditions is satisfied.

1b $\cos\langle N,M^{\perp}\rangle<1$; $\cos\langle N^{\perp},M\rangle<1$.
2b. $H=N+M^{\perp}$ and $N\cap M^{\perp}=\{\mathbb{O}\}$.
3b. $clos(N+M^{\perp})=H$, $\|\mathcal{P}_{N\|M^{\perp}}\|<+\infty$.

PROOF of the lemma is routine, but we include it for the sake of completeness.

The equivalence of the first four assertions follows immediately from the equality $(P_M:N\to M)^*=(P_N:M\to N)$ and the fact that $Ker\,A=\{\mathbb{O}\}\Longleftrightarrow clos\,A^*H=H$.

Implications 1a$\Longleftrightarrow$2a follow from the formula ($\overset{*}{1}$) and implications 2a$\Longleftrightarrow$3a are evident.

To prove the third part of the Lemma use once more the fact that $(P_M|N)^*=P_N|M$ and apply the Banach theorem (an operator is onto if and only if the conjugate operator is an isomorphic imbedding). ●

COROLLARY 2.2. Let θ and B be inner functions. The following assertions are equivalent.

1. $P_\theta|K_B$ is an isomorphism onto its image..
2. $\cos\langle K_B,\theta H^2\rangle<1$.
3. $\|\mathcal{P}_{K_B\|\theta H^2}\|<\infty$.

The operator P_θ maps isomorphically K_B o n t o K_θ iff any of the following equivalent conditions is satisfied:

1a. $\cos\langle K_B,\theta H^2\rangle<1$, $\cos\langle K_\theta,BH^2\rangle<1$.
2a. $H^2=K_B+\theta H^2$, $K_B\cap\theta H^2=\{\mathbb{O}\}$.
3a. $clos(\bar{B}H^2_-+\theta H^2)=L^2(\mathbb{T})$, $\|\mathcal{P}_B^\theta\|<\infty$,

where $\mathcal{P}_B^\theta=\mathcal{P}_{BH^2_-\|\theta H^2_-}$.

PROOF. Apply Lemma 2.1 with $N=K_B$, $M=K_\theta$. When treating the condition 3a one needs to keep in mind that $K_B+H^2_-=\bar{B}H^2_-$. ●

It is easy to compute the number $\cos<K_B,\theta H^2>$ using the following well-known fact: every function g in the Hardy class H^1 can be represented in the form $g=h_1\cdot h_2$ with

$h_1, h_2 \in H^2$ and $\|h_1\|_2^2 = \|h_2\|_2^2 = \|g\|_1$.

LEMMA 2.3. Let φ be a unimodular function on $\mathbb{T}$. Then

$$\cos\langle H_-^2, \varphi H^2\rangle = \mathrm{dist}_{L^\infty}(\varphi, H^\infty),$$

and, in particular, $\cos\langle K_B, \theta H^2\rangle = \mathrm{dist}(\bar{B}\theta, H^\infty)$.

PROOF. $\cos\langle H_-^2, \varphi H^2\rangle =$

$$= \sup\{|\int_{\mathbb{T}} \varphi h_+ \bar{h}_- \,dm| : \|h_\pm\|_2 \le 1,\ h_\pm \in H_\pm^2\} =$$

$$= \sup\{|\int_{\mathbb{T}} \varphi h\,dm : h \in H^1,\ \|h\|_1 \le 1,\ \hat{h}(0) = 0\} = \mathrm{dist}(\varphi, H^\infty).$$

$$\cos\langle K_B, \theta H^2\rangle = \cos\langle BH_-^2, \theta H^2\rangle = \cos\langle H_-^2, \bar{B}\theta H^2\rangle. \quad \bullet$$

The first assertion of Lemma 2.3 essentially coincides with Z.Nehari theorem mentioned in Part I.

We have already pointed out (Part I, §4) that it is possible to obtain Theorem 5 combining well-known theorems of Helson - Szegö andDevinatz - Widom. A proof of Theorem 5 may be found in [18] or extracted from lectures [54]. However, we present here a proof of this theorem to make the exposition selfcontained. This proof is also of interest by another reason: it enables us to consider the Helson-Szegö theorem from a new view-point (as a theorem describing a special class of unimodular functions; see, however, [1] in connection with this view-point). Keeping in mind the unitary equivalence of the Toeplitz operators in the disc and in the half-plane mentioned in Part 1, §6 we shall prove the analog of Theorem 5 for $\mathbb{D}$.

To begin with, we introduce two definitions. If $v \in L^\infty(\mathbb{T})$ then $\tilde{v}$ stands for the harmonic conjugate of v $(\int_{\mathbb{T}} \tilde{v}\,dm = 0)$. From now on we assume all functions from $L^1(\mathbb{T})$ to be harmonically extended into $\mathbb{D}$, a function and its extension being denoted by the same letter. So for a real function v its harmonic conjugate $\tilde{v}$ is uniquely determined by $v + i\tilde{v} \in H^2(\mathbb{D})$ and $\tilde{v}(0) = 0$.

DEFINITION. Let h be an outer function in $H^2(\mathbb{D})$; h is said to satisfy the *Helson-Szegö condition* if there are $u, v \in L^\infty(\mathbb{T})$ with

$$|h^2| = \exp(u + \tilde{v}), \quad \|v\|_\infty < \frac{\pi}{2}. \tag{HS}$$

DEFINITION. A unimodular function φ on $\mathbb{T}$ is called a *Helson-Szegö function* if there are a constant λ , $|\lambda|=1$ and an outer function h satisfying Helson - Szegö condition, such that

$$\varphi=\lambda\frac{\bar h}{h} .$$

THEOREM 5D. Let φ be a unimodular function on $\mathbb{T}$. The following assertions are equivalent.

1. The Toeplitz operator T_φ is invertible.
2. $\mathrm{dist}_{L^\infty}(\varphi,H^\infty)<1$, $\mathrm{dist}_{L^\infty}(\bar\varphi,H^\infty)<1$.
3. There exists an outer function f in $H^\infty=H^\infty(\mathbb{D})$ such that $\|\varphi-f\|_\infty<1$.
4. There exists a Lebesgue measurable branch α of the argument of φ (i.e. $\varphi(\zeta)=e^{i\alpha(\zeta)}$, $\zeta\in\mathbb{T}$) satisfying

$$\mathrm{dist}_{L^\infty}(\alpha,\widetilde{L^\infty(\mathbb{T})}+\mathbb{C})<\frac{\pi}{2} .$$

5. φ is a Helson - Szegö function.

Some details of the proof of this theorem are of independent interest, and so we begin just with them.

LEMMA 2.4. (R.Douglas [54]). Let $\varphi\in L^\infty(\mathbb{T})$, $|\varphi|=1$ a.e. Then the Toeplitz operator T_φ is an isomorphism (onto its image) if and only if $\|H_\varphi\|=\mathrm{dist}(\varphi,H^\infty)<1$.

PROOF. If $f\in H^2$ then clearly

$$\varphi f=H_\varphi f+T_\varphi f\ ,\quad \|f\|^2=\|H_\varphi f\|^2+\|T_\varphi f\|^2,$$

and the result follows. ●

Let $0<\gamma\le 1$. Set

$$A_\gamma=\{\zeta\in\mathbb{C}:|\arg\zeta|<\pi\gamma\} .$$

LEMMA 2.5. 1. If $F\in H^\infty$ and the essential image $F(\mathbb{T})$ of the circle $\mathbb{T}$ is contained in the angle $A_\gamma(0<\gamma\le 1)$ then $F(\mathbb{D})\subset A_\gamma$.

2. If F is analytic in $\mathbb{D}$ and $F(\mathbb{D})\subset A_\gamma$ then F is outer and $F\in H^p$, $p<(2\gamma)^{-1}$.

PROOF. 1. Following J.B.Garnett ([44], p.632, [36], p.199-200), suppose that there exists a point z_0 in $\mathbb{D}$ with $w_0=F(z_0)\notin A_\gamma$. Construct a polynomial P so that

$$P(w_0)=1,\quad \sup_{w\in F(\mathbb{T})}|P(w)|<1/2 .$$

Then $P(F(z_0)) = 1$, but boundary values of the function $P \circ F$ on $\mathbb{T}$ are almost everywhere less than $1/2$. This contradicts the maximum modulus principle.

2. Since $F(\mathbb{D}) \subset A_\gamma$, F has no zeros in $\mathbb{D}$, for otherwise 0 would be an interior point of $F(\mathbb{D})$. Consider the function $f = F^{1/2\gamma}$. Clearly $\operatorname{Re} f \geqslant 0$ in $\mathbb{D}$ and hence f is an outer function (one of numerous well-known ways to see this is as follows: if $\varepsilon > 0$, then $f + \varepsilon$ is evidently an outer function for it is bounded away from zero in $\mathbb{D}$; hence $\log|f(0)+\varepsilon| = \int_{\mathbb{T}} \log|f(\zeta)+\varepsilon|\,dm$ and it suffices to pass to limit, as $\varepsilon \to 0$, using monotone convergence theorem). Consequently the function F is also outer.

The remaining part of the second assertion is due to V.I.Smirnov and is widely known. Here is a proof. If $p < (2\gamma)^{-1}$ then there is a constant C so that $w \in A_{p\gamma} \Longrightarrow |w| \leqslant C \operatorname{Re} w$. Therefore $|F(\zeta)|^p \leqslant C \operatorname{Re} F(\zeta)^p$, $\zeta \in \mathbb{D}$,and, consequently,

$$\int_{\mathbb{T}} |F(r\zeta)|^p dm(\zeta) \leqslant C \int_{\mathbb{T}} \operatorname{Re} F(r\zeta)^p dm(\zeta) = C \operatorname{Re} F(0)^p. \quad \bullet$$

LEMMA 2.6. If the assertion 2 of Theorem 5D is fulfilled then the set $\{ f \in H^\infty(\mathbb{D}) : \|\varphi - f\|_\infty < 1 \}$ consists entirely of outer functions.

PROOF. Let $f, g \in H^\infty(\mathbb{D})$ and

$$\| 1 - \overline{\varphi} f \|_\infty < 1, \qquad \| 1 - \varphi g \|_\infty < 1.$$

These inequalities imply that all values of the functions $\overline{\varphi} f \,|\, \mathbb{T}$, $\varphi g | \mathbb{T}$ lie in A_γ for some γ , $\gamma < 1/2$ and so $fg(\mathbb{T}) \subset A_{2\gamma}$. By Lemma 2.5 fg is an outer function and hence f, g are also outer. $\bullet$

PROOF OF THEOREM 5D. $1 \Longleftrightarrow 2$ by Lemma 2.4, $2 \Longrightarrow 3$ by Lemma 2.6.

$3 \Longrightarrow 4$: Let f be an outer function with $\|\varphi - f\|_\infty = \gamma < 1$. There exists a number λ , $|\lambda| = 1$ such that

$$f | \mathbb{T} = \lambda \exp(\log|f| + i \widetilde{\log|f|}) .$$

The values of the function $\overline{\varphi} f \,|\, \mathbb{T}$ lie in the angle $A_{(\arcsin \gamma)/\pi}$ and so there exists a unique real-valued function α with $\overline{\lambda}\varphi = \exp i\alpha$ and $\| \alpha - \widetilde{\log|f|} \|_\infty < \pi/2$

$4 \Longrightarrow 5$. If $\varphi = \exp i\alpha$ and $\alpha = c + \widetilde{u} + v$ with

$c \in \mathbb{R}$, $u \in L^\infty(\mathbb{T})$, $u(0) = 0$, $\|v\|_\infty < \pi/2$, then we set $\lambda = \exp ic$ and find an outer function h from the equation

$$\widetilde{\log |h^2|} = -\tilde{u} - v + v(0) .$$

We have then $\log |h^2| = -u + \tilde{v}$. Since $\|v\|_\infty < \pi/2$, Lemma 2.5 implies that $\exp(\tilde{v} - iv) \in H^2(\mathbb{D})$, hence $h \in L^2(\mathbb{T})$ and, consequently, h satisfies the Helson - Szegö condition. The formula $\varphi = \lambda \cdot \bar{h}/h$ follows from the construction.

5⇒2. Suppose $\varphi = \bar{h}/h$ with h satisfying the Helson - Szegö condition. Then $\log |h^2| = u + \tilde{v}$, $\widetilde{\log |h^2|} = \tilde{u} + v(0) - v$ and hence $\varphi = \exp(-i(\tilde{u} + v(0) - v))$, where $\|v\|_\infty < \pi/2$. Set

$$f_\varepsilon = \varepsilon e^{-iv(0)} e^{-u - i\tilde{u}} , \ \varepsilon > 0 .$$

Then $f_\varepsilon \in H^\infty$, $f_\varepsilon^{-1} \in H^\infty$. We have:

$$\|\varphi - f_\varepsilon\|_\infty = \|1 - \bar{\varphi} f_\varepsilon\|_\infty = \|1 - |f_\varepsilon| e^{-iv}\|_\infty < 1 ,$$

provided ε is sufficiently small, because $f_\varepsilon \in H^\infty$ and $\|v\|_\infty < \frac{\pi}{2}$. Similarly, $\|\bar{\varphi} - f_\varepsilon^{-1}\| < 1$. ●

REMARKS. 1. Lemma 2.6 and implication 3 ⇒ 2 show that the set $\{f \in H^\infty : \|\varphi - f\|_\infty < 1\}$ either does not intersect the set of outer functions or is contained in it.

2. The famous Helson-Szegö theorem stated below may be easily derived from Theorem 5D.

THEOREM (H.Helson, G.Szegö [38]). Let $w \in L^1(\mathbb{T})$, $w \geq 0$. Then the Riesz projection $\mathbb{P}_+$ ($\mathbb{P}_+(\sum_{n \in \mathbb{Z}} a_n z^n) \overset{def}{=} \sum_{n \geq 0} a_n z^n$) is continuous in the weighted space $L^2(\mathbb{T}, w) = \{f : \int_{\mathbb{T}} |f|^2 w \, dm < \infty\}$ if and only if $w \in (HS)$.

Indeed, the assertion that $\mathbb{P}_+$ is continuous is equivalent to the assertion 2 of Theorem 5D with $\varphi = \bar{h}/h$, h being an outer function satisfying $h \in H^2$, $|h|^2 = w$. ●

Theorem 1.2 combined with Theorems 4 and 5D enables us to list many useful necessary and sufficient conditions for a family of reproducing kernels $(k_\theta(\cdot, \lambda))_{\lambda \in \Lambda}$ to be a basis of the space K_θ . To obtain criteria for such a family to be a basis in its closed linear span, Theorems 1.2 and 2 bis (Part I) and Lemma 2.4 can be used.

3. A criterion in terms of the model operators

Using the implications $1 \Longleftrightarrow 2$ of Theorem 5D and a formula relating Hankel operators and the Functional model, it is possible to add to equivalent assertions 1-5 of Theorem 5D another one expressed in Functional model terms.

Let θ be an inner function and let S stand for the operator of multiplication by z in H^2 (z being the identity function: $z(\zeta) = \zeta$). Consider the model operator

$$T_\theta \stackrel{def}{=} P_\theta S \mid K_\theta .$$

It is well known that this operator admits an H^∞- functional calculus:

$$f(T_\theta) = P_\theta f(S) \mid K_\theta , \quad f \in H^\infty .$$

We have also

$$f(T_\theta) P_\theta = \theta H_{\bar\theta f} .$$

This formula and some of its applications can be found in [18]. Substituting in it $f = B$ we obtain that $P_\theta \mid K_B$ is an isomorphism of K_B onto K_θ if and only if

$$\|\theta(T_B)\| < 1 \qquad \text{and} \ \|B(T_\theta)\| < 1 . \tag{2}$$

Similarly, $P_\theta \mid K_B$ is an isomorphism of K_B onto $P_\theta(K_B)$ if and only if

$$\|\theta(T_B)\| < 1 \tag{3}$$

(combine implications $1 \Longleftrightarrow 2$ in Theorem 5D, theorem of Z. Nehari in § 3 of Part 1 and Lemma 2.4).

Here is a consequence of these assertions.

THEOREM 3.1. Let $\Lambda = \{\lambda_n : n \in \mathbb{Z}\} \subset \mathbb{C}_+$. Suppose $\Lambda \in (C)$ and $\lim_{n\to+\infty} \operatorname{Im} \lambda_n = +\infty$. Then for every positive number a the family of exponents $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ is an unconditional basis in the subspace of $L^2(0,a)$ it generates. The deficiency of this subspace in $L^2(0,a)$ is infinite.

This theorem is a special case of the following one.

THEOREM 3.2.. Let θ be an inner function in $\mathbb{D}$ and let $\theta = B\cdot S$ be the canonical factorization of θ. Let Λ be a subset of $\mathbb{D}$ satisfying the Carleson condition and also the condition $\lim\limits_{\lambda\in\Lambda, |\lambda|\to 1} |\theta(\lambda)| = 0$. Then the following assertions holds.

1. There exists a subset Λ' of Λ with $card(\Lambda\setminus\Lambda') < \infty$ so that the family $\{k_\theta(\cdot,\lambda): \lambda\in\Lambda'\}$ is an unconditional basis of its closed linear span.

2. If $S \neq const$ then $\Lambda' = \Lambda$ and $dim(K_\theta \ominus span\{k_\theta(\cdot,\lambda): \lambda\in\Lambda\}) = \infty$.

PROOF. Assertion 1 is almost immediate. Observe that the rational fractions $\{(1-\bar\lambda z)^{-1}: \lambda\in\Lambda\}$ form an unconditional basis and $\theta(T_{B'})^*(1-\bar\lambda z)^{-1} = \overline{\theta(\lambda)}\,(1-\bar\lambda z)^{-1}$, $\lambda\in\Lambda'$, where B' is the Blaschke product corresponding to the set Λ'. From this follows the inequality

$$\|\theta(T_{B'})\| \leqslant const \sup_{\lambda\in\Lambda'} |\theta(\lambda)|,$$

the right-hand side of which is strictly less than 1 for an appropriate choice of Λ', $card(\Lambda\setminus\Lambda') < \infty$.

The essence of assertion 2 is given by the following argument. Set $\theta_\alpha \overset{def}{=\!=} BS^\alpha$, $\alpha > 0$. We still have $\lim\limits_{\lambda\in\Lambda, |\lambda|\to 1} |\theta_\alpha(\lambda)| = 0$. Hence an application of assertion 1 shows that for some $\Lambda'\subset\Lambda$ with $card(\Lambda\setminus\Lambda') < \infty$ the family $\{k_{\theta_\alpha}(\cdot,\lambda): \lambda\in\Lambda'\}$ forms an unconditional basis in its closed linear span. But if $\alpha' < \alpha$ then $K_{\theta_{\alpha'}} \subset K_{\theta_\alpha}$, $dim(K_{\theta_\alpha} \ominus K_{\theta_{\alpha'}}) = \infty$ and $K_{\theta_{\alpha'}} = P_{\theta_{\alpha'}} K_\theta$. The rest is contained in two elementary lemmas (the first one to be applied to $A = P_{\theta_{\alpha'}} \mid K_{\theta_\alpha}$).

LEMMA 3.3. Let X, Y be linear topological spaces and let A be a continuous linear map from X to Y. If $(x_n)_{n\geqslant 1}$ is a basis in $span_X\{x_n: n\geqslant 1\}$ and $(Ax_n)_{n\geqslant 1}$ is a basis in $span_Y\{Ax_n: n\geqslant 1\}$ then

$$codim\ span_X\{x_n: n\geqslant 1\} \geqslant dim\ Ker A.$$

PROOF. Note that A is one-to-one on the space $span_X\{x_n: n\geqslant 1\}$. ●

LEMMA 3.4. Let an inner function θ and two subsets Λ, Λ_1 of $\mathbb{D}$ satisfy

$$\Lambda\cap\Lambda_1 = \emptyset,\ dim(K_\theta \ominus span\{k_\theta(\cdot,\lambda): \lambda\in\Lambda\}) \geqslant card\,\Lambda_1$$

and suppose Λ_1 is finite. Then

$$\mathrm{span}\{k_\theta(\cdot,\lambda):\lambda\in\Lambda_1\}\cap \mathrm{span}\{k_\theta(\cdot,\lambda):\lambda\in\Lambda\}=\{\mathbb{0}\}.$$

PROOF. It is sufficient to consider the case $\mathrm{card}\,\Lambda_1=1$ (i.e. to check that $k_\theta(\cdot,\mu)\notin \mathrm{span}\{k_\theta(\cdot,\lambda):\lambda\in\Lambda\}$ provided $\mu\notin\Lambda$ and $\mathrm{span}\{k_\theta(\cdot,\lambda):\lambda\in\Lambda\}\neq K_\theta$). Indeed, an induction by the number of the nonzero summands in $\sum_{\lambda_1\in\Lambda_1} c_{\lambda_1} k_\theta(\cdot,\lambda_1)$ enables us to reduce the Lemma to this particular case. But the "base of induction" we need is immediate: if $f\in K_\theta$, $f\perp k_\theta(\cdot,\lambda)$, $\lambda\in\Lambda$, $f\not\equiv 0$, and if n is the multiplicity of zero of f at a point μ , $\mu\notin\Lambda$ then the function g , $g\overset{def}{=\!=}P_+\bar{b}_\mu^n f=\bar{b}_\mu^n f$ (as earlier, $b_\mu(z)=$ $=\frac{\mu-z}{1-\bar{\mu}z}\,\frac{|\mu|}{\mu}$), belongs to K_θ, $g(\mu)\neq 0$ and $g\,|\,\Lambda\equiv 0$. This means that $k_\theta(\cdot,\mu)\notin \mathrm{span}\{k_\theta(\cdot,\lambda):\lambda\in\Lambda\}$. ●

To complete the proof of Theorem 3.2 it suffices now to verify that in the case $S\neq\mathrm{const}$ we can take $\Lambda'=\Lambda$. But we have already established that $\dim(K_\theta\ominus \mathrm{span}\{k_\theta(\cdot,\lambda):\lambda\in\Lambda'\})=\infty$, Λ' being the set existing in virtue of assertion 1. By Lemma 3.4 the family $\{k_\theta(\cdot,\lambda):\lambda\in\Lambda\}$ is also a basis in the subspace it generates. ●

REMARK. Lemma 3.4 is a generalization of some propositions of R.Paley - N.Wiener [59] and N.Levinson [48] concerning the case $\theta(z)=\exp a\frac{z+1}{z-1}$, $a>0$ (i.e. families of exponents in $L^2(0,a)$). This lemma shows also that a family of reproducing kernels (or exponents) neither loses nor gains the property to form a basis of K_θ (or of the subspace of K_θ it generates) if a finite set of its members is replaced by a set of functions of the same sort having the same cardinality. Another consequence (also generalizing some remarks from the books just mentioned; cf.also R.Redheffer [51]): a family $\{k_\theta(\cdot,\lambda):\lambda\in\Lambda\}$ either is a minimal one or $k_\theta(\cdot,\mu)\in \mathrm{span}\{k_\theta(\cdot,\lambda):\lambda\in\Lambda\setminus\{\mu\}\}$ $\forall\mu\in\Lambda$.

Theorems 3.1, 3.2 show that tests to establish whether a family of reproducing kernels (or exponents) is a basis involving conditions(2), (3) may be exploited not only in general theory, but in some concrete questions as well. Here is one more example confirming this.

THEOREM 3.5. Let $\Lambda\subset\mathbb{C}$, $\inf_{\lambda\in\Lambda}\mathrm{Im}\,\lambda>-\infty$. The following assertions are equivalent.

1. The family $\{e^{i\lambda x}\chi_{(0,a)}:\lambda\in\Lambda\}$ is an unconditio-

nal basis in the subspace of $L^2(0,a)$ it generates for some a, $a>0$.

2. $\Lambda+iy\in(C)$ for $y>-\inf_{\lambda\in\Lambda}\operatorname{Im}\lambda$.

This theorem is, of course, a simple consequence of the analogous fact for the unit disc.

THEOREM 3.5D. Let $\Lambda\subset\mathbb{D}$ and let θ be an inner function with $\sup_{\lambda\in\Lambda}|\theta(\lambda)|<1$. The following assertions are equivalent.

1. There exists a positive integer n such that $\{k_{\theta^n}(\cdot,\lambda):\ \lambda\in\Lambda\}$ is an unconditional basis in its closed linear span.

2. $\Lambda\in(C)$.

PROOF. The implication $1\Longrightarrow 2$ follows from Theorem 1.1.

$2\Longrightarrow 1$. Let $B=\prod_{\lambda\in\Lambda} b_\lambda$. Since the fractions $(1-\bar\lambda z)^{-1}$, $\lambda\in\Lambda$ constitute an unconditional basis of the subspace they generate and since $\theta^n(T_B)^*(1-\bar\lambda z)^{-1}=\overline{\theta(\lambda)^n}(1-\bar\lambda z)^{-1}$, it follows that for n sufficiently large we have the inequality

$$\|\theta^n(T_B)\|<1.$$

Combining this with the condition (3) and Theorem 2 bis (Part I) we obtain the desired implication. ●

To clarify better the situation some links between Theorem 3.1 and an interesting paper of P.Koosis [43] (cf.also [46]) are to be pointed out. In Koosis' paper a necessary and sufficient condition is found for all operators

$$f\longmapsto \chi_{(a,+\infty)}f,\qquad a>0 \tag{4}$$

to be compact on the space $\operatorname{span}_{L^2(\mathbb{R}_+)}\{e^{i\lambda_n x}\chi_{\mathbb{R}_+}:\ n\in\mathbb{Z}\}$. The condition reads as follows [*]:

$$\lim_n \operatorname{Im}\lambda_n=+\infty,\quad \lim_{|x|\to\infty}\sum_n\frac{\operatorname{Im}\lambda_n}{|\lambda_n-x|^2}=0.$$

Theorem 3.1 is an easy consequence of this result, for Koosis condition is implied by its hypotheses (i.e. $(\lambda_n)_{n\in\mathbb{Z}}\in(C)$, $\lim_n\operatorname{Im}\lambda_n=+\infty$). It should be noted that under the hypotheses of Theorem 3.1 we can establish with an equal ease that all o p e r a t o r s o f t h e f o r m (4) a r e c o m p a c t . Indeed, each operator of such form is equal to

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[*] It is not hard to see that the same condition is equivalent to compactness of all Hankel operators $H_{\bar B\theta^a}$, $a>0$ where $\theta^a=e^{iaz}$.

$(I-P_{\theta^a})|K_B = \theta^a P_+ \bar{\theta}^a | K_B = \theta^a \cdot \theta^a (T_B)^* \ (\theta^a = exp\, i a z)$,
and the operator $\theta^a (T_B)^*$ is evidently compact, for the eigenvectors of this operator form an unconditional basis and its eigenvalues tend to zero. ●

Note also that the proof of theorem 3.1 presented here is much simpler than that of Koosis' theorem. This is due to the fact that in Theorem 3.1 Λ is assumed to satisfy Carleson condition.

Similar links exist between Theorem 3.2 and the recent paper [39]. In [39] all pairs (B, θ) of inner functions with the following property are identified: θ is singular and the Hankel operator $H_{\bar{B}\theta^a}$ is compact for every positive a.

4. Unconditional bases of reproducing kernels (the general case)

Theorems 1.2 and 5D give a solution of the problem concerning unconditional basis families of reproducing kernels under the additional assumption that the pair (θ, Λ) satisfies condition (1). Now we are going to treat the general case. If condition (1) is not satisfied then (see §1) the orthogonal projection P_θ distorts rational fractions and so $P_\theta \mid K_B$ is no longer an isomorphic imbedding. It is natural to try to "correct" the fractions $k(\cdot, \lambda)$ by means of a non-bounded operator in such a manner that the subsequent application of P_θ should produce no distortion.

Let $G \in H^2$ and let $T_{\bar{G}}$ be the Toeplitz operator whose symbol is $\bar{G}$. If $G \notin H^\infty$ then this operator is unbounded, but in any case its domain contains H^∞. It is evident (and well-known) that

$$T_{\bar{G}}\,(1-\bar{\lambda} z)^{-1} = \overline{G(\lambda)}\,(1-\bar{\lambda} z)^{-1} .$$

Thus $T_{\bar{G}}$ compensates the distortion produced by P_θ provided

$$G(\lambda) = (1-|\theta(\lambda)|^2)^{-1/2}, \quad \lambda \in \Lambda . \tag{5}$$

LEMMA 4.1. If the family $\{k_\theta(\cdot, \lambda) : \lambda \in \Lambda\}$ is an unconditional basis of its closed linear span then

$$\sum_{\lambda\in\Lambda}\frac{1-|\lambda|^2}{1-|\theta(\lambda)|^2}<+\infty\ , \tag{6}$$

and there exists a solution G, $G\in H^2$ of the problem (5).

PROOF. Consider the normed reproducing kernels $x_\lambda = \frac{(1-|\lambda|^2)^{1/2}}{(1-|\theta(\lambda)|^2)^{1/2}}\ \frac{1-\overline{\theta(\lambda)}\,\theta}{1-\bar\lambda z}$, $\lambda\in\Lambda$. If $f\in K_\theta$ then

$$\sum_{\lambda\in\Lambda}\frac{1-|\lambda|^2}{1-|\theta(\lambda)|^2}\,|f(\lambda)|^2=\sum_{\lambda\in\Lambda}|(f,x_\lambda)|^2\leqslant const\,\|f\|^2.$$

Setting here $f=P_\theta\mathbb{1} = 1-\theta\,\overline{\theta(0)}$ and using $|f(\lambda)|\geqslant 1-|\theta(0)|>0$ we obtain (6). Since $\Lambda\in(C)$ (Theorem 1.1), by Theorem A of §2, Part I the problem (5) has a solution in H^2 if and only if the inequality (6) holds. ●

REMARK. The solution of the problem (5) in K_B is unique and is given by the following formula:

$$G(z)=\sum_{\lambda\in\Lambda}\left(\frac{1-|\lambda|^2}{1-|\theta(\lambda)|^2}\right)^{1/2}\frac{(1-|\lambda|^2)^{1/2}}{1-\bar\lambda z}\ \frac{B_\lambda(z)}{B_\lambda(\lambda)}\ . \tag{7}$$

COROLLARY 4.2. Suppose that the assumptions of Lemma 4.1 are satisfied. If, in addition, θ is a singular inner function and μ is the representing measure of θ then

$$\sum_{\lambda\in\Lambda}\left(\int_{\mathbb{T}}\frac{d\mu(\zeta)}{|\zeta-\lambda|^2}\right)^{-1}<\infty\ .$$

Indeed, $1-|\theta(\lambda)|^2 = 1-exp\left(-2\int_{\mathbb{T}}\frac{1-|\lambda|^2}{|\zeta-\lambda|}\,d\mu(\zeta)\right)\leqslant$ $\leqslant 2\int_{\mathbb{T}}\frac{1-|\lambda|^2}{|\zeta-\lambda|^2}\,d\mu(\zeta)$. ●

We have already mentioned that if a family $\{k_\theta(\cdot,\lambda):\lambda\in\Lambda\}$ is an unconditional basis then $dist(\lambda, supp\,\mu)$ necessarily tends to 0 as $|\lambda|\to 1$, $\lambda\in\Lambda$ (see §6 for the proof). Corollary 4.2 shows that in the case of a purely singular inner function θ, moreover, it must tend *at least with some prescribed rapidity*, namely

$$\sum_{\lambda\in\Lambda}(dist(\lambda, supp\,\mu))^2<\infty\ .$$

THEOREM 4.3. Let $\Lambda\subset\mathbb{D}$ and let θ be an inner function. The following assertions are equivalent.

1. The family $\{k_\theta(\cdot,\lambda):\lambda\in\Lambda\}$ is a basis of K_θ (resp., of the subspace it generates).

2. $\Lambda\in(C)$ and there is a function G in H^1 so that the operator $P_\theta T_{\overline{G}}$ may be extended from the linear span of the fractions $(1-\overline{\lambda}z)^{-1}$, $\lambda\in\Lambda$ to an isomorphism of K_B onto K_θ (resp., into K_θ).

PROOF. Implication $1 \Longrightarrow 2$ follows from Theorem 1.1 and Lemma 4.1, for if G is the function from this lemma, then
$P_\theta T_{\overline{G}}(1-\overline{\lambda}z)^{-1}=(1-|\theta(\lambda)|^2)^{-1/2}k_\theta(\cdot,\lambda),\quad \lambda\in\Lambda$;
hence

$$\|P_\theta T_{\overline{G}}(1-\overline{\lambda}z)^{-1}\|=\|(1-\overline{\lambda}z)^{-1}\|,\quad \lambda\in\Lambda$$

and consequently $P_\theta T_{\overline{G}}$ may be extended to an isomorphism (indeed, it takes an unconditional basis to an unconditional basis and does not change the norms of its elements).

To prove that $2 \Longrightarrow 1$ we argue similarly to Theorem 1: the family of fractions $(1-\overline{\lambda}z)^{-1}$, $\lambda\in\Lambda$ is an unconditional basis of K_B, hence any isomorphic image of this family is also an unconditional basis; in particular so is the family
$\overline{G(\lambda)}\,k_\theta(\cdot,\lambda)$, $\lambda\in\Lambda$. ●

REMARK. The property of the function G expressed by assertion 2 is shared by any other function F in H^1 satisfying

$$0<\inf_{\lambda\in\Lambda}\left|\frac{F(\lambda)}{G(\lambda)}\right|\le\sup_{\lambda\in\Lambda}\left|\frac{F(\lambda)}{G(\lambda)}\right|<+\infty .$$

It is clear also that for any such F

$$0<\inf_{\lambda\in\Lambda}|F(\lambda)|(1-|\theta(\lambda)|^2)^{1/2}\le\sup_{\lambda\in\Lambda}|F(\lambda)|(1-|\theta(\lambda)|^2)^{1/2}<+\infty .$$

Unfortunately Theorem 4.3 is too non-constructive, and the situation is unlikely to improve very much even if we try to use some concrete G (e.g. one given by (7) provided (6) is satisfied) when applying this theorem. As for the function (7), it very probably fails to be the most appropriate. For example, in the case (1) (i.e. $\sup_{\lambda\in\Lambda}(1-|\theta(\lambda)|^2)^{-1/2}<\infty$) it is natural to choose $G\equiv\mathbb{1}$ (and so we did in Theorems of sections 1-3). Some facts supporting what we have just said may be found in the next §5.

5. Orthogonal and nearly orthogonal bases of reproducing kernels

It was already mentioned that if $\Lambda\subset\mathbb{D}$ and $\operatorname{card}\Lambda>1$ then the family $\{k_\theta(\cdot,\lambda):\lambda\in\Lambda\}$ cannot be orthogonal. In some cases it is possible, however, to consider reproducing kernels with poles on the unit circle. For example let the function θ admit an analytic continuation through a point λ, $\lambda\in\mathbb{T}$. Then the kernel

$$k_\theta(z,\lambda)=\frac{1-\overline{\theta(\lambda)}\,\theta(z)}{1-\bar\lambda z}=\frac{\overline{\theta(\lambda)}}{\bar\lambda}\,\frac{\theta(\lambda)-\theta(z)}{\lambda-z}$$

evidently lies in $H^2(\mathbb{D})$ and, moreover, in K_θ. A criterion for the inclusion $k_\theta(\cdot,\lambda)\in H^2(\mathbb{D})$, $\lambda\in\mathbb{T}$ was obtained by P.Ahern and D.Clark [26]. Let

$$\theta(z)=z^N\prod_n\frac{\bar a_n}{|a_n|}\,\frac{a_n-z}{1-\bar a_n z}\,exp\Big\{-\int_{\mathbb{T}}\frac{\zeta+z}{\zeta-z}\,d\mu(\zeta)\Big\}$$

be the canonical factorization of an inner function θ and set

$$E_\theta\overset{def}{=\!=}\Big\{\zeta\in\mathbb{T}:\ \sum_n\frac{1-|a_n|^2}{|\zeta-a_n|^2}+\int_{\mathbb{T}}\frac{d\mu(t)}{|t-\zeta|^2}<+\infty\Big\}\ .$$

Roughly speaking, E_θ consists of those points at which the argument of θ is differentiable.

THEOREM (P.Ahern, D.Clark [26]). Let $\lambda\in\mathbb{T}$. Then the fraction $(1-\bar c\,\theta(z))(1-\bar\lambda z)^{-1}$ lies in $H^2(\mathbb{D})$ for some complex number c if and only if $\lambda\in E_\theta$. If $\lambda\in E_\theta$ then this c is in fact unique and is given by $c=\lim_{r\to 1-0}\theta(r\lambda)$.

It should be noted here that Frostman's theorem (cf.[30]) implies that θ has radial limits on a set wider than E_θ, namely on the set

$$\Big\{\zeta\in\mathbb{T}:\ \sum_n\frac{1-|a_n|^2}{|\zeta-a_n|}+\int\frac{d\mu(t)}{|t-\zeta|}<+\infty\Big\}\ .$$

Let now $\theta(\lambda)=\theta(\lambda')=\alpha$, $|\alpha|=1$ and assume $\lambda,\lambda'\in E_\theta$, $\lambda\neq\lambda'$. Then

$$(k_\theta(z,\lambda'),k_\theta(z,\lambda))=k_\theta(\lambda,\lambda')=\frac{1-\overline{\theta(\lambda')}\,\theta(\lambda)}{1-\bar\lambda'\lambda}=0\ . \tag{8}$$

This remarkable property was observed for the first time by D.Clark [29] and later (independently) by D.Georgijević [37]. We are going to illustrate this property by an example; to do this we pass for some time to the upper half-plane. Let $\theta = \theta^{2\pi} = e^{2\pi i z}$. Then $k_\theta(z,t) = i\,\frac{1-\overline{\theta(t)}\,\theta(z)}{z-t} \in K_\theta$ for all t in $\mathbb{R}$. Evidently $\theta(\zeta)=1$ if and only if ζ is an integer. For $\lambda = n \in \mathbb{Z}$ we have

$$k_\theta(z,n) = \frac{e^{2\pi i z}-1}{i(z-n)} = \int_0^{2\pi} e^{izt}\, e^{-int}\,dt\,,$$

and so the kernels $k_\theta(\cdot, n)$ are Fourier-Laplace transforms of the classical orthogonal system of exponents $\{e^{int}: n\in\mathbb{Z}\}$. We see (!) that the reproducing kernels $\{k_\theta(\cdot, n): n\in\mathbb{Z}\}$ form a complete orthogonal system in K_θ.

It turns out that this example may be generalized to a class of inner functions θ. The construction was performed by D.Clark [29] in connection with the investigation of spectra of one-dimensional perturbations of the model operator T_θ.

Let θ be an inner function and $\alpha\in\mathbb{T}$. Substituting θ for z in the Poisson kernel $\frac{1-|z|^2}{|\alpha-z|^2}$ we obtain a nonnegative harmonic function in the disc, which can be represented by a Poisson integral:

$$\frac{1-|\theta(z)|^2}{|\alpha-\theta(z)|^2} = \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\overline{\zeta} z|^2}\, d\sigma_\alpha(\zeta), \quad |z|<1\,. \tag{9}$$

The measure σ_α is nonnegative and singular with respect to the Lebesgue measure since $\lim\limits_{r\to 1-0}\theta(r\zeta)\neq\alpha$ almost everywhere on $\mathbb{T}$. On the other hand it is well-known that the radial limits of the Poisson integral of a singular measure are equal to $+\infty$ almost everywhere with respect to this measure. Hence

$$\lim_{r\to 1-0}\theta(r\zeta) = \alpha, \quad \sigma_\alpha\text{-a.e.}$$

Thus measures σ_α and σ_β are mutually singular if $\alpha\neq\beta$. The equality (9) can be given another form:

$$\frac{1-|\theta(z)|^2}{1-|z|^2} = \int_{\mathbb{T}} \left|\frac{1-\alpha\,\overline{\theta(z)}}{1-\overline{z}\zeta}\right|^2 d\sigma_\alpha(\zeta)\,.$$

Since $\lim\limits_{r\to 1-0} k_\theta(r\zeta, z) = (1-\alpha\,\overline{\theta(z)})(1-\overline{z}\zeta)^{-1}$, the last equality means that the restriction map $\mathcal{U}$: $f \mapsto f|\operatorname{supp}(\sigma_\alpha)$ from H^2 to $L^2(\sigma_\alpha)$ preserves the norms of the reproducing kernels $k_\theta(\cdot,\lambda)$, $|\lambda|<1$. In fact $\mathcal{U}$ can be extended to an isometry of K_θ onto $L^2(\sigma_\alpha)$ (see Clark [29] for the details).

Let $\Lambda \subset \mathbb{T}$. Then the family $\{k_\theta(z,\lambda): \lambda\in\Lambda\}$ is orthogonal in K_θ if and only if $\Lambda\subset E_\theta$ and $\theta|\Lambda\equiv\alpha$, $\alpha\in\mathbb{T}$. It turns out that every such orthogonal family is the family of eigenfunctions of a unitary operator U_α and that this U_α is a one-dimensional perturbation *) of the model operator $P_\theta S|K_\theta$. The action of this unitary operator is described by the formula

$$U_\alpha f = z\left(f-(f,K_0)\frac{K_0}{\|K_0\|^2}\right)+ w(f,K_0)\frac{k_0}{\|k_0\|^2},$$

where

$$K_0 = P_\theta 1 = 1-\overline{\theta(0)}\theta,\ k_0 = z^{-1}(\theta(z)-\theta(0)),\ w=\frac{\alpha-\theta(0)}{1-\overline{\theta(0)}\alpha}.$$

Restricting this formula to the support set of σ_α and using the fact that $\theta=\alpha$ a.e. with respect to σ_α we obtain

$$\mathcal{U}U_\alpha f = z\,\mathcal{U}f.$$

Hence U_α is equivalent to the operator of multiplication by z in $L^2(d\sigma_\alpha)$. This reasoning proves the following theorem of Clark.

THEOREM (D.Clark [29]). The space K_θ has an orthogonal basis consisting of reproducing kernels $\{k_\theta(z,\lambda):\lambda\in\Lambda\}$, $\Lambda\subset\mathbb{T}$ if and only if for some α, $\alpha\in\mathbb{T}$ the measure σ_α is purely atomic. ●

Unfortunately it is not easy to use this criterion. There exists, however, a simpler sufficient condition: if the set $\mathbb{T}\setminus E_\theta$ is at most countable then for any α, $\alpha\in\mathbb{T}$, the family $\{k_\theta(z,\lambda): \theta(\lambda)=\alpha,\ \lambda\in E_\theta\}$ is a complete orthogonal system in K_θ. This condition is also due to Clark. It is satisfied, for example, for inner functions $\theta(z) = \exp\left\{-\int_{\mathbb{T}}\frac{\zeta+z}{\zeta-z}\,d\mu(\zeta)\right\}$ such that the set $\operatorname{supp}(\mu)$ is at most coun-

*) It was the investigation of such perturbation that led D.Clark to all his results. A "vector-valued" theory of the same sort is developped in [71], [72].

table.

Using orthogonal bases consisting of reproducing kernels corresponding to points of the unit circle it is possible to construct unconditional reproducing kernel bases with members corresponding to points of $\mathbb{D}$. For example suppose that for a given θ a family $\{k_\theta(z,\lambda_n)\}_{n\in\mathbb{Z}}$ with $|\lambda_n|=1, \theta(\lambda_n)=\alpha, n\in\mathbb{Z}$ oonstitutes an orthogonal basis in K_θ. Choose for each n a point μ_n in $\mathbb{D}$ so close to λ_n that

$$\sum_n \| \|k_\theta(\cdot,\lambda_n)\|^{-1}k_\theta(\cdot,\lambda_n)-\|k_\theta(\cdot,\mu_n)\|^{-1}k_\theta(\cdot,\mu_n)\|^2<1 .$$

Then $(k_\theta(\cdot,\mu_n))_{n\in\mathbb{Z}}$ is clearly an unconditional basis in K_θ. This method to construct bases is, of course, merely a generalization of the Paley - Wiener method..

There exist however inner functions θ such that the space K_θ contains no reproducing kernels corresponding to points of the unit circle, but yet has an unconditional basis consisting of reproducing kernels. To construct such a θ it is sufficient to produce a Blaschke product B whose zeros form a Carleson set $\Lambda=\{\lambda_n : n\in\mathbb{Z}\}$, but

$$\sum_n \frac{1-|\lambda_n|^2}{|\zeta-\lambda_n|^2}=+\infty$$

for $\zeta\in\mathbb{T}$. (Given such a B take simply $\theta=B$. Then by Theorem A the family $(k(\cdot,\lambda_n))_{n\in\mathbb{Z}}$ is an unconditional basis in K_B and by the theorem of P.Ahern - D.Clark the set E_B is empty). We shall show even that there exists a subset Λ of the unit disc such that $\Lambda\in(C)$ and

$$\sum_{\lambda\in\Lambda}\frac{1-|\lambda|}{|\zeta-\lambda|}=+\infty \tag{10}$$

for all ζ, $\zeta\in\mathbb{T}$. Let $(k_n)_{n\geqslant 2}$ be a sequence of positive integers with the properties

$$\sum_n k_n 2^{-n}<\infty , \quad \sum_n k_n 2^{-n}\log k_n=\infty .$$

(Take, for example, $k_n=[(n\log^2 n)^{-1}2^n]$, $[x]$ being the greatest integer less than or equal to x). For each n choose k_n equidistant points on the circle $\{\zeta\in\mathbb{C} : |\zeta|=r_n \overset{def}{=\!=} 1-2^{-n}\}$, and let Λ be the set of all choosen points. We claim that Λ has the desired properties.

The rarity condition (i.e. $\mathcal{D}(\lambda,\varepsilon(1-|\lambda|))\cap \cap\mathcal{D}(\lambda',\varepsilon(1-|\lambda'|))=\emptyset$ for $\lambda,\lambda'\in\Lambda$, $\lambda\neq\lambda'$ and for sufficiently small ε, $\varepsilon>0$) as well as the fact that $\sum_{\lambda\in\Lambda}(1-|\lambda|)\delta_\lambda$ is a Carleson measure (i.e. $\sum_{\lambda\in Q}(1-|\lambda|)\leqslant \leqslant const\cdot\ell$ for every rectangle $Q=\{\zeta:1-\ell\leqslant|\zeta|<1,$ $\arg\zeta\in I\}$, $I\subset\mathbb{T}$, $|I|=\ell$, $\ell>0$) are easily checked. To verify (10) take $\zeta\in\mathbb{T}$ and estimate separately the summands with $|\lambda|=r_n$, $r_n=1-2^{-n}$. Note that for every such λ except possibly two we have $|\zeta-\lambda|\leqslant const|r_n\zeta-\lambda|$, and hence

$$\sum_{\lambda\in\Lambda,|\lambda|=r_n}\frac{1-|\lambda|}{|\zeta-\lambda|}\geqslant const\, 2^{-n}\sum_{\lambda\in\Lambda,|\lambda|=r_n}\frac{1}{|r_n\zeta-\lambda|}\geqslant$$

$$\geqslant const\, 2^{-n}\sum_{k=1}^{[k_n]}\frac{k_n}{k}\geqslant const\cdot k_n 2^{-n}\log k_n . \quad \bullet$$

6. Interpolation by K_θ-functions. The H^p- spaces

In this section we are mainly concerned with applications of our results on exponential bases and bases of reproducing kernels to interpolation theory, and with some variants of these results for the H^p- spaces.

We begin with the second subject, restricting ourselves by the case $1<p<\infty$. For these values of p the theory turns out to be a duplicate (with minor variations) of the H^2-theory already discussed, the reason being, of course, the L^p-continuity of the Riesz projection $\mathbb{P}_+$. We recall that $\mathbb{P}_+ f \stackrel{def}{=\!=} \stackrel{def}{=\!=} \sum_{n\geqslant 0}\hat f(n)z^n$, and thus $\mathbb{P}_+$ maps $L^p=L^p(\mathbb{T})$ onto the Hardy space $H^p\stackrel{def}{=\!=}\{f\in L^p:\hat f(n)=0,\ \ n<0\}$. Setting $(H^p)_-=\{f\in L^p:\hat f(n)=0,\ n\geqslant 0\}$ we see that for $p\in(1,\infty)$ L^p is the direct sum of H^p and H^p_-, and so using the duality $\langle f,g\rangle=\int_{\mathbb{T}} f\bar g\,dm$ we may identify (in the anti-linear manner) the conjugate space $(H^p)^*$ with $H^{p'}$, $\frac{1}{p}+\frac{1}{p'}=1$. It is clear that the main formulae of §§ 1-4 remain valid in the H^p-setting also.

Define for an inner function θ

$$K^p_\theta=(\theta H^{p'})^\perp\ ,\ \ k_\theta(\cdot,\lambda)=\frac{1-\overline{\theta(\lambda)}\,\theta}{1-\bar\lambda z}\ .$$

Then $P_\theta \overset{def}{=} \theta \mathbb{P}_- \bar\theta$ is the projection onto K_θ along θH^p, $H^p = K_\theta + \theta H^p$ and k_θ the reproducing kernel of the space $K_\theta^{p'}$ (which may be naturally considered as the conjugate space $(K_\theta^p)^* = H^{p'}/(K_\theta^p)^\perp = H^{p'}/\theta H^{p'}$); indeed,

$$f(\lambda) = \langle f, k_\theta(\cdot,\lambda)\rangle \; ; \; \lambda \in \mathbb{D} \;, \; f \in K_\theta^{p'}$$

and $k_\theta(\cdot,\lambda) \in K_\theta^p \; (\lambda \in \mathbb{D})$. It is also clear that $k_\theta(\cdot,\lambda) = P_\theta(1-\bar\lambda z)^{-1}$, $\lambda \in \mathbb{D}$.

We shall be interested mainly in the case when the functions $k_\theta(\cdot,\lambda_n)$ "do not lie very far from rational fractions", i.e.

$$\sup_n |\theta(\lambda_n)| < 1. \tag{11}$$

As before we shall discuss u n c o n d i t i o n a l b a s e s of the form $\{k_\theta(\cdot,\lambda) : \lambda \in \Lambda\}$, $\Lambda \subset \mathbb{D}$ but now using the general difinition of this notion which we have mentioned in the first pages of our paper (p. 217)

LEMMA 6.1. Suppose that the family $\{k_\theta(\cdot,\lambda_n) : \lambda_n \in \Lambda\}$ is an unconditional basis of the subspace of H^p it generates, $1<p<\infty$ and assume that (11) holds. Then $\Lambda \in (C)$.

PROOF. Note that, in H^p, the Carleson condition (C) is still necessary and sufficient for the system $\{(1-\bar\lambda_n z)^{-1} : \lambda_n \in \Lambda\}$ to be uniformly minimal. It remains to apply to $A = P_\theta$, $x_n = (1-\bar\lambda_n z)^{-1}$ the lemma about the uniformly minimal families proved in § 2 Part I. ●

At this point some widely known facts concerning the geometry of families of rational functions $\{(1-\bar\lambda_n z)^{-1} : n \geq 1\}$ in the H^p-metric, $1<p<\infty$, should be recalled. (For a more detailed exposition see [8], [4], [17], [18]). One of these facts has already been used (namely, that the condition (C) is equivalent to the uniform minimality), others (to be used later on) are as follows. The Carleson condition (C) is equivalent to each of assertions listed below:

a) the family $\{(1-\bar\lambda_n z)^{-1} : \lambda_n \in \Lambda\}$ is an unconditional basis of the subspace of H^p it generates;

b) The family $\{(1-|\lambda_n|)^{1/p'}(1-\bar\lambda_n z)^{-1} : \lambda_n \in \Lambda\}$ is an unconditional basis (in its closed linear span) isomorphic to the standard unit vector basis of ℓ^p ;

c) $\mathcal{J} H^p = \ell^p$, where $\mathcal{J} f = \{f(\lambda_n)(1-|\lambda_n|)^{1/p} : \lambda_n \in \Lambda\}$

One more condition equivalent to a)-c) worth mentioning (seems to be present in the literature only in an implicit form, if at all):

d) $\mathcal{J}H^p \supset \ell^p$, i.e. any interpolation problem $\mathcal{J}f = a$ with the data a in ℓ^p has a solution in H^p.

To verify that d) is equivalent to a)-c) note that the inclusion $\mathcal{J}H^p \supset \ell^p$ and the closed graph theorem imply that the problem mentioned is not merely solvable, but is solvable with an estimate: there exists a constant C so that $\forall a \in \ell^p \ \exists f \in H^p$: $\mathcal{J}f = a$, $\|f\|_{H^p} \leq c\|a\|_{\ell^p}$. Taking as a the unit vectors of the space ℓ^p we obtain the uniform minimality of the family $\{(1-\bar{\lambda}_n z)^{-1} : \lambda_n \in \Lambda\}$ in $H^{p'}$, i.e. the Carleson condition (C). ●

Also well-known is the general duality between the problems concerning bases and interpolation, cf. [17], [18] for the details. In our setting it is expressed by the following lemma.

LEMMA 6.2. Let $\Lambda \subset \mathbb{D}$, Θ be an inner function, $1 < p < \infty$. The following assertions are equivalent.

1. The family $\{k_\Theta(\cdot, \lambda_n) : \lambda_n \in \Lambda\}$ is an unconditional basis of the subspace of H^p it generates.

2. The space of restrictions $K_\Theta^{p'} \mid \Lambda$ is an ideal space (that is, from $f \in K_\Theta^{p'}$ and $|a_n| \leq |f(\lambda_n)|$, $\lambda_n \in \Lambda$ it follows that there exists a function g in $K_\Theta^{p'}$ interpolating $\{a_n\}$: $g(\lambda_n) = a_n$, $\lambda_n \in \Lambda$). ●

In fact (and this will be the essence of theorem 6.3 below) an ideal space mentioned in the lemma will turn out to be simply a weighted ℓ^p-space (just as for the problem of free interpolation in the whole space H^p). Now we mention only that the interpolation by $K_\Theta^{p'}$-functions is nothing else as the interpolation by functions analytically continuable trough the points of $\mathbb{T} \setminus \operatorname{spec}\Theta$ and satisfying some estimates in $\mathbb{C} \setminus \mathbb{T}$, cf. [73], [18]. That is why the condition $\lim\limits_{\lambda \in \Lambda, |\lambda| \to 1} \operatorname{dist}(\lambda, \operatorname{spec}\Theta) = 0$ mentioned on p. 268 is necessary for the family $\{k_\Theta(\cdot, \lambda) : \lambda \in \Lambda\}$ to form an unconditional basis (see also the next corollary and theorem 6.3).

For $p = 2$ no additional work is needed to give a precise theorem connecting interpolation and reproducing kernels bases. We present both a general assertion concerning the spaces $K_\Theta = K_\Theta^2$ and an assertion concerning the most interesting particular case $\Theta = \Theta^a$, connected with exponential bases $\{e^{i\lambda_n x}\chi_{(0,a)} : \lambda_n \in \Lambda\}$.

COROLLARY D. Let $\Lambda \subset \mathbb{D}$, Θ be an inner function. The following assertions are equivalent.

1. The family $\{k_\Theta(\cdot, \lambda_n) : \lambda_n \in \Lambda\}$ is an unconditional

basis of the subspace of H^2 it generates.

2. $\mathcal{J}_\theta K_\theta = \ell^2$, where $\mathcal{J}_\theta f=\left\{f(\lambda_n)\left(\frac{1-|\lambda_n|^2}{1-|\theta(\lambda_n)|^2}\right)^{\frac{1}{2}}:\lambda_n\in\Lambda\right\}$.

If the condition (11) satisfied, then we have another equivalent assertion:

3. $\mathcal{J}_\theta K_\theta \supset \ell^2$.

COROLLARY $\mathbb{C}_+$. Let $a>0$, $\Lambda\subset\mathbb{C}_\sigma$, $\sigma>0$. The following assertions are equivalent.

1. The family $\{e^{i\lambda_n x}\chi_{(0,a)}:\lambda_n\in\Lambda\}$ is an unconditional basis of the subspace of $L^2(0,a)$ it generates.

2. $\mathcal{J}_a E_a^2=\ell^2$, where E_a^2 is the space of all entire functions of exponential type less than or equal to $a/2$ and square summable on $\mathbb{R}$, and $\mathcal{J}_a f\overset{def}{=}\{c_n f(\lambda_n):\lambda_n\in\Lambda\}$, $c_n=(\operatorname{Im}\lambda_n)^{1/2}\exp(-1/2\,a\operatorname{Im}\lambda_n)$.

3. $\mathcal{J}_a E_a^2\supset\ell^2$.

To check these corollaries one needs only to add to what has already been explained the (evident) fact that for any unconditional basis $\{x_n\}$ in a Hilbert space the space of Fourier coefficients $\{(x,x_n/\|x_n\|)\}$ coincides with ℓ^2. ●

Passing to the main result of this section we recall the Muckenhoupt condition (A_p) in terms of which the reproducing kernel bases in K_θ^p will be described. This condition, imposed for $1<p<\infty$ on a positive function w on $\mathbb{T}$, looks as follows:

$$\sup_I\left(\frac{1}{mI}\int_I w\,dm\right)\left(\frac{1}{mI}\int_I w^{-\frac{1}{p-1}}\,dm\right)^{p-1}<\infty \qquad (A_p),$$

where the $\sup$ is taken over all intervals (arcs) of $\mathbb{T}$.

THEOREM 6.3. Let $1<p<\infty$, $\Lambda\subset\mathbb{D}$, θ be an inner function in $\mathbb{D}$ and suppose that the condition (11) holds. The following assertions are equivalent.

1. The family $\{k_\theta(\cdot,\lambda_n):\lambda_n\in\Lambda\}$ is an unconditional basis of K_θ^p.

2. The family $\{k_\theta(\cdot,\lambda_n)(1-|\lambda_n|)^{1/p'}:\lambda_n\in\Lambda\}$ is a basis of K_θ^p equivalent to the standard unit vector basis of ℓ^p.

3. $\Lambda\in(C)$ and the operator $P_\theta\,|\,K_B^p$ is an isomorphism of K_B^p onto K_θ^p (here $B=\prod_{\lambda_n\in\Lambda} b_{\lambda_n}$, the Blaschke product corresponding to the set Λ).

4. $\Lambda\in(C)$ and there exist real functions u and v and a real number c so that $u\in L^\infty(\mathbb{T})$ and

$$\overline{B}\Theta = exp(u+ic-i\tilde{v}),\ exp(\tfrac{p}{2}v)\in(A_p)\ . \tag{12}$$

5. If $f\in K_\theta^{p'}$, $f(\lambda_n)=0$ $(\lambda_n\in\Lambda)$ then $f\equiv 0$; and $\mathcal{J}^{p'}K_\theta^{p'}=\ell^{p'}$, where $\mathcal{J}^{p'}f=\{f(\lambda_n)(1-|\lambda_n|)^{1/p'}:\lambda_n\in\Lambda\}$.

6. $\mathcal{J}^{p'}K_\theta^{p'}\supset\ell^{p'}$ and if $f\in K_\theta^{p'}$, $f(\lambda_n)=0$ $(\lambda_n\in\Lambda)$ then $f\equiv 0$.

PROOF. It is clear that $3\Longleftrightarrow 2\Longrightarrow 1$, $2\Longleftrightarrow 5$ and that the implication $6\Longrightarrow 5$ has in fact already been proved (the reverse implication $5\Longrightarrow 6$ being evident). It remains to check that $1\Longrightarrow 2$ and $3\Longleftrightarrow 4$.

$1\Longrightarrow 2$. If $\{x_n\}$ is an unconditional basis in L^p-metric then "integrating over signs" we obtain

$$\int\Big|\sum a_n x_n\Big|^p \asymp \int\Big(\sum|a_n|^2|x_n|^2\Big)^{p/2};$$

the symbol $\asymp$ means that each of the integrals majorizes another one multiplied by a constant independent of the coefficients a_n. Setting $x_n=k_\theta(\cdot,\lambda_n)$ and taking into account the condition (11), Lemma 6.1 and the assertion b) concerning unconditional bases of rational fractions we get

$$\int_{\mathbb{T}}|a_n k_\theta(\cdot,\lambda_n)|^p \asymp \int_{\mathbb{T}}\Big(\sum|a_n|^2|k_\theta(\cdot,\lambda_n)|^2\Big)^{p/2}\asymp$$

$$\asymp\int_{\mathbb{T}}\Big(\sum|a_n|^2\frac{1}{|1-\bar\lambda_n z|^2}\Big)^{p/2}\asymp\int_{\mathbb{T}}\Big|\sum a_n\frac{1}{1-\bar\lambda_n z}\Big|^p\asymp$$

$$\asymp\sum|a_n|^p(1-|\lambda_n|)^{-p/p'}.$$

This relation between the first and the last term just means that the assertion 2 holds.

$3\Longleftrightarrow 4$. Similary to the case $p=2$ (see § 3, Part I) the operator $P_\theta\,|\,K_B^p$ has the same metric properties as the Toeplitz operator $T_{\overline{B}\theta}$ in the space H^p. The criterion of the form (12) for such an operator to be invertible is the subject-matter of the paper [53]. ●

Of course, the material of this section suggests some natural questions. We have skipped them in the hope that they have been noted by the reader who had the patience to reach this point. May be, the reader even knows already how to answer them.

PART III.

EXPONENTIAL BASES AND ENTIRE FUNCTIONS.

1. Generating functions, BMO and theorems 6, 7, 8.

In this section we investigate some properties of the generating functions corresponding to subsets of the upper half-plane and give the proofs of theorems 6, 7, 8. First recall some definitions from the theory of entire functions.

Let F be an entire function of exponential type. The 2π-periodic function h_F defined on $\mathbb{R}$ by the formula

$$h_F(\varphi) = \varlimsup_{r\to\infty} \frac{\log|F(re^{i\varphi})|}{r}, \qquad \varphi\in\mathbb{R}$$

is called the indicator of F.

The indicator diagram of F is by definition the convex set G_F such that

$$h_F(\varphi) = \sup_{\zeta\in G_F} \operatorname{Re}(\zeta e^{-i\varphi}).$$

The set $G_F^* = \{\bar\zeta : \zeta \in G_F\}$ is called the conjugate diagram of F.

The background material concerning the above notions is contained in [13] (ch. I, §§15-17 and §§19-20). We have already explained the reason for our interest in the class $\mathcal{E}_a$, $a>0$, of all entire functions F of exponential type with $G_F^* = [0, ia]$ in Section 5 of Part I. More precisely we shall be interested in the subclass $\mathcal{M}_a$ of the class $\mathcal{E}_a$ consisting of functions F, $F \in \mathcal{E}_a$, satisfying the Muckenhoupt condition (A_2) on $\mathbb{R}$:

$$\sup_{I\in\mathcal{J}}\left(\frac{1}{|I|}\int_I |F|^2 dx\right)\left(\frac{1}{|I|}\int_I |F|^{-2} dx\right) < \infty . \qquad (A_2)$$

Here $\mathcal{J}$ is the set of all bounded intervals of the real axis Recall that the condition (A_2) is equivalent to the Helson-Szegö condition (HS), see Part I, § 4.

LEMMA 1.1. Let w be a positive function on the real axis satisfying the Helson-Szegö condition (HS). Then there exists a number $p = p_w$, $1 < p < \infty$, such that

$$\int_{\mathbb{R}} \frac{w^p(x)}{1+x^2}\, dx < \infty .$$

PROOF. The hypothesis implies that $w = \exp(u + \tilde{v})$, where u, $v \in L^\infty(\mathbb{R})$, $\|v\|_\infty < \pi/2$. It remains to use the following well-known theorem due to A.Zygmund (see [9], ch, YII § 2, th. 2.11 (I)): if $\|v\|_\infty \leqslant 1$ and $0 < \lambda < \pi/2$, then

$$\int_{\mathbb{R}} \exp(\lambda |\tilde{v}(x)|) \frac{dx}{1+x^2} < \infty . \qquad \bullet$$

We denote by $\mathcal{C}$ the set of all entire functions f of exponential type such that

$$\int_{\mathbb{R}} \frac{\log^+|f(x)|}{1+x^2}\, dx < \infty,$$

where $u^+ \overset{def}{=\!=} max(u, 0)$. From lemma 1.1 it follows that

$$\int_{\mathbb{R}} \frac{|F(x)|^2}{1+x^2}\, dx < \infty$$

for $F \in \mathcal{M}_a$, $a > 0$. Hence by the M.Cartwright theorem (see [13], ch. Y, § 4) we may conclude that $\mathcal{M}_a \subset \mathcal{C}$.

The class $\mathcal{C}$ can be characterized as the set of all entire functions f of exponential type with $f|\mathbb{C}_+$, $f|\mathbb{C}_-$ belonging to the Nevanlinna classes in the corresponding half-planes (i.e. to the images of the usual Nevanlinna class in the unit disc $\mathbb{D}$ under the conformal mappings $\mathbb{C}_\pm \to \mathbb{D}$). Hence, if $f \in \mathcal{C}$ then

$$f|\mathbb{C}_+ = cSBf_e ,$$

where $c \in \mathbb{C}$, $|c| = 1$, f_e is an outer function in $\mathbb{C}_+$, B is the Blaschke product corresponding to the zeros of $f|\mathbb{C}_+$ and S is the quotient of two singular inner function in $\mathbb{C}_+$. Because of the analyticity of the function f on the real axis we have $S = \exp i\gamma z$, $\gamma \in \mathbb{R}$ (to see this recall the formula for the singular inner function from § 1, Part I). An analogous factorization formula holds also in the lower half--plane $\mathbb{C}_-$.

We state now a useful connection between the class $\mathcal{M}_a$ and unimodular Helson-Szegö functions on $\mathbb{R}$.

THEOREM 1.2. Let Λ ($\Lambda \subset \mathbb{C}_\delta$, $\delta > 0$) be a Blaschke set, let B denote the corresponding Blaschke product and let $\theta^a = \exp(iaz)$, $a > 0$. The following assertions are equivalent.

1. There exists a function of the class $\mathcal{M}_a$ with simple

zeros whose zero-set is Λ.

2. The restriction $B\bar{\theta}^a | \mathbb{R}$ is a Helson-Szegö function, i.e. there exists a unimodular constant c and an outer (in $\mathbb{C}_+$) function h such that $|h^2| | \mathbb{R} \in (HS)$ and

$$Bh = c\bar{h}\theta^a, \qquad \text{a.e. on } \mathbb{R} .$$

PROOF. 1 $\Rightarrow$ 2. Let F be an entire function mentioned in the assertion 1 and let h be the outer function in $\mathbb{C}_+$ with $|h(x)| = |F(x)|$, $x \in \mathbb{R}$. By the definition of the class $\mathcal{M}_a$, $h_F(\pi/2) = 0$ and hence the canonical factorization of F in $\mathbb{C}_+$ contains no singular inner factor, i.e.

$$F|\mathbb{C}_+ = c_+ B \cdot h , \quad |c_+| = 1.$$

An analogous reasoning for the half-plane $\mathbb{C}_-$ shows (take into account that $h_F(-\pi/2) = a$)

$$F|\mathbb{C}_- = c_- \theta^a h^*, \quad |c_-| = 1,$$

where $h^*(z) = \overline{h(\bar{z})}$ (the outer function in $\mathbb{C}_-$ with $|h^*(x)| = |F(x)|$, $x \in \mathbb{R}$). Comparing the last equality with the preceding one we obtain the assertion 2.

2 $\Rightarrow$ 1. Let h be the function from the assertion 2. It is useful to note that $h(z+i)^{-1} \in H^2_+$ because of lemma 1.1. We define a function F on $\mathbb{C} \setminus \mathbb{R}$ by the equalities:

$$F = \begin{cases} Bh & (\text{on } \mathbb{C}_+), \\ c\,\theta^a h^* & (\text{on } \mathbb{C}_-). \end{cases}$$

In fact, however, the function F admits an extension onto the whole plane $\mathbb{C}$ as an entire function. This is an immediate consequence of the following simple lemma.

LEMMA 1.3. Let f_+ and f_- be analytic functions in the upper and lower half-planes respectively, let Δ be an interval, and let

$$\sup_{0<y<1} \int_\Delta |f_\pm(x \pm iy)|\, dx < +\infty.$$

If $\lim\limits_{y\to 0+} f_+(x+iy) = \lim\limits_{y\to 0+} f_-(x-iy)$ a.e. on Δ then f_+ can be analytically continued through Δ, the continuation coinciding with f_-.

The proof is easy. It can be found for example in [42].

A more general theorem is proved in [48].

To prove that $F\in\mathcal{M}_a$ we use the Cauchy formula and the fact that $|F(z)|\leq \exp\{a\,|\operatorname{Im} z|\}\cdot|h(\bar z)|$, $\operatorname{Im} z>0$, (this inequality follows easily from the definition of F). We have

$$|h(x)|=|F(x)|=$$
$$=\left|\frac{1}{2\pi i}\int\limits_{|\zeta-x|=1}\frac{F(\zeta)}{\zeta-x}\,d\zeta\right|\leq\frac{1}{2\pi}\int\limits_{\{|\zeta-x|=1\}\cap\mathbb{C}_+}|h(\zeta)||d\zeta|+\frac{e^a}{2\pi}\int\limits_{\{|\zeta-x|=1\}\cap\mathbb{C}_-}|h^*||d\zeta|$$

for $x\in\mathbb{R}$.

Since $(z+i)^{-1}h\in H^2_+$ and the "arc length" on $\mathbb{C}_+\cap\{|z-x|=1\}$ is obviously a Carleson measure then from the Carleson imbedding theorem ([18]) it follows that

$$\int\limits_{\mathbb{C}_+\cap\{|\zeta-x|=1\}}\frac{|h(\zeta)|^2}{|\zeta+i|^2}|d\zeta|\leq \mathrm{const}\int\limits_{\mathbb{R}}\frac{|h(x)|^2}{1+x^2}\,dx.$$

This together with Schwarz's inequality imply that

$$|h(x)|\leq\frac{1+e^a}{2\pi}\int\limits_{\mathbb{C}_+\cap\{|\zeta-x|=1\}}|h(\zeta)||d\zeta|\leq$$
$$\leq\frac{1+e^a}{2\pi}\left(\mathrm{const}\int\limits_{\mathbb{R}}\frac{|h(x)|^2}{1+x^2}\,dx\right)^{1/2}\left(\int\limits_{\mathbb{C}_+\cap\{|\zeta-x|=1\}}|\zeta+i|^2|d\zeta|\right)^{1/2}\leq \mathrm{const}\,|x+i|.$$

From the facts that h is real, $|B(\zeta)|<1$ for $\operatorname{Im}\zeta>0$ and the function $\log|\zeta+i|$ is a harmonic function in $\mathbb{C}_+$ representable as a Poisson integral it follows that

$$\log|F(\zeta)|\leq\log|h(\zeta)|=\frac{1}{\pi}\int\limits_{\mathbb{R}}\frac{\operatorname{Im}\zeta}{|\zeta-t|^2}\log|h(t)|\,dt\leq \mathrm{const}+\log|\zeta+i|$$

for $\zeta\in\mathbb{C}_+$.

Therefore

$$|F(z)|\leq \mathrm{const}\,|z+i|,\quad \operatorname{Im} z>0. \tag{1}$$

Similarly we can prove that

$$|F(z)|\leq \mathrm{const}\,|z-i|\exp\{a|\operatorname{Im} z|\},\ \operatorname{Im} z<0. \tag{2}$$

The inequalities (1) and (2) imply that F is an entire function of exponential type. Hence $F\in\mathcal{C}$ because $|F|=|h|$ on $\mathbb{R}$. To prove that $F\in\mathcal{M}_a$ it remains to show that $G_F=[0,-ia]$. It follows from (1) and (2) that $G_F\subset[0,-ia]$. But

$$h_F\left(\frac{\pi}{2}\right)=\overline{\lim_{y\to+\infty}}\ \frac{\log|F(iy)|}{y}=\overline{\lim_{y\to+\infty}}\ \frac{\log|B(iy)|}{y}+\lim_{y\to+\infty}\frac{\log|h(iy)|}{y}=0,$$

$$h_F\left(-\frac{\pi}{2}\right)=\overline{\lim_{y\to+\infty}}\ \frac{\log|F(-iy)|}{y}=a+\lim_{y\to+\infty}\frac{\log|h(iy)|}{y}=a,$$

and therefore $G_F=[0,-ia]$.

The function F does not vanish on $\mathbb{C}_-$ and its zeros in $\mathbb{C}_+$ are in Λ. Let us show that F does not vanish on $\mathbb{R}$. Since the functions $|F|^2|\mathbb{R}$ and $|F^{-2}||\mathbb{R}$ satisfy the Helson-Szegö condition it follows from lemma 1.1 that

$$\int_{\mathbb{R}}\frac{1}{|F(x)|^2}\frac{dx}{1+x^2}<+\infty .$$

Hence F does not vanish on $\mathbb{R}$. ●

From theorem 1.2 it is easy to deduce theorem 7 stated in § 5, Part I.

THEOREM 7. Let $\Lambda=\{\lambda_n : n\in\mathbb{Z}\}\subset\mathbb{C}_\delta$, $\delta>0$, and $a>0$. The family of exponentials $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ is an unconditional basis in $L^2(0,a)$ if and only if $\Lambda\in(C)$ and $F_\Lambda\in\mathcal{M}_a$.

PROOF. It is sufficient to check (see theorems 1 and 4 in part I) that $F_\Lambda\in\mathcal{M}_a$ if and only if the Toeplitz operator $T_{\bar\theta^a B}$ is invertible. By theorem 5 $T_{\bar\theta^a B}$ is invertible if and only if $\bar\theta^a B$ is a Helson-Szegö function. It remains to apply theorem 1.2. ●

One more application of theorem 1.2 permits us to prove the necessity part*) of theorem 6 formulated in § 5 Part I. Recall the statement of this part of theorem 6. Let $\Lambda\subset\mathbb{C}_\delta$, $\delta>0$, and B be a Blaschke product with simple zeros whose zero-set coincides with Λ and α_Λ be a continuous branch of the argument of $B\bar\theta^a$ defined by

$$\alpha_\Lambda(x)=2\int_0^x\sum_{\lambda\in\Lambda}\frac{\operatorname{Im}\lambda}{|\lambda-t|^2}\,dt-ax,\quad x\in\mathbb{R}.$$

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*) Recall that the sufficiency of the same conditions for a family of exponentials to form an unconditional basis was already noted in §6, Part I immediately after the statement of theorem 6.

LEMMA 1.4. Let the family $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ be an unconditional basis in $L^2(0,a)$, $\Lambda=\{\lambda_n, n\in\mathbb{Z}\}$. Then

$$\operatorname{dist}(\alpha_\Lambda, \widetilde{L^\infty}+C) < \frac{\pi}{2}.$$

PROOF. If $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ is an unconditional basis in $L^2(0,a)$ then the Toeplitz operator $T_{\bar\theta^a B}$ is invertible, and therefore $B\bar\theta^a$ is a Helson-Szegö function. By theorem 1.2 $Bh = c\,\theta^a \bar h$ where h is an outer part of an entire function of the class $\mathcal{M}_a$ in the upper half-plane. Since functions of the class $\mathcal{M}_a$ do not vanish on $\mathbb{R}$ the function $x\mapsto \log|h^2(x)|$ is infinitely differentiable. The Hilbert transform preserves the local smoothness and thus the function $\widetilde{\log|h^2|}$ is continuous on $\mathbb{R}$. It remains to use the fact that two continuous branches of the argument of a unimodular function differ by a constant function. ●

The generating function F_Λ is uniquely determined by its zero set (it was noticed in § 5 of part I). Moreover there exists a simple formula which expresses F_Λ in terms of Λ.

LEMMA ON ZEROS OF FUNCTIONS OF CARTWRIGHT CLASS (cf. [13], [27]). Let $F\in \mathcal{C}\cap\mathcal{E}_a$, $\Lambda=\{\lambda\in\mathbb{C}: F(\lambda)=0\}$ and let all zeros of F be simple. Then

1. $$\lim_{r\to+\infty}\frac{n_+(r)}{r} = \lim_{r\to-\infty}\frac{n_-(r)}{r} = \frac{a}{2\pi}, \tag{3}$$

where $n_+(r) \overset{def}{=} \operatorname{Card}\{\lambda\in\Lambda: |\lambda|\le r, \operatorname{Re}\lambda>0\}$, $n_-(r)\overset{def}{=}\operatorname{Card}\{\lambda\in\Lambda:|\lambda|\le r, \operatorname{Re}\lambda<0\}$;

2. There exists $$\lim_{r\to+\infty}\sum_{|\lambda|\le r,\lambda\in\Lambda}\frac{1}{\lambda}. \tag{4}$$

The proof of this lemma uses delicate methods of the theory of entire functions and we refer for the proof to the books [13], [27]. Note that the conditions (3) and (4) can be considered as simple necessary conditions on $\Lambda=\{\lambda_n: n\in\mathbb{Z}\}$ for $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ to be an unconditional basis in $L^2(0,a)$. Let us suppose that Λ satisfies the conditions (3) and (4) of the lemma. Integrating by parts we obtain from (3) that $\sum_{\lambda\in\Lambda\setminus\{0\}}|\lambda|^{-2}<+\infty$. Therefore it follows from the K.Weierstrass factorization theorem (cf. [13], ch. 1, § 4, lemma 3) that the infinite product $\prod_{\lambda\in\Lambda}\left(1-\frac{z}{\lambda}\right)e^{z/\lambda}$ converges absolutely and uniformly on compact subsets of the complex plane. It follows from (4) that there exists a limit

$$\Pi = \text{v.p.}\prod_{\lambda\in\Lambda}\left(1-\frac{z}{\lambda}\right) \overset{\text{def}}{=\!=} \lim_{R\to+\infty}\prod_{\lambda\in\Lambda,|\lambda|\le R}\left(1-\frac{z}{\lambda}\right).$$

It is known that $G_\Pi = [-i\frac{a}{2}, i\frac{a}{2}]$ if Λ is the zero set of a function of class $\mathcal{C}$. This implies the following formula for the generating function F_Λ

$$F_\Lambda = e^{i\frac{a}{2}z}\,\text{v.p.}\prod_{\lambda\in\Lambda}\left(1-\frac{z}{\lambda}\right).$$

Note that if $\Lambda\subset\mathbb{C}_\delta$, $\delta>0$, and Λ satisfies the Blaschke condition (B) then the infinite product in the formula for F_Λ converges if and only if there exists a limit

$$\lim_{R\to\infty}\sum_{\lambda\in\Lambda,|\lambda|\le R}\frac{1}{\lambda}.$$

To prove this it is sufficient to use the fact that the condition (B) and the condition $\Lambda\subset\mathbb{C}_\delta$, $\delta>0$, imply the convergence of $\sum_{\lambda\in\Lambda}|\lambda|^{-2}$. This remark permits us to weaken the hypothesis of the if-part of theorem 7. The condition $F_\Lambda\in\mathcal{M}_a$ can be replaced by conditions (3), (4) and the following condition: the function

$$x\longmapsto\prod_{\lambda\in\Lambda}\left|1-\frac{x}{\lambda}\right|^2,\quad x\in\mathbb{R},$$

satisfies the Helson-Szegö condition on the real line. Together with the condition $\Lambda\in(C)$ this implies that $F_\Lambda\in\mathcal{M}_a$ where a is defined by (3).

To prove theorem 8 we need the following lemma.

LEMMA 1.5. Let $\{\lambda_n\}$ be a sequence of real numbers such that $\inf_{m\ne n}|\lambda_n-\lambda_m| \overset{\text{def}}{=\!=} 3\delta>0$ and let $y>0$. Then the function

$$x\mapsto\sum_{n\in\mathbb{Z}}\log\left(1-\frac{y^2}{(x-\lambda_n)^2+y^2}\right)$$

belongs to $BMO(\mathbb{R})$.

PROOF. Put $\Delta_n \overset{\text{def}}{=\!=} \{x\in\mathbb{R}:|x-\lambda_n|<\delta\}$, $n\in\mathbb{Z}$. By the hypothesis of the lemma $\operatorname{dist}(\Delta_n,\Delta_m)\geqslant\delta$ if $n\ne m$. If $x\notin\Delta_n$ then $y^2((x-\lambda_n)^2+y^2)^{-1}\leqslant y^2(\delta^2+y^2)^{-1}$. There exists a number $C>0$ depending only on δy^{-1} such that $\log(1-t)\geqslant -Ct$ for $0\leqslant t\leqslant\frac{y^2}{\delta^2+y^2}$. Whence it follows that

$$0 \geqslant \sum_{n\in\mathbb{Z},\, x\notin\Delta_n} \log\left(1-\frac{y^2}{(x-\lambda_n)^2+y^2}\right) \geqslant -C \sum_{n\in\mathbb{Z},\, x\notin\Delta_n} \frac{y^2}{(x-\lambda_n)^2+y^2} .$$

It is clear that

$$\sum_{n\in\mathbb{Z},\, x\notin\Delta_n} \frac{y^2}{(x-\lambda_n)^2+y^2} \leqslant \sum_{K=0}^{\infty} \sum_{\substack{n \\ 2^K\delta\leqslant|x-\lambda_n|\leqslant 2^{K+1}\delta}} \frac{y^2}{4^K\delta^2+y^2} \leqslant$$

$$\leqslant \sum_{K=0}^{\infty} \frac{1}{4^K\left(\frac{\delta}{y}\right)^2+1} \operatorname{Card}\{n\in\mathbb{Z} : 2^K\delta\leqslant|x-\lambda_n|\leqslant 2^{K+1}\delta\} \leqslant \sum_{K\geqslant 0} \frac{2^{K+1}}{4^K\left(\frac{\delta}{y}\right)^2+1} < +\infty.$$

These estimates imply that

$$-\sum_{n\in\mathbb{Z}} \log\left(1-\frac{y^2}{|x-\lambda_n|^2+y^2}\right) = u(x) + \sum_{n\in\mathbb{Z}} \log^+ \frac{\delta^2}{(x-\lambda_n)^2} ,$$

where $u\in L^\infty(\mathbb{R})$. The function $\log^+ \frac{\delta^2}{x^2}$ belongs to BMO and the distances between the supports Δ_n of its translates $\log^+ \frac{\delta^2}{(x-\lambda_n)^2}$ are at least δ. It follows that the sum $v(x) = \sum_{n\in\mathbb{Z}} \log^+ \frac{\delta^2}{(x-\lambda_n)^2}$ belongs to BMO. To prove this we use the description of BMO in terms of mean oscillations.

If $I\in\mathcal{J}$ and $|I|<\delta$ then $v|I = \log^+ \frac{\delta^2}{(x-\lambda_n)^2}|I$ for some $n\in\mathbb{Z}$. If $|I|\geqslant\delta$ then

$$\frac{1}{|I|}\int_I |v(x)|\,dx \leqslant \frac{1}{|I|}\cdot\frac{|I|}{3\delta}\int_{-\delta}^{\delta} \log^+ \frac{\delta^2}{x^2}\,dx < +\infty. \qquad \bullet$$

THEOREM 8. Let $\lambda_n\in\mathbb{R}$, $n\in\mathbb{Z}$. The family of exponentials $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ is a Riesz basis in $L^2(0,a)$ if and only if

1. $\inf\limits_{n\neq m} |\lambda_n-\lambda_m|>0$;
2. $N_\Lambda - \frac{a}{2\pi}x \in \mathcal{P}_{1/4}$.

PROOF. The "only if" part. Let $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ be a Riesz basis in $L^2(0,a)$. Then $\Lambda+iy\in(C)$ for any $y>0$ and so $\inf\limits_{n\neq m}|\lambda_n-\lambda_m|>0$. Since $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ is a Riesz basis then there exists the generating function

$$F_\Lambda(z) = e^{i\frac{a}{2}z}\cdot \text{v.p.}\prod_{n\in\mathbb{Z}}\left(1-\frac{z}{\lambda_n}\right).$$

Let $F_{\Lambda+iy}$ be the generating function for the set $\Lambda+iy$. The functions F_Λ and $F_{\Lambda+iy}$ obviously satisfy

$$F_{\Lambda+iy}(z)\,F_\Lambda(-iy)=F_\Lambda(z-iy).$$

Our first purpose is to prove that the function $x\mapsto \log|F_\Lambda(x)|$ belongs to BMO. To prove this we consider the difference

$$\log|F_\Lambda(x)|^2-\log|F_{\Lambda+iy}(x)|^2=\log|F_\Lambda(-iy)|^2-\frac{a}{2}y+\sum_n\log\left(1-\frac{y^2}{|x-\lambda_n|^2+y^2}\right).$$

The sum on the right-hand side of the formula belongs to BMO by lemma 1.5. The function $x\mapsto\log|F_{\Lambda+iy}(x)|^2$ belongs to BMO because $|F_{\Lambda+iy}|^2\,|\mathbb{R}\in(HS)$ by theorem 7. Therefore $\log|F_\Lambda|^2|\mathbb{R}\in$ $\in\mathrm{BMO}$.

Let now c be a complex number such that $|c|=1$ and $F_\Lambda^2(i)c>0$. Then $f=F_\Lambda^2\cdot c$ is an outer function and

$$\log f(z)=\frac{1}{\pi i}\int_{\mathbb{R}}\frac{tz+1}{t-z}\,\frac{\log|F_\Lambda(t)|^2}{1+t^2}\,dt=iaz+\log c+2\sum_n\log\left(1-\frac{z}{\lambda_n}\right).\quad(5)$$

This formula enables us to compute the values of $\widetilde{\log|F_\Lambda|^2}$ on the real line. Note that

$$\lim_{y\to 0+}\log\left(1-\frac{x+iy}{\lambda_n}\right)=\log\left|1-\frac{x}{\lambda_n}\right|-\pi i\begin{cases}\chi_{[\lambda_n,+\infty)}(x) & \text{if } \lambda_n>0,\\ -\chi_{(-\infty,\lambda_n]}(x) & \text{if } \lambda_n<0.\end{cases}$$

It follows that

$$\log f(x)=\log|F_\Lambda(x)|^2+i(ax+\arg c-2\pi N_\Lambda(x)).$$

Thus

$$\widetilde{\log|F_\Lambda|^2}(x)=ax-2\pi N_\Lambda(x)+\arg c.$$

By theorem 7 $|F_\Lambda(x+iy)|^2\in(A_2)$ for any $y>0$. This implies that $N_\Lambda(x)-\frac{a}{2\pi}x\in\mathcal{P}_{1/4}$.

The "if" part. The most difficult step of the proof is to show that the generating function corresponding to Λ exists.

Suppose that the function $\alpha(x)=\frac{a}{2}x-\pi N_\Lambda(x)-c$ belongs to $\mathcal{P}_{1/4}$. Here c is a complex number such that the harmonic continuation of α to the half-plane $\mathbb{C}_+$ (we denote this continuation by the same letter α) vanishes at the point i.

It is obvious that $\alpha, \tilde{\alpha} \in BMO$. Using the fact that the Hilbert transform preserves the local smoothness it is easy to see that $\tilde{\alpha}$ is infinitely differentiable on $\mathbb{R}\setminus\Lambda$. In a neighbourhood of a point $\lambda \in \Lambda$ the following equality holds

$$-\tilde{\alpha}(x) = \log\left|1-\frac{x}{\lambda}\right| + \beta_\lambda(x), \tag{6}$$

where β_λ is a differentiable function in a neighbourhood of λ.

Consider an outer function on $\mathbb{C}_+$

$$F = \exp(-\tilde{\alpha} + i\alpha).$$

It is easy to see that the function $f(z) = F(z)e^{-i\frac{a}{2}z}e^{ic}$ is real on $\mathbb{R}$ and differentiable (cf. (6)). By the symmetry principle f can be analytically continued into $\mathbb{C}_-$ and so f is an entire function.

Let us show that $F \in \mathcal{E}_a$. From the fact that $\alpha \in \mathcal{P}_{1/4}$ it follows that there exists a positive number d such that the restriction of $|F^2|$ to the line $\{\zeta \in \mathbb{C} : \operatorname{Im}\zeta = d\}$ satisfies the Helson-Szegö condition (HS). The equality $f(z) = \overline{f(\bar{z})}$ implies that $|F^2(\zeta)| = |F^2(\bar{\zeta})|e^{ad}$ if $\operatorname{Im}\zeta = -d$. Hence the restriction of $|F^2|$ to the line $\{\zeta : \operatorname{Im}\zeta = -d\}$ also satisfies the Helson-Szegö condition. It is also clear that $F|\mathbb{C}_+$, $F|\mathbb{C}_-$ belong to the Nevanlinna classes in $\mathbb{C}_+$ and $\mathbb{C}_-$. Therefore $F|\mathbb{C}_{-d}$ belongs to the Nevanlinna class in $\mathbb{C}_{-d}$. The inner part of F in $\mathbb{C}_{-d}$ has no singular factors because $F|\mathbb{C}_+$ is an outer function. Lemma 1.1 applied to $\mathbb{C}_{-d}$ implies that the function $(z+2id)^{-1}\cdot F$ belongs to the Hardy class H^2 in $\mathbb{C}_{-d}$. Applying the method used in the proof of theorem 1.2 we obtain that $|F(z)| \leqslant \mathrm{const}\cdot|z+i|$ if $\operatorname{Im} z \geqslant 0$. Since

$$F(z) = \overline{F(\bar{z})}\, e^{iaz} e^{2ic}, \qquad \operatorname{Im} z < 0. \tag{7}$$

We obtain that $|F(z)| \leqslant \mathrm{const}\cdot|z-i|e^{a|\operatorname{Im} z|}$. These inequalities show that F is of exponential type. Moreover it is clear that $h_F(\pi/2) = 0$ (because $F|\mathbb{C}_+$ is outer) and that $h_F(-\pi/2) = a$ (cf. (7)). Thus $F \in \mathcal{E}_a$. Put $F^*(z) = F(z-id)\cdot F(-id)^{-1}$. It is easy to see that F^* is the generating function for $\Lambda + id$. Moreover, $|F^*|^2|\mathbb{R}$ satisfies the Helson-Szegö condition. By theorem 7 we can conclude that $\{e^{i(\lambda_n + id)}\}_{n\in\mathbb{Z}}$ is a Riesz basis in $L^2(0,a)$. ●

REMARK. Let $\Lambda \subset \mathbb{R}$, $\inf\limits_{n\neq m}|\lambda_n - \lambda_m| > 0$ and

$\Lambda(x)-\frac{a}{2\pi}x \in \mathcal{P}_{1/4}$. Then the harmonic continuation $\mathcal{U}(x)$ of the function $\Lambda-\frac{a}{2\pi}x$ into the upper half-plane satisfies the following condition:

for any positive y there exist a real number c and $u, v \in L^\infty(\mathbb{R})$ such that

$$\mathcal{U}(x+iy)=c+\tilde{u}(x)+v(x)$$

and $\|v\|_\infty < 1/4$.

Indeed, if the above equality holds for some $y>0$, $c\in\mathbb{R}$, $u,v\in L^\infty(\mathbb{R})$ then it follows from (2) that $|F_\Lambda(x+iy)|^2\in(A_2)$ and so $|F_{\Lambda+iy}(x)|^2\in(A_2)$. Since the translation $\Lambda+iy\to\Lambda+iy'$ induces an isomorphism in $L^2(0,a)$, $|F_{\Lambda+iy}(x)|^2\in(A_2)$ for any $y>0$. ●

If $\{e^{i\lambda x}\chi_{[0,a]}:\lambda\in\Lambda\}$ is an unconditional basis in $L^2(0,a)$ ($\Lambda\subset\mathbb{C}_\delta$, $\delta>0$) then, as we saw in § 2 of part II, the angle between the subspaces K_{θ^a} and K_B of H^2_+ is non-zero and they span H^2_+ . Consider the subspaces $\theta_a H^2_- = H^2_- \oplus K_{\theta_a}$ and BH^2_+ . Now it is possible to obtain an explicit formula for the projection $\mathcal{P}_{\theta H^2_-\|BH^2_+}$ onto θH^2_- along BH^2_+ using the generating function $F\overset{def}{=}F_\Lambda$.

THEOREM 1.6. For $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ to be an unconditional basis in $L^2(0,a)$ it is necessary and sufficient that $\Lambda\in(C)$, $\theta H^2_- + BH^2_+$ is dense in $L^2(\mathbb{R})$ and the projection $\mathcal{P}_{\theta H^2_-\|BH^2_+}$ is bounded. If $F=F_\Lambda$ and $\mathcal{M}_g$ is the multiplication by g operator on $L^2(\mathbb{R})$ then

$$\mathcal{P}_{\theta H^2_-\|BH^2_+}=\mathcal{M}_F\mathbb{P}_-\mathcal{M}_{1/F} .$$

PROOF. The first part of the theorem easily follows from corollary 2.2 of part II. It remains to prove the formula for the projection. It is easy to see that the operator $\mathcal{M}_F\mathbb{P}_-\mathcal{M}_{1/F}$ is bounded in $L^2(\mathbb{R})$ if and only if $\mathbb{P}_-$ is bounded in the weighted space $L^2(|F|^2dx)$ and this is equivalent (by the Hunt-Muckenhoupt-Wheeden theorem) to the fact that $|F|^2\in(A_2)$. We check the formula on a dense subset of $L^2(\mathbb{R})$. Since the function $(z+i)^{-1}$ is outer $H^2_+=\mathrm{Span}\,(e^{iaz}(z+i)^{-1}: a\geqslant 0)$ by P.Lax's theorem. Denote by $\mathcal{X}$ the linear span of functions $e^{iaz}(z+i)^{-1}$, $a\geqslant 0$. It is clear that $|f(x)|\leqslant\frac{C_f}{|x+i|}$ where $C_f>0$ if $f\in\mathcal{X}$. Since $|F|^2\in(A_2)$

$$\int_{\mathbb{R}}\frac{|F(x)|^2}{1+x^2}dx+\int_{\mathbb{R}}\frac{|F(x)|^{-2}}{1+x^2}dx<+\infty .$$

At last by theorem 1.1.

$$F = Bh = c\bar{h}\,\theta_a$$

where h is an outer function. Let $f = Bg$ where $g \in \mathcal{X}$. We have

$$\mathcal{M}_F \mathbb{P}_- \mathcal{M}_{1/F} f = \mathcal{M}_F \mathbb{P}_- \frac{g}{h} = 0 \qquad \text{because}$$

obviously, $gh^{-1} \in H^2_+$.

If $f = \theta_a \bar{g}$ where $g \in \mathcal{X}$ then

$$\mathcal{M}_F \mathbb{P}_- \mathcal{M}_{1/F} f = \mathcal{M}_F \mathbb{P}_- \frac{1}{c\bar{h}\,\theta_a}\,\theta_a \bar{g} = F\frac{\bar{g}}{c\bar{h}} = f. \qquad \bullet$$

2. Theorems on perturbations of unconditional bases.

We begin this Section with the deducing the theorems of S.A.Avdonin and V.È. Kacnelson (for the statements see §7 PartI). The following lemma reduces the general case to the examination of bases of exponentials with only real frequencies.

LEMMA 2.1. Let $\Lambda = \{\lambda_n : n \in \mathbb{Z}\} \subset \mathbb{R}$ and let $(\delta_n)_{n\in\mathbb{Z}}$ be an arbitrary bounded sequence of real numbers. Further, let us assume that the set Λ is separated and let $\Lambda^* \overset{def}{=\!=} \{\lambda_n^* : n \in \mathbb{Z}\}$, $\lambda_n^* \overset{def}{=\!=} \lambda_n + i\delta_n$. Then the family of exponentials $(e^{i\lambda_n t})_{n\in\mathbb{Z}}$ forms a Riesz basis in the space $L^2(0,a)$ if and only if the family $(e^{i\lambda_n^* t})_{n\in\mathbb{Z}}$ does.

THE PROOF can be easily obtained from theorem 7. Let $y > 10 \sup_n |\delta_n|$. We shall examine the following ratio

$$|F_{\Lambda^*+iy}(x)|^2 \cdot |F_{\Lambda+iy}(x)|^{-2} = \prod_{n\in\mathbb{Z}} \frac{|\lambda_n + iy|^2}{|\lambda_n^* + iy|^2} \prod_{n\in\mathbb{Z}} \left| \frac{\lambda_n + i\delta_n + iy - x}{\lambda_n + iy - x} \right|^2.$$

It is clear that

$$\left| \frac{\lambda_n + i\delta_n + iy - x}{\lambda_n + iy - x} \right|^2 = \frac{(\lambda_n - x)^2 + (y+\delta_n)^2}{(\lambda_n - x)^2 + y^2} = 1 + \frac{\delta_n(2y+\delta_n)}{(\lambda_n - x)^2 + y^2}.$$

Since $y > 10 \sup_n |\delta_n|$, we have

$$\frac{\delta_n(2y+\delta_n)}{(\lambda_n - x)^2 + y^2} \leqslant \frac{2\delta_n y + \delta_n^2}{y^2} < \frac{1}{2}.$$

Further, let λ_m be the point of Λ nearest to the fixed point x , $x \in \mathbb{R}$, and let $d = \inf_{k \neq n} |\lambda_k - \lambda_n|$. Then

$$\sum_{n\in\mathbb{Z}} \frac{\delta_n(2y+\delta_n)}{(\lambda_n-x)^2+y^2} \leqslant \frac{1}{2} + \sum_{n\neq m} \frac{\delta_n(2y+\delta_n)}{(\lambda_n-x)^2+y^2} \leqslant$$

$$\leqslant \frac{1}{2} + y^2 \sum_{n\neq m} \frac{1}{(\lambda_n-x)^2} \leqslant \frac{1}{2} + y^2 \sum_{k=0}^{\infty} \frac{4}{4^k d^2} \operatorname{Card}\{n : 2^{k-1}d < |\lambda_n - x| \leqslant 2^k d\} \leqslant$$

$$\leqslant \frac{1}{2} + 4y^2 \sum_{k=0}^{\infty} \frac{1}{2^k d^2} = \frac{1}{2} + \frac{8y^2}{d^2} .$$

This yields

$$\log|F_{\Lambda^*+iy}(x)|^2 - \log|F_{\Lambda+iy}(x)|^2 \in L^\infty(\mathbb{R}). \qquad \bullet$$

Let $(\delta_n)_{n\in\mathbb{Z}}$ be a bounded sequence of real numbers and $\Lambda = \{\lambda_n : n \in \mathbb{Z}\} \subset \mathbb{R}$. We denote

$$\Delta_x(R) \stackrel{\text{def}}{=} \sum_{x-R\leqslant\lambda_k\leqslant x+R} \delta_k ,$$

and let $\lambda_n^* = \lambda_n + \delta_n$, $\Lambda^* = \{\lambda_n^* : n \in \mathbb{Z}\}$ is a "real perturbation" of the set Λ .

Lemma 2.1 allows to phrase the Avdonin's theorem as follows.

THEOREM 2.2. Let $\Lambda = \{\lambda_n : n \in \mathbb{Z}\}$ be a separated subset of the real line. Suppose that Λ is a zero set of a STF with the width of the indicator diagram equal to 2π . Let us assume that the set Λ^* is separated and

$$\varlimsup_{R\to+\infty} \sup_{x\in\mathbb{R}} \frac{|\Delta_x(R)|}{2R} < \frac{1}{4} .$$

Then the family $(e^{i\lambda_n^* t})_{n\in\mathbb{Z}}$ forms a Riesz basis in the space $L^2(0, 2\pi)$.

THE PROOF of the theorem in its essential features follows that of the Kadec $1/4$ -theorem expounded in Section 5, Part I.

Let F_Λ be a generating function for the set Λ . The following formula is true (see §1)

$$\log|F_\Lambda(x+iy)|^2 = -2\widetilde{d_{\Lambda+iy}}(x) + c,$$

where $c \in \mathbb{R}$, $y > 0$. According to the definition of the STF the function $x \mapsto \log|F_\Lambda(x+iy)|^2$ $(x \in \mathbb{R})$ is bounded and therefore $d_{\Lambda+iy} \in \widetilde{L^\infty} + \mathbb{C}$. In order to use Theorem 6, let us compute the difference

$$\alpha_{\Lambda+iy}(x)-\alpha_{\Lambda^*+iy}(x)=2\sum_{n\in\mathbb{Z}}\int_0^x\left\{\frac{y}{(t-\lambda_n)^2+y^2}-\frac{y}{(t-\delta_n-\lambda_n)^2+y^2}\right\}dt=$$

$$=2\sum_{n\in\mathbb{Z}}\int_{x-\delta_n}^{x}\frac{y}{(t-\lambda_n)^2+y^2}\,dt+\text{const}.$$

By the mean value theorem we have

$$\int_{x-\delta_n}^{x}\frac{y}{(t-\lambda_n)^2+y^2}\,dt=\frac{y}{(x-\lambda_n)^2+y^2}\cdot\delta_n\left(1+O\left(\tfrac{1}{y}\right)\right),\quad y\to+\infty,$$

uniformly with respect to x , $x\in\mathbb{R}$. It remains only to verify that

$$\varlimsup_{y\to+\infty}\ \sup_{x\in\mathbb{R}\setminus\Lambda}\left|\sum_{n\in\mathbb{Z}}\frac{y}{(x-\lambda_n)^2+y^2}\,\delta_n\right|<\frac{\pi}{4}.$$

If $x\in\mathbb{R}\setminus\Lambda$, then

$$\sum_{n\in\mathbb{Z}}\frac{y}{(x-\lambda_n)^2+y^2}\,\delta_n=\int_0^{+\infty}\frac{y}{t^2+y^2}\,d\Delta_x(t)=$$

$$=\int_0^{+\infty}\frac{\Delta_x(t)}{t}\cdot\frac{2yt^2}{(t^2+y^2)^2}\,dt=\int_0^{+\infty}\frac{\Delta_x(yt)}{2yt}\cdot\frac{4t^2}{(1+t^2)^2}\,dt.$$

Let R_0 be a such positive number that $\sup\limits_{x\in\mathbb{R}}\dfrac{|\Delta_x(R)|}{2R}<\dfrac{1}{4}$ if $R\geqslant R_0$. Then we have

$$\sup_{x\in\mathbb{R}}\left|\int_{R_0y^{-1}}^{\infty}\frac{\Delta_x(yt)}{2yt}\,\frac{4t^2}{(1+t^2)^2}\,dt\right|<\int_0^{\infty}\frac{t^2}{(1+t^2)^2}\,dt=\frac{\pi}{4},$$

and

$$\sup_{x\in\mathbb{R}}\left|\int_0^{R_0y^{-1}}\frac{\Delta_x(yt)}{2yt}\,\frac{4t^2}{(1+t^2)^2}\,dt\right|\leqslant\sup_{x\in\mathbb{R}}\sum_{x-R_0\leqslant\lambda_k\leqslant x+R_0}|\delta_k|\cdot\frac{1}{y}\int_0^{\infty}\frac{2t}{(1+t^2)^2}\,dt\leqslant$$

$$\leqslant\frac{1}{y}\sup_{k\in\mathbb{Z}}|\delta_k|\cdot\mathrm{Card}\{k: x-R_0\leqslant\lambda_k\leqslant x+R_0\}.$$

This expression obviously tends to zero as $y\to\infty$. ●

PROOF OF THE V.È. KACNELSON'S THEOREM. We shall deduce this theorem from Theorem 2.2. Lemma 2.1 permits to consider only real frequencies in this case also. Let Λ be a subset of the real line. Suppose that Λ is the zero set of a STF with the width of the indicator diagram equal to 2π . Let $\rho_n = \inf\{|\lambda_n - \lambda_m| : m \in \mathbb{Z}\setminus\{n\}\}$. In the Kacnelson's theorem "perturbations" δ_n were supposed to satisfy the condition

$$|\delta_n| \leqslant d\rho_n \quad \text{, where} \quad 0 \leqslant d < \frac{1}{4}.$$

Lemma 2.3 (see below) shows that for zeros of a STF the sequence $(\rho_n)_{n\in\mathbb{Z}}$ must be bounded, say by a constant ρ , $\rho > 0$. But then the inequality

$$\frac{|\Delta_x(R)|}{2R} < \frac{1}{4}$$

obviously is valid, if $R \gg \rho$. ●

LEMMA 2.3. Let $\Lambda = \{\lambda_n : n \in \mathbb{Z}\}$ be a subset of the real line coinciding with the zero-set of a STF, and $\rho_n = \inf\{|\lambda_n - \lambda_m| : m \neq n\}$. Then the sequence $(\rho_n)_{n\in\mathbb{Z}}$ is bounded.

PROOF. Put $S = \prod_{n\in\mathbb{Z}} \left(1 - \frac{z}{\lambda_n}\right)$ and suppose the width of the indicator diagram of S is equal to a . Then

$$\lim_{y\to\infty} \frac{S'(x+iy)}{S(x+iy)} = -i\,\frac{a}{2}\,, \tag{8}$$

uniformly with respect to x , $x \in \mathbb{R}$. A simple proof of this fact can be found in an interesting paper of B.Ja.Levin and I.V.Ostrovskii [15], containing many other useful facts conserning the structure of zero-sets of STF's (see the remark to lemma 2 on the page 89 in [15]). Computing the imaginary part of the equality

$$\frac{S'(z)}{S(z)} = (\log S(z))' = \sum_{n\in\mathbb{Z}} \frac{1}{z-\lambda_n}\,,$$

we obtain from the formula (8):

$$\lim_{y\to\infty} \sup_{x\in\mathbb{R}} \left| \sum_{n\in\mathbb{Z}} \frac{y}{(x-\lambda_n)^2 + y^2} - \frac{a}{2} \right| = 0. \tag{9}$$

If the sequence $(\rho_n)_{n\in\mathbb{Z}}$ is unbounded, then for any R , $R > 0$, there exists a number n , $n \in \mathbb{Z}$, such that $\rho_n > 2R$. In this case the interval $(\lambda_n - R, \lambda_n + R)$ contains only one point of the set Λ . Let $x \in (\lambda_n - R, \lambda_n + R)$, $y = \sqrt{R}$. Then

$$\sum_{k\neq n} \frac{y}{(x-\lambda_k)^2 + y^2} \leqslant y \sum_{k\neq n} \frac{1}{(x-\lambda_k)^2} \leqslant$$

$$\leqslant y\sum_{m=0}^{\infty}\frac{\mathrm{Card}\{k:2^m R\leqslant|x-\lambda_k|\leqslant 2^{m+1}R\}}{4^m R^2}\leqslant \mathrm{const}\,\frac{y}{R}=\mathrm{const}\,\frac{1}{\sqrt{R}}\,.$$

Since $\dfrac{y}{(x-\lambda_n)^2+y^2}\leqslant\dfrac{1}{y}=\dfrac{1}{\sqrt{R}}$, we have

$$\sup_{x\in(\lambda_n-R,\lambda_n+R)}\sum_{m\in\mathbb{Z}}\frac{y}{(x-\lambda_m)^2+y^2}\leqslant\frac{\mathrm{const}}{\sqrt{R}}$$

for $y=\sqrt{R}$. But this contradicts (9) if R is large enough.●

We consider now a "perturbation theorem" for unconditional bases of exponents in a more general setting dropping the assumption $\sup\limits_{\lambda\in\Lambda}\mathrm{Im}\,\lambda<+\infty$.

Let us introduce some notation. Suppose that $\Lambda=\{\lambda_n:n\in\mathbb{Z}\}\subset\subset\mathbb{C}_\delta$, $\delta>0$, and the exponentials $(e^{i\lambda_n t})_{n\in\mathbb{Z}}$ form an unconditional basis in the space $L^2(0,a)$. For every integer n consider the disc

$$D(\lambda_n,\delta_n)=\{\zeta\in\mathbb{C}:|\zeta-\lambda_n|\leqslant\delta_n\}.$$

We shall be interested in the restrictions to be imposed on $(\delta_n)_{n\in\mathbb{Z}}$ ensuring that any family $(e^{i\lambda_n^* t})_{n\in\mathbb{Z}}$ with $\lambda_n^*\in D(\lambda_n,\delta_n)$ forms an unconditional basis in $L^2(0,a)$. Denote by the symbol $P_z(t)$ the Poisson kernel $\frac{1}{\pi}\frac{\mathrm{Im}\,z}{|z-t|^2}$, $\mathrm{Im}\,z>0$.

THEOREM 2.4. Let $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ be an unconditional basis in the space $L^2(0,a)$ and

$$\inf_{y>0}\{\|2\pi\sum_{n\in\mathbb{Z}}\delta_n P_{\lambda_n+iy}\|_\infty+\mathrm{dist}\,(\alpha_{\Lambda+iy},\widetilde{L^\infty}+\mathbb{C})\}<\frac{\pi}{2}\,.$$

Suppose also that $\lambda_n^*\in D(\lambda_n,\delta_n)$, $n\in\mathbb{Z}$, and $\{\lambda_n^*:n\in\mathbb{Z}\}\in(C)$. Then the family $(e^{i\lambda_n^* x})_{n\in\mathbb{Z}}$ forms an unconditional basis in the space $L^2(0,a)$.

THE PROOF is based on Theorem 6. Let $\lambda\in\mathbb{C}_\delta$ (i.e. $D(\lambda,\delta)\subset\mathbb{C}_+$), and estimate the difference

$$\int_0^x\frac{\mathrm{Im}\,\lambda^*}{|\lambda^*-t|^2}\,dt-\int_0^x\frac{\mathrm{Im}\,\lambda}{|\lambda-t|^2}\,dt.$$

To do this we note the formula

$$\int_0^x\frac{1}{\bar\lambda-t}\,dt=\log\left(\frac{1}{1-\frac{x}{\bar\lambda}}\right)$$

where the right hand side is meant as principal value of the logarithm. Taking imaginary parts we obtain

$$\int_0^x \frac{\operatorname{Im}\lambda}{|\lambda-t|^2}\,dt = \widetilde{\log \frac{1}{|1-\frac{x}{\lambda}|}} + \arg\left(\frac{1}{1-\frac{t}{\lambda}}\right).$$

Let us consider two cases. At first let $\operatorname{Re}\lambda^* = \operatorname{Re}\lambda$, $\lambda^* = \lambda + i\eta$, $|\eta| \leqslant \delta$. Then

$$\int_0^x \frac{\operatorname{Im}\lambda^*}{|\lambda^*-t|^2}\,dt - \int_0^x \frac{\operatorname{Im}\lambda}{|\lambda-t|^2}\,dt = -\widetilde{\log\left|\frac{x-\lambda^*}{x-\lambda}\right|} + \arg\left(1-\frac{i}{\lambda^*}\right)^{-1} - \arg\left(1-\frac{i}{\lambda}\right)^{-1}.$$

It is not difficult to see that

$$\left|\frac{x-\lambda^*}{x-\lambda}\right|^2 = 1 + \frac{(\operatorname{Im}\lambda^*)^2 - (\operatorname{Im}\lambda)^2}{|x-\lambda|^2} = 1 + \frac{\eta(2\operatorname{Im}\lambda+\eta)}{|x-\lambda|^2}.$$

Since $|\eta| = \delta < \operatorname{Im}\lambda$, we have $|\eta||2\operatorname{Im}\lambda+\eta| < 3\delta\operatorname{Im}\lambda$. Hence the convergence of the series $\sum_{n\in\mathbb{Z}} \frac{\delta_n \operatorname{Im}\lambda_n}{|x-\lambda_n|^2}$ implies $\alpha_{\Lambda^*} - \alpha_\Lambda \in \widetilde{L}^\infty + \mathbb{C}$.

Now let $\operatorname{Im}\lambda^* = \operatorname{Im}\lambda$, i.e. $\lambda^* = \lambda + \eta$, $\eta \in (-\delta, \delta)$. Then

$$\int_0^x \frac{\operatorname{Im}\lambda}{|\lambda-t|^2}\,dt - \int_0^x \frac{\operatorname{Im}\lambda^*}{|\lambda^*-t|^2}\,dt = \int_0^x \frac{\operatorname{Im}\lambda}{|\lambda-t|^2}\,dt - \int_{-\eta}^{x-\eta} \frac{\operatorname{Im}\lambda}{|\lambda-t|^2}\,dt =$$

$$= \int_{x-\eta}^x \frac{\operatorname{Im}\lambda}{|\lambda-t|^2}\,dt - \int_{-\eta}^0 \frac{\operatorname{Im}\lambda}{|\lambda-t|^2}\,dt.$$

Clearly,

$$\int_{x-\eta}^x \frac{\operatorname{Im}\lambda}{|\lambda-t|^2}\,dt = \frac{\eta\cdot\operatorname{Im}\lambda}{|\lambda-x|^2}\left(1+O\left(\frac{1}{\operatorname{Im}\lambda}\right)\right)$$

uniformly with respect to x, $x \in \mathbb{R}$.

Let now arbitrary perturbations λ_n^* be given. It is easy to see that any of them can be obtained in two steps: at first we shift the point λ_n along the real axis up to some point λ_n' and then along the imaginary axis up to the point λ_n^*. Taking a number y large enough, we have

$$\operatorname{dist}_{L^\infty}(\alpha_{\Lambda'+iy} - \alpha_{\Lambda+iy}, \mathbb{C}) \leqslant \sup_{x\in\mathbb{R}} 2\left(1+O\left(\tfrac{1}{y}\right)\right) \sum_n \delta_n P_{\lambda_n+iy}(x).$$

For the shifts along the imaginary axis the inclusion $\alpha_{\Lambda^*+iy} - \alpha_{\Lambda'+iy} \in \widetilde{L}^\infty + \mathbb{C}$ is valid. It remains to refer to Theorem 6. ●

COROLLARY 2.5. Let $\Lambda \subset \mathbb{C}_\delta$, $\delta > 0$, and suppose the family of the exponentials $(e^{i\lambda t})_{\lambda\in\Lambda}$ forms an unconditional basis in the space $L^2(0,a)$. Then there exists a number ε, $\varepsilon>0$, such that any choice of a single point λ_n^* from every disc $\Delta_n(\lambda_n,\varepsilon)$ gives rise to an unconditional basis in $L^2(0,a)$.

PROOF. Obvious. ●

Note that Corollary 2.5 is a generalization of the Duffin and Schaeffer theorem [35], cited in Section 7, Part I.

3. The set of frequencies does not lie in a strip of finite width. Complementation up to an unconditional basis.

Are there unconditional bases in the space $L^2(0,a)$ consisting of exponentials $(e^{i\lambda x})_{\lambda\in\Lambda}$, if $\sup\limits_{\lambda\in\Lambda} \operatorname{Im}\lambda=+\infty$? The affirmative answer to this question was obtained by S.A.Vinogradov. His reasoning was improved later on by V.I.Vasjunin. One more question which naturally arises is as follows: it is possible to complement any unconditional basis of exponentials $(e^{i\lambda x})_{\lambda\in\Lambda}$ (in their linear span) up to an unconditional basis $(e^{i\lambda x})_{\lambda\in\Lambda'}$, $\Lambda'\supset\Lambda$, in the whole space $L^2(0,a)$? We do not know now (1980), whether this is true, but we shall find a sufficient condition (V.I.Vasjunin), ensuring above-mentioned possibility to "enlarge" the basis. In particular, any family $(e^{i\lambda x})_{\lambda\in\Lambda}$ under conditions $\Lambda\in(C)$, $\Lambda\subset\mathbb{C}_\delta$, $\delta>0$ and $\lim\limits_{\lambda\in\Lambda}\operatorname{Im}\lambda=+\infty$ can be complemented up to an unconditional basis of exponentials in the whole space $L^2(0,a)$.

Before we shall formulate and prove the corresponding theorems, let us descuss some heuristic considerations. For an affirmative answer to the first formulated question it is obviously, necessary and sufficient the existence of an *interpolating Blaschke product* B (i.e. such that the set of its zeros is a Carleson set) and an outer function F such that $\|\Theta_a\bar{B}-cF\|<1$, $c\in\mathbb{C}$, $|c|=1$. But then the set $\{f\in H^\infty:\|\Theta_a\bar{B}-f\|_\infty<1\}$ consists of functions of the form cf_e, where $c\in\mathbb{C}$, $|c|=1$ and f_e is outer (see Remark 1 after Theorem 4 $\mathbb{D}$ from Section 2, Part II). Consider functions F, $F\in H^\infty$, such that the module of the difference $\Theta_a\bar{B}-F$ is a constant $\alpha>0$ on $\mathbb{R}$. It is well-known that such functions F exist if

$\alpha > \operatorname{dist}(\theta_a \bar{B}, H^\infty)$ (see [18] p.262 or [1]). If $|\theta_a \bar{B} - F| = \alpha$ on $\mathbb{R}$, then $\theta_a - BF = \alpha B^*$, where B^* is a Blaschke product. The S.A.Vinogradov's idea is to inverse this reasoning. Let us take a suitable Blaschke product B^* whose zeros form a Carleson set, and let $0 < \alpha < 1$. Then

$$\theta_a - \alpha B^* = BF,$$

where F is an outer function. In this case the Toeplitz operator $T_{\theta_a \bar{B}}$ is invertible, of course, and Theorems 2 and 3 may be used; but the main difficulty is to show that the Blaschke product B is interpolating if the product B^* is. Zeros of B can be controlled by means of Rouché's theorem. Therefore if the imaginary parts of zeros of the product B^* are unbounded, then the zeros of the product B are unbounded too. Let us turn now to the exact formulations.

Let B^* be a Blaschke product with zeros a_n, $n = 1, 2, \dots$, and let $b_n = b_{a_n} \overset{\text{def}}{=} \frac{z - a_n}{z - \bar{a}_n}$, $B_n^* \overset{\text{def}}{=} B^* b_n^{-1}$. For a given pair of numbers α, $\delta \in (0,1)$, consider a set $\mathcal{B}(\alpha, \delta)$ of Blaschke products B^* satisfying the following conditions:

$$\inf_{\operatorname{Im} z > 0} \{ |b_n(z)| + |B_n^*(z)| \} \geq \delta; \tag{I0}$$

$$\inf_n \operatorname{Im} a_n \geq \frac{24}{\alpha \delta^2}. \tag{II}$$

Let the symbol D_n denote the disc $\{\zeta \in \mathbb{C} : |b_n(\zeta)| \leq \frac{\delta}{2}\}$. It is clear that the set $\mathcal{B}(\alpha, \delta)$ consists of interpolating Blaschke products, whose zeros lie high enough above the real line. The less is the constant δ, the higher have zeros to lie.

THEOREM 3.1. (V.I.Vasjunin, S.A.Vinogradov). Let $B^* \in \mathcal{B}(\alpha, \delta)$, and $\log \frac{1}{\alpha} > \frac{2}{\delta^2}$. Then the function $\theta - \alpha B^*$ has exactly one zero in each disc D_n and admits the factorization $\theta - \alpha B^* = BF$, where the function F is outer and B is an interpolating Blaschke product. In particular the family $\{e^{i\lambda x} : B(\lambda) = 0\}$ forms an unconditional basis in the space $L^2(0,1)$.

PROOF. Obviously $|B_n^*(\zeta)| \geq \frac{\delta}{2}$, if $\zeta \in D_n$. Hence $|B^*(\zeta)| = |b_n(\zeta)| |B_n^*(\zeta)| \geq \frac{\delta^2}{4}$ on the boundary of the disc D_n. Let $G = \mathbb{C}_+ \setminus \bigcup_{n=1}^{\infty} D_n$. Then $|B^*(\zeta)| \geq \frac{\delta^2}{4}$ in G by the minimum principle.

The lower point of the disc $\{\zeta \in \mathbb{C} : |(\zeta - a)/(\zeta - \bar{a})| \leq \delta/2\}$ lies at the distance $\operatorname{Im} a \, \frac{1 - \delta/2}{1 + \delta/2} \geq \frac{\operatorname{Im} a}{3}$ above the real

line, therefore $D_n \subset \{\zeta : \operatorname{Im}\zeta \geqslant s\}$, $s \overset{def}{=} 8/\alpha\delta^2$.
Let $\zeta \in \partial D_n$. Then

$$|\theta(\zeta)| = e^{-\operatorname{Im}\zeta} \leqslant e^{-s} < \alpha\frac{\delta^2}{8} < \alpha|B^*(\zeta)|$$

(because $e^{-s} < \frac{1}{s}$ when $s > 0$), hence by Rouché's theorem the function $\theta - \alpha B^*$ has exactly one zero in the each disc D_n, $n = 1, 2, \dots$. This estimate shows also that the Blaschke product B has no other zeros in the half-plane $\{\zeta : \operatorname{Im}\zeta \geqslant s\}$.

Let us check now that the product B has no zeros in the strip $\{\zeta : 0 < \operatorname{Im}\zeta < \log 1/\alpha\}$. Indeed

$$|B(\zeta)F(\zeta)| = |\theta(\zeta) - \alpha B^*(\zeta)| > e^{-\operatorname{Im}\zeta} - \alpha > 0$$

if $\operatorname{Im}\zeta < \log\frac{1}{\alpha}$.

So non-controlled zeros of B can lie only in the strip $\{\zeta : \log 1/\alpha \leqslant \operatorname{Im}\zeta < s\}$. Note that if $\lim_n \operatorname{Im} a_n = +\infty$, this strip d o e s contain infinitely many zeros (see Theorem 2.4, Part II, Section 2).

Let us suppose now that $B(\lambda) = 0$ and $\log 1/\alpha \leqslant \operatorname{Im}\lambda < s$. From the system of equations

$$e^{i\lambda} - \alpha B^*(\lambda) = 0$$

$$ie^{i\lambda} - \alpha B^{*\prime}(\lambda) = B'(\lambda)F(\lambda),$$

we have

$$|B'(\lambda)| = \frac{1}{|F(\lambda)|}\left|ie^{i\lambda} - \alpha B^{*\prime}(\lambda)\right| = \frac{\alpha}{|F(\lambda)|}\left|iB^*(\lambda) - B^{*\prime}(\lambda)\right|.$$

Let $B_\lambda = B \cdot b_\lambda^{-1}$, then $2\operatorname{Im}\lambda|B'(\lambda)| = |B_\lambda(\lambda)|$. It is useful to remember a trivial estimation

$$|f'(\lambda)| \leqslant \frac{\|f\|_\infty}{2\operatorname{Im}\lambda} .$$

Summarizing this information we obtain

$$|B_\lambda(\lambda)| = \frac{\alpha}{|F(\lambda)|}\cdot 2\operatorname{Im}\lambda \cdot |iB^*(\lambda) - B^{*\prime}(\lambda)| \geqslant$$

$$\geqslant \frac{\alpha}{1+\alpha}\left[2\operatorname{Im}\lambda|B^*(\lambda)| - 1\right] \geqslant \frac{\alpha}{1+\alpha}\left(\frac{\delta^2}{2}\log 1/\alpha - 1\right) > 0$$

because $\log 1/\alpha > 2/\delta^2$. Therefore the inequality taking part in the Carleson condition holds at every zero of B contained in the strip $\{\zeta : \log 1/\alpha \leqslant \operatorname{Im} \zeta < s\}$. Since the remaining zeros are in the discs D_n and B is an interpolating Blaschke product, the product B^* also is interpolating.●

Now we shall show that refining the reasonings from the proof of the preceding theorem we can obtain that the generating function F_Λ, $\Lambda = \{\lambda : B(\lambda) = 0\}$ will be a GSTF (S.A.Vinogradov). Note that it is not difficult of course to give examples of GSTF with the zero-set contained in no strip of finite width. However, it is much more difficult to combine this property with the carlesonity. But at first we give an auxiliary definition.

Let $\mathcal{U}_\infty$ be the set of all unimodular functions φ on $\mathbb{R}$ representable in the form

$$\varphi = c\frac{\bar{h}}{h},$$

where $c \in \mathbb{T}$, h is an invertible element of the algebra H^∞ $(h \in (H^\infty)^{-1})$. It is clear that $\mathcal{U}_\infty$ is a group with respect to the pointwise multiplication of functions. It is easy to see that the mapping $(c, h) \mapsto c\bar{h}h^{-1}$ is an isomorphism of the group $\mathbb{T} \times (H^\infty)^{-1}$ onto $\mathcal{U}_\infty$.

LEMMA 3.2. Let $\Lambda \subset \mathbb{C}_\delta$, $\delta > 0$, and B be a Blaschke product with the zero set Λ. Then the generating function F_Λ is a GSTF with the width of the indicator diagram equal to a iff the function $\bar{B}\theta_a$ belongs to $\mathcal{U}_\infty$.

THE PROOF of the lemma is provided by Theorem 1.2 and the definition of a GSTF. ●

THEOREM 3.3 (S.A.Vinogradov). There exists a set Λ, $\Lambda \subset \mathbb{C}_+$, such that $\Lambda \in (C)$, $\sup\limits_{\lambda\in\Lambda} \operatorname{Im} \lambda = +\infty$ and F_Λ is a generalized sine-type function.

PROOF. Let $\alpha \in (0, 1)$ and let B^* be an auxiliary Blaschke product, whose choice will be specified later. We find the required Blaschke product from the equation

$$\theta - \alpha B^* = B f_e,$$

where f_e is an outer function and $\theta = e^{iz}$. Note that $f_e \in (H^\infty)^{-1}$ because $1-\alpha \leqslant |f_e| \leqslant 1+\alpha$ on $\mathbb{R}$. Since $|B^* - \alpha\theta| = |\theta - \alpha B^*|$ on $\mathbb{R}$, there exists a Blaschke product C such that

$$B^* - \alpha\theta = C f_e.$$

The first equality yields

$$B\bar{\theta} = \frac{1-\alpha B^*\bar{\theta}}{f_e}$$

and the second one provides

$$1-\alpha \bar{B}^*\theta = \bar{B}^* C f_e .$$

Hence

$$B\bar{\theta} = \frac{B^*}{C}\cdot\frac{\bar{f}_e}{f_e} .$$

Therefore to get the inclusion $B\bar{\theta}\in\mathcal{U}_\infty$ we have to find a Blaschke product B^* such that $B^*\bar{C}\in\mathcal{U}_\infty$. In addition B must be interpolating. By Theorem 3.1 it is really so if $B^*\in\mathcal{B}(\alpha,\delta)$ and $\log 1/\alpha > 2/\delta^2$.

Let $(a_n)_{n\geqslant 1}$ be a sequence of zeros of the function B^* . We suppose that $\lim\limits_n \operatorname{Im} a_n = +\infty$ and the discs $D_n = \{\zeta\in\mathbb{C} : \left|\frac{\zeta-a_n}{\zeta-\bar{a}_n}\right| \leqslant \delta/2\}$ do not intersect. Since $\log\frac{1}{\alpha} > \frac{2}{\delta^2}$ implies $\alpha < \delta^2/4$, we have

$$|B^*-\alpha\theta| \geqslant |B^*| - \alpha \geqslant \frac{\delta^2}{4} - \alpha > 0$$

in the domain $G \stackrel{def}{=} \mathbb{C}_+ \setminus \bigcup_{n=1}^{\infty} D_n$. Hence the product C has no zeros in G . By Rouché's theorem the product C has exactly one zero, say c_n , in each disc D_n. The Rouché's theorem allows to control the behaviour of the points c_n as $n\to\infty$. In fact $|B_n^*(\zeta)| \geqslant \delta/2$ if $\zeta\in D_n$. Therefore the estimate

$$|B^*(\zeta)| = \left|\frac{\zeta-a_n}{\zeta-\bar{a}_n}\right| |B^*(\zeta)| \geqslant \frac{\delta}{2}\left|\frac{\zeta-a_n}{\zeta-\bar{a}_n}\right|$$

is valid in D_n . On the other hand

$$\inf_{\zeta\in D_n} \operatorname{Im}\zeta = \frac{1-\delta/2}{1+\delta/2}\operatorname{Im} a_n > \frac{1}{3}\operatorname{Im} a_n .$$

Hence

$$|B^*(\zeta)| - |\alpha\theta(\zeta)| \geqslant \frac{\delta}{2}\left|\frac{\zeta-a_n}{\zeta-\bar{a}_n}\right| - e^{-\frac{1}{3}\operatorname{Im} a_n}, \quad \zeta\in D_n .$$

So by Rouché's theorem we have

$$\left|\frac{c_n-a_n}{c_n-\bar{a}_n}\right| \leqslant \frac{2}{\delta}\, e^{-\frac{1}{3}\operatorname{Im} a_n}.$$

Since $\left|\frac{\zeta-a}{\zeta-\bar{a}}\right| < \varepsilon$ implies $|\zeta-a| \leqslant \frac{2\varepsilon}{1-\varepsilon}\operatorname{Im} a$ and

$\frac{2}{\delta} e^{-1/3 \operatorname{Im} a_n} \leq \frac{2}{\delta} e^{-\frac{8}{\alpha\delta^2}} \leq \frac{2}{e^8} < \frac{1}{5}$, we get

$$|c_n - a_n| \leq \frac{8}{\delta} \operatorname{Im} a_n e^{-\frac{1}{3}\operatorname{Im} a_n}.$$

Writing the explicit expressions for B^* and C we have

$$\frac{B^*}{C}(x) = \prod_{n=1}^{\infty} \frac{\varepsilon_n}{\varepsilon_n'} \prod_{n=1}^{\infty} \frac{1-x/a_n}{1-x/c_n} \overline{\prod_{n=1}^{\infty} \frac{1-x/c_n}{1-x/a_n}} .$$

So it is sufficient to check that the argument of the product $\prod_{n=1}^{\infty} \frac{1-x/a_n}{1-x/c_n}$ belongs to the space $\operatorname{Re} H^\infty \overset{\text{def}}{=\!=} \{\operatorname{Re} f : f \in H^\infty\}$.
This follows from the formula

$$\log\left(\frac{1-x/a_n}{1-x/c_n}\right) = \log\frac{c_n}{a_n} + \log\left(1 + \frac{c_n - a_n}{x - c_n}\right)$$

which implies that the logarithm of our product belongs to $\overline{H^\infty}$ if

$$\sum_{n=1}^{\infty} \operatorname{Im} a_n e^{-\frac{1}{3}\operatorname{Im} a_n} < +\infty . \qquad \bullet$$

REMARK. The method used for the proof of Theorem 3.3 allows in fact to obtain a stronger result. Namely, one can construct such Blaschke product B with Carleson set of zeros Λ , $\sup_{\lambda\in\Lambda} \operatorname{Im}\lambda = +\infty$, that the unimodular function $B\bar{\theta}$ belongs to the subgroup of $\mathcal{U}_\infty$ consisting of functions of the form $\bar{h}/h$, $h \in \exp(H^\infty) \overset{\text{def}}{=\!=} \{h : h = e^g,\ g \in H^\infty\}$. In this case the logarithm of the outer part of the generating function F_Λ will be uniformly bounded in the upper half-plane. To prove this it is sufficient to note that in the preceding example $\arg B^*\bar{C} \in \operatorname{Re} H^\infty$ and $\log f_e \in H^\infty$. Indeed

$$\log f_e = \log\frac{B^* - \alpha\theta}{C} = \log\frac{B^*}{C} + \log(1 - \alpha\overline{B^*}\theta)$$

(the equality holds on $\mathbb{R}$ and obviously implies $\log f_e \in L^\infty(\mathbb{R})$). $\bullet$

In conclusion let us prove the theorem on the "complementing up to a basis" mentioned at the beginning of the Section.

THEOREM 3.4 (V.I.Vasjunin). Let $(a_n)_{n\in\mathbb{Z}}$ be a Carleson sequence of points of the upper half-plane satisfying $\lim_n \operatorname{Im} a_n = +\infty$. Then for any positive number a the family $(a_n)_{n\in\mathbb{Z}}$ can be complemented up to such a family $(\lambda_n)_{n\in\mathbb{Z}}$ that the exponentials $(e^{i\lambda_n t})_{n\in\mathbb{Z}}$ form an unconditional basis in the space $L^2(0,a)$.

THE PROOF follows immediately from Corollary 2.5. Let us remember that by Theorem 3.1 for the Blaschke product with zeros $(a_n)_{n\in\mathbb{Z}}$ there exists a number α , $\alpha \in (0,1)$, such that

$$\theta_a - \alpha B^* = B\cdot F,$$

where B is an interpolating Blaschke product and F is an outer function. Let b_n be a zero of B , which is close to the zero a_n . Then $\lim|b_n - a_n| = 0$ because $\operatorname{Im} a_n \to +\infty$ (see the application of Rouché's theorem in the proof of Theorem 3.2). Therefore by Corollary 2.5 we can return the zero b_n into the point a_n for each n , may be except for a finite set of n . But a finite set of zeros causes no difficulty because we can move them into any free place. ●

4. The equiconvergence of harmonic and non-harmonic Fourier series.

Suppose that $\Lambda = \{\lambda_n : n\in\mathbb{Z}\} \subset \mathbb{C}_\gamma$, $\gamma \in \mathbb{R}$, and the family of exponentials $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ forms an unconditional basis in the space $L^2(-\pi,\pi)$. Let $(h_n)_{n\in\mathbb{Z}}$ be the "coordinate family" (the dual sequence) for this basis:

$$(e^{i\lambda_m x}, h_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\lambda_m x}\,\overline{h_n(x)}\,dx = \begin{cases} 1 & n = m,\\ 0 & n \neq m.\end{cases}$$

Then to each function f , $f\in L^2(-\pi,\pi)$ corresponds the non-harmonic Fourier series

$$\sum_{n\in\mathbb{Z}} (f, h_n)\, e^{i\lambda_n x}.$$

It is natural to consider together with the non-harmonic Fourier series the harmonic one:

$$f \sim \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx}\,; \qquad \hat{f}(n) \stackrel{\text{def}}{=\!=} \frac{1}{2\pi}\int_{-\pi}^{\pi} f\, e^{-inx}\,dx,\ \ n\in\mathbb{Z}.$$

The main theorem of this Section demonstrates that as to the convergence inside the interval $(-\pi,\pi)$, a non-harmonic Fourier series behaves in the same way as the corresponding harmonic one.

THEOREM 4.1. Let $\Lambda=\{\lambda_n : n\in\mathbb{Z}\}\subset\mathbb{C}_\gamma$, $\gamma\in\mathbb{R}$, and let a family of exponentials $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ form an unconditional basis in the space $L^2(-\pi,\pi)$. Then the equality

$$\lim_{R\to\infty}\sup_{|x|\leq\pi}(\pi-|x|)^{1/2}\Big|\sum_{|n|\leq R}\hat{f}(n)e^{inx}-\sum_{|\lambda_n|\leq R}(f,h_n)e^{i\lambda_n x}\Big|=0 \qquad (12)$$

holds for each function f, $f\in L^2(-\pi,\pi)$.

REMARKS. 1. The initial formulation of the Theorem guaranteed only the equiconvergence of the harmonic and non-harmonic Fourier series uniformly on compact subsets of the interval $(-\pi,\pi)$. A.M.Sedletskii has amiably informed one of the authors that proposition (12) was recently proved by him assuming the set of frequencies lies in a strip of finite width parallel to $\mathbb{R}$. Our method turned out to lead to this more general proposition too. The method of A.M.Sedletskii differs from ours.

We refer the interested reader to the paper [24] containing a lot of other useful facts about bases of exponentials. In particular it is shown there that it is impossible to improve the weight $(\pi-|x|)^{1/2}$ in (12).

2. Without loss of generality one may suppose that $\Lambda\subset\mathbb{C}_2$. Indeed, suppose Theorem 4.1 is proved for such sets Λ. Consider then the set of frequencies $\Lambda-iy$, $y>0$. It is clear that the dual sequence for the family of exponentials $(e^{i(\lambda_n-iy)x})_{n\in\mathbb{Z}}$ coincides with the family $(e^{-yt}h_n(t))_{n\in\mathbb{Z}}$. Then the non-harmonic Fourier series for the function f with respect to the new family is

$$f\sim e^{yt}\sum_n (f\cdot e^{-ys},h_n)e^{i\lambda_n t}.$$

By assumption this series is equiconvergent with the Fourier series $e^{yt}\sum_n \widehat{fe^{-ys}}(n)\,e^{int}$. Let $S_N(f,t)$ denote the partial sum $\sum_{|n|\leq N}\hat{f}(n)\,e^{int}$ of the Fourier series of f. Then we have

$$e^{yt}S_N(fe^{-ys},t)-S_N(f,t)=$$

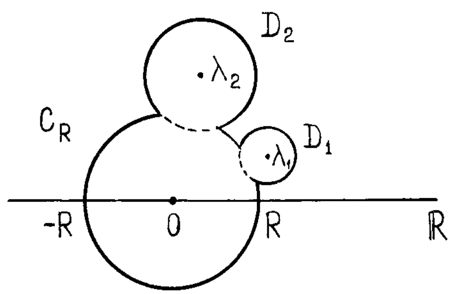
$$= e^{yt}\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin(N+1/2)(t-s)}{2\sin\frac{t-s}{2}}f(s)\left[e^{-ys}-e^{-yt}\right]ds = O(1). \qquad \bullet$$

3. By technical reasons it is convenient to replace the partial sum in the formula (12) by the integral $\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin R(t-s)}{t-s}f(s)\,ds$. Simple estimations of the Dirichlet kernel show that the such replacement causes an error at most $O(1)\cdot\|f\|_2 \quad (R \longrightarrow +\infty)$.

4. Since the family of exponentials $(e^{i\lambda_n x})_{n\in\mathbb{Z}}$ forms an unconditional basis in $L^2(0,a)$, Λ is a Carleson set. Then there exists a positive number ε , so small that discs

$$D_n \overset{def}{=} \{\zeta\in\mathbb{C}_+ : |\zeta-\lambda_n| \leqslant \varepsilon \operatorname{Im}\lambda_n\}, \quad n\in\mathbb{Z},$$

are disjoint. Let R be an arbitrary positive number, $D(0,R) = \{\zeta\in\mathbb{C} : |\zeta| < R\}$, and C_R be a closed curve forming the boundary of the domain $D(0,R)\cup\{D_n : D_n \cap D(0,R) \neq \emptyset\}$. (See the diagram below).



At the end of the section we shall demonstrate that it is possible to replace the sum $\sum_{|\lambda_n|\leqslant R}(f,h_n)e^{i\lambda_n x}$ by the sum $\sum_{\lambda_n\in \operatorname{Int} C_R}(f,h_n)e^{i\lambda_n x}$ not violating the condition (12).

THE PROOF OF THEOREM 4.1 follows in its idea a plan, proposed by N.Levinson [48] . Though we prove a more general result, than the Levinson's one, our proof is technically simpler, because we use estimates of entire functions satisfying the condition (A_2) on $\mathbb{R}$. We have chosen the interval $(-\pi,\pi)$ instead of $(0,2\pi)$, for the sake of symmetry. Let F be the generating function for our set of frequencies. Then $G_F = [-\pi i, \pi i]$ and

$$F = \text{v.p.} \prod_{n\in\mathbb{Z}} \left(1-\frac{z}{\lambda_n}\right).$$

Clearly (see [48])

$$\frac{F}{(z-\lambda_n)F'(\lambda_n)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{itz}\, \overline{h_n(t)}\, dt, \quad n\in\mathbb{Z}. \tag{I3}$$

Let B be the Blaschke product with the zero-set Λ, let h be the outer part of $F|\mathbb{C}_+$ and $h^*(z) \overset{def}{=} \overline{h(\bar z)}$, $\operatorname{Im} z<0$. Then

$$F(z)=\begin{cases} B(z)e^{-\pi i z}h(z) & \text{, if } \operatorname{Im} z\geqslant 0 \\ e^{\pi i z}h^*(z) & \text{, if } \operatorname{Im} z<0. \end{cases}$$

Note that $|h|^2|\mathbb{R}$ satisfies the Helson-Szegö condition. The Blaschke product B satisfies the following condition

$$\inf_{R>0}\ \inf_{\zeta\in C_R} |B(\zeta)|>0.$$

This inequality is an immediate consequence of the Carleson condition $\inf_n |B_n(\lambda_n)|>0$. Our choice of C_R is aimed just at the lower estimate of B (on C_R) .

The "algebraic" base of our proof is the following lemma due to N.Levinson which may be derived from the book [48] .

LEMMA (N.Levinson). For an arbitrary function f, $f\in L^2(-\pi,\pi)$, for any positive number R and for each t, $|t|<\pi$, the following formula holds:

$$\sum_{\lambda_n\in \operatorname{Int} C_R} (f,h_n)e^{i\lambda_n t} - \frac{1}{\pi}\int_{-R}^{R} \frac{\sin R(t-s)}{t-s} f(s)\,ds =$$

$$= \frac{1}{2\pi i}\int_{C_R} \frac{e^{i\zeta t}}{F(\zeta)}\,d\zeta \left\{ \int_{\mathbb{R}} \frac{F(x)\hat f(x)}{x-\zeta}\,dx \right\}.$$

Here $\hat f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ixt} f(t)\,dt$ is the Fourier transformation of f .

REMARK. Theorem 4.2 will follow from Levinson's lemma, if we prove the inequality

$$\sup_{|t|<\pi} (\pi-|t|)^{1/2} \left| \frac{1}{2\pi i}\int_{C_R} \frac{e^{i\zeta t}\,d\zeta}{F(\zeta)} \left\{ \int_{\mathbb{R}} \frac{F(x)\hat f(x)}{x-\zeta}\,dx \right\} \right| \leqslant \text{const}\cdot\|f\|_2 \tag{I4}$$

Indeed, then

$$\sup_{|t|<\pi}(\pi-|t|)^{1/2}\left|\sum_{\lambda_n\in \operatorname{Int} C_R}(f,h_n)e^{i\lambda_n t}-\sum_{|n|\leq R}\hat{f}(n)e^{int}\right|\leq \mathrm{const}\cdot\|f\|_2 .$$

It remains to note that $\operatorname{span}\{e^{i\lambda_n x}\chi_{[-\pi,\pi]}: n\in\mathbb{Z}\}=L^2(-\pi,\pi)$ and the equiconvergence holds for the exponentials $e^{i\lambda_n x}, n\in\mathbb{Z}$. ●

PROOF OF LEVINSON'S LEMMA. According to the Cauchy's formula

$$\frac{1}{2\pi i}\int_{C_R}\frac{e^{i\zeta t}}{G(\zeta)(x-\zeta)}\,d\zeta=\sum_{\lambda_n\in \operatorname{Int} C_R}\frac{e^{i\lambda_n t}}{G'(\lambda_n)(x-\lambda_n)}-\frac{e^{ixt}}{G(x)}\chi_{[-R,R]}(x)$$

(assuming $x\neq\pm R$). Hence

$$G(x)\frac{1}{2\pi i}\int_{C_R}\frac{e^{i\zeta t}}{G(\zeta)(x-\zeta)}\,d\zeta=\sum_{\lambda_n\in \operatorname{Int} C_R}\frac{e^{i\lambda_n t}G(x)}{G'(\lambda_n)(x-\lambda_n)}-e^{ixt}\chi_{[-R,R]}(x);$$

this implies, in particular, that the left part of the preceding formula belongs to $L^2(\mathbb{R})$. Computing the Fourier transform of the left and right parts and using the inversion formula we get

$$\int_{\mathbb{R}}e^{ixs}G(x)\,dx\cdot\frac{1}{2\pi i}\int_{C_R}\frac{e^{i\zeta t}}{G(\zeta)(x-\zeta)}\,d\zeta=\sum_{\lambda_n\in \operatorname{Int} C_R}e^{i\lambda_n t}\,\overline{h_n(s)}\,\chi_{[-\pi,\pi]}(s)-\int_{-R}^{R}e^{ix(t-s)}dx.$$

Multiplying this by $\frac{1}{2\pi}f(s)$, integrating over the interval $[-\pi,\pi]$ and interchanging the integrals we obtain

$$\int_{\mathbb{R}}\hat{f}(x)G(x)\left\{\frac{1}{2\pi i}\int_{C_R}\frac{e^{i\zeta t}}{G(\zeta)(x-\zeta)}d\zeta\right\}dx=\sum_{\lambda_n\in \operatorname{Int} C_R}(f,h_n)e^{i\lambda_n t}-\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin R(t-s)}{t-s}f(s)\,ds.$$

Now we need only to note that

$$\int_{\mathbb{R}}\hat{f}(x)G(x)\left\{\frac{1}{2\pi i}\int_{C_R}\frac{e^{i\zeta t}}{G(\zeta)(x-\zeta)}d\zeta\right\}dx=\frac{1}{2\pi i}\int_{C_R}\frac{e^{i\zeta t}}{G(\zeta)}d\zeta\int_{\mathbb{R}}\frac{\hat{f}(x)G(x)}{x-\zeta}\,dx.$$

Let us justify now the interchanging of the integrals. The function $x\mapsto|G(x)|^2$, $x\in\mathbb{R}$, satisfies the Muckenhoupt's condition (A_2) and hence $(x-\zeta)^{-1}G(x)\in L^2(\mathbb{R})$ if $\zeta\notin\mathbb{R}$. If we remove in the preceding formula the part of C_R lying in the strip $\Pi_\varepsilon\overset{\text{def}}{=}\{z\in\mathbb{C}:|\operatorname{Im} z|\leq\varepsilon\}$, the formula will follow from Fubini's theorem. Now we use an estimate which can be easily verified:

$$\left|\frac{1}{2\pi i}\int_{C_R\cap\Pi_\varepsilon}\frac{e^{i\zeta t}}{G(\zeta)}\frac{d\zeta}{(x-\zeta)}\right|\leq C\cdot\min\left(1,\frac{\varepsilon}{|x-R|}\right),\quad C>0,$$

and note that according to the Sohotski's formulae the function $\int_{\mathbb{R}} \frac{\hat{f}(x)G(x)}{x-\zeta}\,dx$ is uniformly bounded on C_R and has only two points of discontinuity (jump-points) namely $\pm R$. The passage to the limit $\varepsilon \to +0$ completes the proof. ●

We shall need estimates of the Poisson's integrals $P_\zeta(W)$ of functions W satisfying the Muckenhoupt's condition (A_2). Let us denote by h the outer function (in $\mathbb{C}_+$) satisfying $|h(x)| = W(x)$, $x \in \mathbb{R}$.

LEMMA 4.2. The following assertions are equivalent.

1) $W \in (A_2)$;

2) There exist functions u and v, $u, v \in L^\infty(\mathbb{R})$ with

$$\log W(x) = u(x) + \tilde{v}(x), \qquad \|v\|_\infty < \pi/2;$$

3) The outer function h maps the upper half-plane into an angle with vertex at the origin and with size less than $\pi/2$;

4) $\sup\limits_{\zeta \in \mathbb{C}_+} P_\zeta(W) P_\zeta\left(\frac{1}{W}\right) < +\infty$;

5) There exists a constant C, $C > 1$, such that for $\zeta \in \mathbb{C}_+$

$$|h(\zeta)| \leqslant P_\zeta(W) \leqslant C|h(\zeta)|,$$

$$\frac{1}{|h(\zeta)|} \leqslant P_\zeta\left(\frac{1}{W}\right) \leqslant C\,\frac{1}{|h(\zeta)|}.$$

The proof can be found in the known paper [40] (see Theorem 2). Note that the assertion 3) of Lemma implies that the restriction of the outer function h on any line $\{\operatorname{Im} z = y\}$, $y > 0$, satisfies the Muckenhoupt's condition if the restriction of h on the real line does.

The proof of the following lemma is contained in [40] also.

LEMMA 4.3. Suppose that $W \in (A_2)$. Then there exists a constant C, $C > 0$, such that for any ζ, $\zeta \in \mathbb{C}_+$, the inequality

$$P_\zeta(W) \leqslant \frac{C}{2\operatorname{Im}\zeta} \int\limits_{|t-\operatorname{Re}\zeta| \leqslant \operatorname{Im}\zeta} W(t)\,dt$$

is valid.

Now we prove the inequality (14). For this aim we divide the contour C_R into two parts $C_R^+ \stackrel{def}{=} C_R \cap \mathbb{C}_+$ and

$C_R^- \overset{def}{=} C_R \cap \mathbb{C}_-$ and prove (14) for each contour separately. Let us begin with the estimate for the boundary C_R^- (the case of C_R^+ being analogous).

If $\operatorname{Im} \zeta < 0$, then the function $(z-\zeta)^{-1} F \hat{f}$ obviously belongs to the Hardy class H^1 in the strip $\{\zeta \in \mathbb{C} : 0 < \operatorname{Im} \zeta < 1\}$:

$$\int_{\mathbb{R}} \frac{|F(x+iy)||\hat{f}(x+iy)|}{|x+iy-\zeta|} dx \leqslant \left(\int_{\mathbb{R}} \frac{|F(x+iy)|^2}{|x+iy-\zeta|} dx \right)^{1/2} \cdot \left(\int_{\mathbb{R}} |\hat{f}(x+iy)|^2 dx \right)^{1/2} < +\infty.$$

So according to the Cauchy formula we obtain the identity

$$\int_{\mathbb{R}} \frac{F(x)\hat{f}(x)}{x-\zeta} dx = \int_{\mathbb{R}} \frac{F(x+i)\hat{f}(x+i)}{x+i-\zeta} dx,$$

from which the inequality

$$\left| \int_{\mathbb{R}} \frac{F(x)\hat{f}(x)}{x-\zeta} dx \right| \leqslant e^{2\pi} \|f\|_2 \left(\int_{\mathbb{R}} \frac{|h(x+i)|^2}{|x-(\zeta-i)|^2} dx \right)^{1/2}$$

follows immediately.

Remind, that h is the outer part $F|\mathbb{C}_+$. Applying the assertion 5) of Lemma 4.2, we obtain

$$\int_{\mathbb{R}} \frac{|h(x+i)|^2}{|x-(\zeta-i)|^2} dx \leqslant \frac{c}{1+|\operatorname{Im}\zeta|} |h(2i+\bar{\zeta})|^2.$$

Hence

$$(\pi-|t|)^{1/2} \left| \frac{1}{2\pi i} \int_{C_R^-} \frac{e^{i\zeta t}}{F(\zeta)} d\zeta \cdot \int_{\mathbb{R}} \frac{F(x)\hat{f}(x)}{x-\zeta} dx \right| \leqslant$$

$$\leqslant \mathrm{const} \cdot \|f\|_2 \cdot (\pi-|t|)^{1/2} \int_{C_R^-} \frac{e^{|\operatorname{Im}\zeta t|}}{|h^*(\zeta)|} \cdot \frac{|h(2i+\bar{\zeta})|}{\sqrt{1+|\operatorname{Im}\zeta|}\, e^{\pi|\operatorname{Im}\zeta|}} |d\zeta|;$$

observing, that

$$(\pi-|t|)^{1/2} \int_{C_R^-} e^{-(\pi-|t|)|\operatorname{Im}\zeta|} \frac{|d\zeta|}{\sqrt{\operatorname{Im}\zeta}} = \int_{(\pi-|t|)C_R^-} \frac{e^{-|\operatorname{Im}\zeta|}}{\sqrt{\operatorname{Im}\zeta}} |d\zeta| = O(1),$$

we see, that it is sufficient to prove the inequality

$$|h(2i+\bar{\zeta})| \leqslant \mathrm{const}|h^*(\zeta)|, \quad \operatorname{Im}\zeta < 0. \tag{15}$$

It is obvious, that h is an entire function: $h(z) = e^{\pi i z} \cdot F(z) B^{-1}(z)$, $z \in \mathbb{C}$. Since zeroes of the product B lie in the half-plane $\mathbb{C}_2$, the function h proves to be outer in $\mathbb{C}_{-1}$. Further, if $x \in \mathbb{R}$, then

$$|h(x-i)| = \frac{e^{\pi}}{|B(x-i)|} \cdot e^{\pi} |h^*(x-i)|.$$

The function $x \longmapsto |h^*(x-i)|^2$ satisfies the condition (A_2) (see assertion 3) of the Lemma 4.2). Zeros of the product B satisfy Carleson's condition (C), outside small discs D_n, $n \in \mathbb{Z}$, lying in the half-plane $\mathbb{C}_-$, the inequality $|B(z)| \geqslant \gamma > 0$, $\operatorname{Im} z \geqslant 0$ is valid. According to the symmetry principle $B(z) = \frac{1}{\overline{B(\bar z)}}$, $\operatorname{Im} z < 0$. Hence

$$\gamma e^{2\pi} |h^*(x-i)| \leqslant |h(x-i)| \leqslant e^{2\pi} |h^*(x-i)|$$

and, therefore, the function $x \longmapsto |h(x-i)|^2$ satisfies the condition (A_2) as well.

Consider an auxiliary function g defined in the upper half-plane:

$$g(\zeta) = h^2(\zeta - i).$$

Remembering, that $h^*(\zeta) = \overline{h(\bar\zeta)}$, we can rewrite inequality (I5) in the following way:

$$|g(3i+\zeta)| \leqslant \text{const}\, |g(i+\zeta)|, \quad \operatorname{Im}\zeta > 0.$$

To prove this inequality let us use Lemma 4.3:

$$|g(3i+\zeta)| \leqslant P_{3i+\zeta}(|g|) \leqslant \frac{C}{2(3+\operatorname{Im}\zeta)} \int\limits_{|t-\operatorname{Re}\zeta| \leqslant 3+\operatorname{Im}\zeta} |g(t)|\,dt \leqslant \text{const}\cdot P_{i+\zeta}(|g|) \leqslant \text{const}\cdot |g(i+\zeta)|$$

(the inequality 5) of Lemma 4.2 is used in the last inequality). Thus the inequality (I5) is proved.

Some words about changes needed to estimate the contribution of the contour C_R^+ into the integral in the left-hand side of (14). Since the function $(z-\zeta)^{-1} G \hat f$ belongs to the Hardy class H^1 in the strip $\{\zeta : -1 \leqslant \operatorname{Im}\zeta < 0\}$ for $\zeta \in \mathbb{C}_+$, we have

$$\left| \int\limits_{\mathbb{R}} \frac{F(x)\hat f(x)}{x-\zeta}\,dx \right| = \left| \int\limits_{\mathbb{R}} \frac{F(x-i)\hat f(x-i)}{x-i-\zeta}\,dx \right| \leqslant$$

$$\leqslant e^{2\pi}\|f\|_2\left(\int_{\mathbb{R}}\frac{|h^*(x-i)|^2}{|x-(i+\zeta)|^2}\,dx\right)^{1/2}\leqslant const\cdot\|f\|_2\,|h(\zeta+2i)|.$$

Since $F=Be^{-\pi i z}h$ in the half-plane $\mathbb{C}_+$ and $|B|\geqslant\gamma>$ >0 on C_R , an estimate

$$(\pi-|t|)^{1/2}\left|\frac{1}{2\pi i}\int_{C_R^+}\frac{e^{i\zeta t}}{F(\zeta)}\,d\zeta\int_{\mathbb{R}}\frac{F(x)\hat{f}(x)}{x-\zeta}\,dx\right|\leqslant$$

$$\leqslant\|f\|_2\cdot const\cdot\sup_{\zeta\in\mathbb{C}_+}\frac{|h(\zeta+2i)|}{|h(\zeta)|}(\pi-|t|)^{1/2}\int_{C_R^+}\frac{e^{|\operatorname{Im}\zeta t|}}{|B(\zeta)|e^{\pi|\operatorname{Im}\zeta|}}\,\frac{|d\zeta|}{\sqrt{\operatorname{Im}\zeta}}\leqslant$$

$$\leqslant const\,\|f\|_{L^2(-\pi,\pi)}$$

holds.

To finish the proof of Theorem 4.1, we have to prove the assertion from Remark 4.

LEMMA 4.4. If $f\in L^2(-\pi,\pi)$, then $|(f,h_n)|\leqslant$ $\leqslant const\sqrt{\operatorname{Im}\lambda_n}\,e^{-\pi\operatorname{Im}\lambda_n}\|f\|_2$.

PROOF. From the formulas (13), Parseval identity and Schwarz inequality we have

$$|(f,h_n)|\leqslant\|f\|_2\left\{\int_{\mathbb{R}}\frac{\operatorname{Im}\lambda_n}{|x-\lambda_n|^2}\,|F(x)|^2dx\right\}^{1/2}\frac{1}{\sqrt{\operatorname{Im}\lambda_n}\,|F'(\lambda_n)|}\,.$$

Further, $F=hBe^{-\pi i z}$ in the half-plane $\mathbb{C}_+$. Hence $|F'(\lambda_n)|=|h(\lambda_n)||B'(\lambda_n)|\,e^{\pi\operatorname{Im}\lambda_n}$. Now we need only to note that

$$|B'(\lambda_n)|=(2\operatorname{Im}\lambda_n)^{-1}|B_n(\lambda_n)|\geqslant\gamma\,(2\operatorname{Im}\lambda_n)^{-1}$$

and applying the assertion 5) from Lemma 4.2 completes the proof. ●

For any positive number R consider the set N_R of integers n such that $\lambda_n\notin D(0,R)$ but $D(0,R)\cap D_n\neq$ $\neq\emptyset$. Let us show that

$$\sup_R\sup_{|t|\leqslant\pi}\left|(\pi-|t|)^{1/2}\sum_{n\in N_R}(f,h_n)e^{i\lambda_n t}\right|\leqslant const\cdot\|f\|_2\,.$$

Due to Lemma 4.4 the inequality

$$(\pi-|t|)^{1/2}\left|\sum_{n\in N_R}(f,h_n)e^{i\lambda_n t}\right|\leqslant const\cdot\|f\|_2\sum_{n\in N_R}(\pi-|t|)^{1/2}\sqrt{\operatorname{Im}\lambda_n}\,e^{-(\pi-|t|)\operatorname{Im}\lambda_n}$$

is valid. The discs D_n are disjoint and their radii are proportional to the distance from the centre λ_n to the real line. Therefore the number of indices n, $n \in N_R$, with $2^k \leqslant \leqslant \operatorname{Im} \lambda_n \leqslant 2^{k+1}$, $k = 0, 1, \dots$, is uniformly bounded. We need only to prove an elementary inequality:

$$\sup_{y>0} \sum_{n \geqslant 0} (2^n y)^{1/2} e^{-2^n y} < +\infty .$$

It is clear that without loss of generality supremum in this inequality can be taken with respect to the set $\{ y : y = 2^m, m \in \mathbb{Z} \}$. Then

$$\sum_{n \geqslant 0} (2^{n+m})^{1/2} e^{-2^{n+m}} \leqslant \sum_{n=-\infty}^{\infty} 2^{n/2} e^{-2^n} < +\infty . \qquad \bullet$$

PART IY.

THE REGGE PROBLEM IN THE THEORY OF DIFFERENTIAL OPERATORS

An investigation of the completeness and bases problem for a family of eigen-functions of a differential operator containing the spectral parameter in the boundary condition is our main task in Part IY. As we shall see, the approach, which has been utilized in the preceding parts, is useful in this part as well. We intend here to demonstrate the approach in one more special situation rather than achieve results of maximal generality. Subtle results of differential operator theory form only a scenery for our exposition. So this Part can be addressed to the reader who is, possibly, for the first time, getting acquainted with the problem of eigen-function expansions.

Let $a>0$ and let ρ be a positive function on $[0,a]$. It is assumed that $\rho(x)\equiv 1$ if $x>a_\rho$ for some number a_ρ in $(0,a)$ and that

$$\max\left(\int_0^a \rho^2 dx,\ \int_0^a \rho^{-2} dx\right)<+\infty \tag{1}$$

Let

$$L^2_\rho(0,a)\overset{def}{=}\left\{f:\|f\|^2=\int_0^a |f(x)|^2\rho^2(x)\,dx<+\infty\right\}.$$

Now the spectral problem (the Regge problem) for a second order differential operator $L=-\rho^{-2}\frac{d^2}{dx^2}$ in $L^2_\rho(0,a)$ containing a spectral parameter in the boundary condition can be stated as follows. Let $\sigma(\rho)$ be the set of all complex numbers k such that the equation

$$-y''=k^2\rho^2 y\ ;\quad y'(0)=0,\ \ y'(a)+iky(a)=0 \tag{2}$$

has a non-zero solution $y(x,k)$. The question is - does the family $(y(x,k))_{k\in\sigma(\rho)}$ of all such solutions form a complete family or even a Riesz basis in $L^2_\rho(0,a)$? This problem came from the scattering theory on a "transparent" compact barrier for acoustic waves spreading in a medium with a constant refraction coefficient.

The plan of our exposition in Part IY is the following. In

§1 we give a brief outline of the Lax-Phillips approach to the scattering theory for wave equation. In §2 the relationship of this theory and the Regge problem is discussed. In conclusion in §3 we formulate our main result and consider an important example.

In what follows we assume the reader's familiarity with backgrounds of the theory of self-adjoint operators. A detailed exposition of the theory is given in [67] , [68] .

1. Lax-Phillips approach to the scattering theory.

It is well known that the wave equation for the semiinfinite string with the free end $x=0$ and the local propagation speed ρ^{-1} , $\rho(x)\equiv 1$ for $x>a_\rho$, is defined by

$$\begin{cases} \rho^2(x)\,u_{tt}=u_{xx}\ ; & u_x(0,t)=0 \\ u(x,0)=u_0(x)\ ; & u_t(x,0)=u_1(x),\ x\in\mathbb{R}_+ \end{cases} \tag{3}$$

The pair $\mathcal{U}(0)=(u_0,u_1)$ is called " the Cauchy data ", or simply "data", of the problem (2). The evolution operator U_t of (2) transforms (by definition) the data $\mathcal{U}(0)$ into the data $\mathcal{U}(t)=(u(x,t),u_t(x,t))$ related to the moment t . A natural Hilbert space of data is a Hilbert space E of all data with finite energy:

$$E=\Big\{(u_0,u_1):\ \int_0^\infty |u_1|^2\rho^2dx<+\infty,\quad u_0'\in L^2(\mathbb{R}_+)\Big\}$$

$$<\mathcal{U},\mathcal{V}>_E\ \stackrel{def}{=}\ \frac{1}{2}\int_0^\infty\Big\{u_{0x}\overline{v}_{0x}+\rho^2u_1\overline{v}_1\Big\}\,dx\ . \tag{4}$$

To be more precise we are to consider the space E as a space of equivalence classes identifying $\mathcal{U}$ and $\mathcal{V}$ iff $u_1=v_1$, $u_0-v_0\equiv$ const.

Let now $L^2_\rho \stackrel{def}{=} \{f:\ \int_0^\infty |f|^2\rho^2\,dx<+\infty\}$ be a Hilbert space with the inner product $<f,g>_{L^2_\rho}=\int_0^\infty f\overline{g}\rho^2dx$.

LEMMA 1.1. The operator $L=-\rho^{-2}\dfrac{d^2}{dx^2}$ with the domain

$$\mathcal{D}(L)=\{f\in L^2_\rho:\ Lf\in L^2_\rho,\quad f'(0)=0\}$$

is a self-adjoint non-negative operator in L^2_ρ .

PROOF. The set $D(L)$ is dense in L^2_ρ . Indeed, if φ is any smooth function with a compact support in $(0,+\infty)$ then $\varphi\in D(L)$ because according to (1) we have

$$\int_0^\infty |L\varphi|^2\rho^2 dx=\int_0^\infty \frac{|\varphi''|^2}{\rho^2}dx\le \sup_{x\in\mathbb{R}_+}|\varphi''(x)|^2\cdot\int_{supp(\varphi)}\frac{1}{\rho^2}dx<+\infty .$$

Clearly, $f|[a_\rho,+\infty)\in L^2(a_\rho,+\infty)$, $f''|[a_\rho,+\infty)\in L^2(a_\rho,+\infty)$ if $f\in D(L)$ ($\rho(x)\equiv 1$ for $x>a_\rho$) . It follows from the well-known inequality

$$\int_{a_\rho}^\infty |f'|^2dx\le\Big(\int_{a_\rho}^\infty |f|^2dx\Big)^{1/2}\cdot\Big(\int_{a_\rho}^\infty |f''|^2dx\Big)^{1/2}$$

(take the Fourier transform for the proof) that $\lim\limits_{x\to+\infty} f'(x)=0$ for any f in $D(L)$.

It is also clear that $\int_0^x |f''(t)|dt<+\infty$ for any f in $D(L)$ and for $x\in(0,+\infty)$. Indeed,

$$\int_0^x |f''(t)|dt\le\Big(\int_0^x\frac{1}{\rho^2}|f''|^2dt\Big)^{1/2}\cdot\Big(\int_0^x\rho^2dt\Big)^{1/2}\le\|Lf\|_{L^2_\rho}\Big(\int_0^x\rho^2dt\Big)^{1/2}<+\infty .$$

The integration by parts shows now the operator L is symmetric.

To prove $L=L^*$ it is sufficient to check that $D(L^*)\subset D(L)$. Let $v\in D(L^*)$. Then

$$|<Lu,v>_{L^2_\rho}|=\Big|\int_0^\infty u''\cdot\bar v\,dx\Big|\le const.\|u\|_{L^2_\rho}$$

for any u in $D(L)$. This means, in particular, that

$$\Big|\int_0^\infty u''\bar v dx\Big|\le const.\sup_{t\in supp(u)}|u(t)|\cdot\Big(\int_{supp(u)}\rho^2dt\Big)^{1/2}$$

and therefore the distribution v'' coincides with a σ-finite measure μ on $[0,+\infty)$. Then

$$\Big|\int_0^\infty u\,d\mu\Big|\le const.\Big(\int_0^\infty|u\rho|^2dt\Big)^{1/2}$$

and therefore $d\mu = p\,dt$ is absolutely continuous with $\int_0^\infty \frac{1}{\rho^2}|p|^2 dt < +\infty$. It is clear now that $\rho^{-2}\cdot v'' \in L^2_\rho$.
To prove that $v'(0) = 0$ we should only remark that

$$\int_0^\infty u''\bar{v}\,dx = -\int_0^\infty \bar{v}'\,du = \bar{v}'(0)\,u(0) - \int_0^\infty \bar{v}''u\,dx .$$

This, obviously, implies that

$$|v'(0)|\cdot|u(0)| \le const.\|u\|_{L^2_\rho} + |<u, Lv>_{L^2_\rho}| \le const.\,\|u\|_{L^2_\rho} .$$

Therefore the assumption $v'(0) \ne 0$ implies that the functional $u \mapsto u(0)$ is bounded in L^2_ρ. ●

THEOREM 1.2. The operator

$$\mathcal{L} = i\begin{pmatrix} \mathbb{O} & -I \\ L & \mathbb{O} \end{pmatrix},\ \mathcal{D}(\mathcal{L}) = \{\mathcal{U} \in E : u_0'' \in L^2(\mathbb{R}_+),\ u_0'(0) = 0,\ u_1' \in L^2(\mathbb{R}_+)\}$$

is self-adjoint. The family $(U_t)_{t\in\mathbb{R}}$ of evolution operators coincides with the strongly continuous unitary group $U_t = exp(it\mathcal{L})$. For every $\mathcal{U} = (u_0, u_1)$ in $\mathcal{D}(\mathcal{L})$ the formula $\mathcal{U}(t) = U_t\mathcal{U}$ defines a function $u_0(x,t)$ satisfying (3).

PROOF. Our first task is to check that the operator $\mathcal{L}$ defined above is self-agjoint. A simple calculation shows the operator $\mathcal{L}$ is symmetric on a dense set of smooth data:

$$<\mathcal{L}\mathcal{U}, \mathcal{V}>_E = \frac{1}{2}\int_0^\infty -iu_1'\cdot\bar{v}_0'\,dx + \frac{1}{2}\int_0^\infty iu_0''\cdot\bar{v}_1\,dx =$$

$$= -\frac{1}{2}\int_0^\infty i\bar{v}_0'\,du_1 + \frac{1}{2}\int_0^\infty i\,\bar{v}_1\,du_0' =$$

$$= \frac{1}{2}\int_0^\infty u_1 i v_0''\,dx + \frac{1}{2}\int_0^\infty iu_0'\bar{v}_1'\,dx = <\mathcal{U}, \mathcal{L}\mathcal{V}>_E .$$

Let $\mathcal{V} \in \mathcal{D}(\mathcal{L}^*)$. Then, clearly,

$$|<\mathcal{L}\mathcal{U}, \mathcal{V}>_E| \le const.\,\|\mathcal{U}\|_E$$

for every $\mathcal{U}$ in $\mathcal{D}(\mathcal{L})$. Let $\mathcal{U} = (u_0, 0) \in \mathcal{D}(\mathcal{L})$. Then

$$\left|\int_0^\infty u_0''\bar{v}_1\,dx\right| \le const.\left(\int_0^\infty |u_0'|^2 dx\right)^{1/2}$$

and therefore $v_1' \in L^2(\mathbb{R})$. Let now $\mathcal{U} = (0, u_1) \in \mathcal{D}(\mathcal{L})$.

Then it follows that

$$\left|\int_0^\infty u_1'\cdot v_0'\,dx\right| = \left|\int_0^\infty u_1''\cdot v_0\,dx\right| \le const.\ \|u_1\|_{L^2_\rho} .$$

Since the operator L is self-adjoint in L^2_ρ (see Lemma 1.1), we have $L v_0 \in L^2_\rho$, $v_0'(0) = 0$ and therefore $v \in \mathcal{D}(\mathcal{L})$.

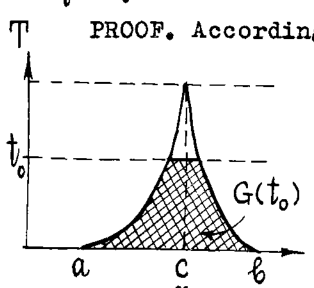
By the Stone theorem the operators $U_t = exp(i\mathcal{L}t)$ are unitary and $U_{t+s} = U_t U_s$. We may now define

$$\mathcal{U}(t) \overset{def}{=\!=} U_t\,\mathcal{U}(0) .$$

To prove the last statement of the theorem one should only remark that $U_t\mathcal{D}(\mathcal{L}) \subset \mathcal{D}(\mathcal{L})$ (see theorem VIII.7 of [67]) and therefore $\frac{\partial}{\partial t}\mathcal{U}(t) = i\mathcal{L}\mathcal{U}(t)$. ●

THEOREM 1.3 (Huygens principle, see [47]). Let $I = (a,b)$ be an interval on $\mathbb{R}_+$, let $x_0 \in I$ and let $\mathcal{U} \in \mathcal{D}(\mathcal{L})$ and $\mathcal{U}|I \equiv 0$. Then $\mathcal{U}(t)(x_0) = 0$ for sufficiently small t .

PROOF. According to (1) $\rho(x) > 0$ a.e. on $\mathbb{R}_+$. So the function $\int_a^x \rho(s)\,ds$ is, obviously, strictly increasing and the function $\int_x^b \rho(s)\,ds$ decreases. A point c on the picture can be found from the equation $\int_a^c \rho(s)ds = \int_c^b \rho(s)ds$. Let $t_1(x) = \int_a^x \rho(s)\,ds$ for $a < x < c$ and let $t_2(x) = \int_x^b \rho(s)\,ds$ for $c < x < b$. At last, $T = t_1(c) = t_2(c)$. We show that the energy of the wave with the boundary values $\mathcal{U}$ vanishes in the domain $G(t_0)$. To do this let $t_1(x_1) = t_0 = t_2(x_2)$. We have



$$0 = \iint_{G(t_0)} \left[\rho^2(x)\,u_{tt}u_t - u_{xx}u_t\right] dx\,dt .$$

Clearly, for $0 < t < t_0$,

$$-\int_{t_1^{-1}(t)}^{t_2^{-1}(t)} u_{xx}u_t\,dx = \int_{t_1^{-1}(t)}^{t_2^{-1}(t)} u_x u_{tx}\,dx - u_t u_x \Big|_{t_1^{-1}(t)}^{t_2^{-1}(t)} .$$

Therefore

$$\iint\limits_{G(t_0)} -u_{xx}u_t\,dx\,dt = \iint\limits_{G(t_0)} u_x u_{tx}\,dx\,dt + \int_a^{t_1^{-1}(t_0)} u_t(x,t_1(x))\,u_x(x,t_1(x))\,t_1'(x)\,dx -$$
$$- \int_{t_2^{-1}(t)}^{b} u_t(x,t_2(x))\,u_x(x,t_2(x))\,t_2'(x)\,dx\ .$$

But

$$\iint\limits_{G(t_0)} \{\rho^2(x)u_{tt}\cdot u_t + u_{xt}\cdot u_t\}\,dx\cdot dt = \frac{1}{2}\iint\limits_{G(t_0)} \frac{d}{dt}\{\rho^2 u_t^2 + u_x^2\}\,dx\,dt =$$
$$= \frac{1}{2}\int_a^{t_1^{-1}(t_0)} [\rho^2(x)u_t^2(x,t_1(x)) + u_x^2(x,t_1(x))]\,dx + \frac{1}{2}\int_{t_1^{-1}(t_0)}^{t_2^{-1}(t_0)} [\rho^2(x)u_t^2(x,t_0) + u_x^2(x,t_0)]\,dx$$
$$+ \int_{t_2^{-1}(t_0)}^{b} [\rho^2(x)\,u_t^2(x,t_2(x)) + u_x^2(x,t_2(x))]\,dx\ .$$

We get therefore

$$0 = \frac{1}{2}\int_{t_1^{-1}(t_0)}^{t_2^{-1}(t_0)} [\rho^2(x)\,u_t^2(x,t_0) + u_x^2(x,t_0)]\,dx +$$
$$+ \frac{1}{2}\int\limits_{[a,t_1^{-1}(t_0)]\cup[t_2^{-1}(t_0),b]} (\rho(x)\,u_t(x,t(x)) + u_x(x,t(x)))^2\,dx\ . \quad \bullet$$

A remarkable property of the unitary group $(U_t)_{t\in\mathbb{R}}$ is that it has a pair of orthogonal invariant subspaces $(\mathcal{D}_+,\mathcal{D}_-)$ in E satisfying

$$U_t\,\mathcal{D}_+ \subset \mathcal{D}_+,\ t>0\,; \quad U_t\,\mathcal{D}_- \subset \mathcal{D}_-,\ t<0\ .$$

For example, let

$$\mathcal{D}_+ = \left\{ \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} : -u_1 = u_0',\ \ u_0' \in L^2(\mathbb{R}_+);\ u_0(x) \equiv const,\ x<a \right\},$$

$$\mathcal{D}_- = \left\{ \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} :\ u_1 = u_0',\ \ u_0' \in L^2(\mathbb{R}_+);\ u_0(x) \equiv const,\ x<a \right\}.$$

Then $u_0(x+t)$ is called an *incoming wave* and $u_0(x-t)$ is called an *outgoing wave*. Clearly

$$U_t\begin{pmatrix} u_0 \\ -u_0' \end{pmatrix} = \begin{pmatrix} u_0(x-t) \\ -u_0'(x-t) \end{pmatrix}$$

if $u_0(x) \equiv const$ for $x < a$, and if $t > 0$. So we may imagine the space $\mathcal{D}_-$ as the space of incoming waves and $\mathcal{D}_+$ as the space of outgoing waves. It were P.Lax and R.Phillips who have stressed the importance of these invariant subspaces for the first time [47] . They advanced a new approach (L-Ph-approach) to the scattering theory for unitary groups which have invariant subspaces of this type [47] . Let $K \stackrel{def}{=\!=} E \ominus \{\mathcal{D}_+ \oplus \mathcal{D}_-\}$. The scattering matrix arising in L-Ph-approach turns out a characteristic function for the strong continuous semigroup of contractions [69]

$$Z_t \stackrel{def}{=\!=} P_K U_t | K, \quad t > 0 \ .$$

The following lemma describes the data in K .

LEMMA 1.4. Let $u \in E$. Then $u \in K$ if and only if $u_0(x) \equiv const$ for $x > a$ and $u_1(x) \equiv 0$ for $x > a$.

PROOF. Let $G_a = \{ g : \int_0^\infty |g'|^2 dx < +\infty \ , \ g(x) \equiv const$ for $x < a \}$. Then clearly

$$\begin{pmatrix} g \\ 0 \end{pmatrix} = \frac{1}{2}\left\{\begin{pmatrix} g \\ -g' \end{pmatrix} + \begin{pmatrix} g \\ g' \end{pmatrix}\right\} \in \mathcal{D}_+ \oplus \mathcal{D}_- \ ; \ \begin{pmatrix} 0 \\ g' \end{pmatrix} = \frac{1}{2}\left\{\begin{pmatrix} g \\ g' \end{pmatrix} - \begin{pmatrix} g \\ -g' \end{pmatrix}\right\} \in \mathcal{D}_+ \oplus \mathcal{D}_- \ .$$

Therefore $u \in K$ if and only if

$$< \begin{pmatrix} g \\ 0 \end{pmatrix}, u >_E \ = \ < \begin{pmatrix} 0 \\ g' \end{pmatrix}, u >_E \ = 0$$

for every g in G_a . The du Bois-Reymond lemma implies that this is equivalent to the statement of the lemma. ●

The semigroup $(Z_t)_{t \geq 0}$ is unitary equivalent to the semigroup $P_{K_S} e^{ixt} | K_S$, $t \geq 0$. Here S is an inner function in $\mathbb{C}_+$ and K_S stands for $H_+^2 \ominus S H_+^2$. The function S is called a c h a r a c t e r i s t i c f u n c t i o n for $(Z_t)_{t \geq 0}$. In the scattering theory S is known as a reflection coefficient. We shall return to its physical meaning a bit later.

It is remarkable that the unitary correspondence between the semigroups can be given by explicit formulae. To do this we have to find a family of generalized eigen-functions for $(U_t)_{t\in\mathbb{R}}$ or equivalently for $\mathcal{L}$. In its turn this can be done with the help of so-called Jost solutions $y(x,\lambda)$:

$$-y'' = \lambda^2 \cdot \rho^2 \cdot y\,, \quad y(a_\rho,\lambda)=1\,, \quad y'(a_\rho,\lambda)=-i\lambda\,.$$

The existence and uniqueness of the Jost solution $y(x,\lambda)$ is implied by the standard existence theorem of the differential equations theory. Moreover, the well-known iteration method leads, obviously, to the conclusion that $\lambda \longmapsto y(x,\lambda)$ is an entire function for every x in $\mathbb{R}$. Let now $a>a_\rho$. Then a Jost solution corresponding to a point a is defined by

$$y_a(x,\lambda) = e^{i\lambda(a-a_\rho)}\cdot y(x,\lambda)\,.$$

Clearly,

$$y_a(a,\lambda) = 1\,, \quad y_a'(a,\lambda) = -i\lambda$$

and $y_a(x,\lambda) = e^{-i\lambda(x-a)}$ for $x \geq a$.
It follows from the uniqueness of the Jost solution that

$$\overline{y_a(x,-\overline{\lambda})} = y_a(x,\lambda)\,. \tag{5}$$

A linear combination of the Jost solutions

$$\varphi_a(x,\lambda) = y_a(x,-\lambda) + \overline{S_a(\lambda)}\cdot y_a(x,\lambda)$$

satisfies the boundary condition $\varphi_a'(0,\lambda)=0$ if

$$\overline{S_a(\lambda)} = -\frac{y_a'(0,-\lambda)}{y_a'(0,\lambda)} = -e^{-2i\lambda(a-a_\rho)}\,\frac{y'(0,-\lambda)}{y'(0,\lambda)}\,.$$

It is clear that $|S_a(\lambda)|=1$ for $\lambda\in\mathbb{R}$ (see (5)). A simple computation shows that $\mathcal{L}\,\Phi_a(x,\lambda) = \lambda\,\Phi_a(x,\lambda)$ for

$$\Phi_a(x,\lambda) = \begin{pmatrix} 1/i\lambda \cdot \varphi_a(x,\lambda) \\ \varphi_a(x,\lambda) \end{pmatrix}.$$

Let E_0 be a dense subset of data in E which have a compact support in $\mathbb{R}_+$. We define a mapping $\mathcal{T}_-$ by the following formula. For $\mathcal{U}\in E_0$ let

$$\mathcal{T}\mathcal{U}=\frac{1}{\sqrt{2\pi}}\langle \mathcal{U},\Phi_a(\cdot,\lambda)\rangle_E\equiv\frac{1}{2\sqrt{2\pi}}\int_0^\infty\{u_{0x}\cdot\overline{\frac{1}{i\lambda}\varphi_{ax}}+\rho^2u_1\overline{\varphi}_a\}\,dx=$$

$$=\frac{1}{2\sqrt{2\pi}}\int_0^\infty(i\lambda u_0+u_1)\rho^2\overline{\varphi}_a\,dx.$$

THEOREM 1.5. The closure of the operator $\mathcal{T}:E_0\to L^2(\mathbb{R})$

$$\mathcal{T}\mathcal{U}=\frac{1}{2\sqrt{2\pi}}\int_0^\infty(i\lambda u_0(x)+u_1(x))\rho^2\overline{\varphi_a(x,\lambda)}\,dx$$

defines an isometry of E onto $L^2(\mathbb{R})$. The following formulae hold

$$\mathcal{T}\mathcal{D}_-=H^2_-\ ,\quad \mathcal{T}\mathcal{D}_+=S_aH^2_+$$
$$\mathcal{T}U_t=e^{i\lambda t}\cdot\mathcal{T}.$$

The function S_a is an inner function in $\mathbb{C}_+$ and

$$\mathcal{T}K=H^2_+\ominus S_aH^2_+\,;\quad \mathcal{T}Z_t\mathcal{U}=P_{K_{S_a}}e^{i\lambda t}\mathcal{T}\mathcal{U},\quad \mathcal{U}\in K.$$

PROOF. Let $\mathcal{U}\in\mathcal{D}_-\cap E_0$. Then $\mathcal{U}=(u,u')$ and $u(x)\equiv 0$ for $x\leqslant a$. It follows that

$$\mathcal{T}\mathcal{U}=\frac{1}{2\sqrt{2\pi}}\int_a^{+\infty}(i\lambda u+u')\{e^{-i\lambda(x-a)}+S_a(\lambda)e^{i\lambda(x-a)}\}\,dx=$$

$$=\frac{1}{2\sqrt{2\pi}}\int_a^{+\infty}(e^{i\lambda x}u)'\{e^{-2i\lambda x}e^{i\lambda a}+S_a(\lambda)e^{-i\lambda a}\}\,dx=$$

$$=\frac{i\lambda}{\sqrt{2\pi}}\int_a^\infty u\,e^{i\lambda(a-x)}dx=\frac{1}{\sqrt{2\pi}}\int_a^{+\infty}u'e^{i\lambda(a-x)}\,dx.$$

Hence, by the Parseval theorem and by (3) we have

$$\|\mathcal{T}\mathcal{U}\|_{L^2(\mathbb{R})}=\left(\int_a^{+\infty}|u'|^2dx\right)^{1/2}=\|\mathcal{U}\|_E.$$

An analogous computation shows that for $\mathcal{U}\in\mathcal{D}_+\cap E_0$

$$\mathcal{T}u = \frac{1}{2\sqrt{2\pi}} \int_a^{+\infty} (i\lambda u - u')\{e^{-i\lambda(x-a)} + S_a(\lambda) e^{i\lambda(x-a)}\} dx =$$

$$= S_a(\lambda) \cdot \frac{1}{\sqrt{2\pi}} \int_a^{+\infty} e^{i\lambda(x-a)} u' dx$$

and again we have $\|\mathcal{T}u\|_{L^2(\mathbb{R})} = \|u\|_E$. Now it follows by the Paley-Wiener theorem

$$\mathcal{T}\mathcal{D}_- = H^2_- , \quad \mathcal{T}\mathcal{D}_+ = S_a H^2_+ .$$

Let now u be a smooth function in E_0 . Clearly

$$< \mathcal{L}u, \Phi_a(\cdot, \lambda) >_E = \lambda < u, \Phi_a(\cdot, \lambda) >_E$$

and consequently

$$\frac{d}{dt} < U_t u, \Phi_a >_E = < i\mathcal{L}u, \Phi_a >_E = i\lambda < U_t u, \Phi_a >_E .$$

Therefore the boundary condition $U_0 = I$ implies

$$< U_t u, \Phi_a >_E = e^{i\lambda t} < u, \Phi_a >_E .$$

By theorem 1.2

$$\mathrm{span}\,(U_t \mathcal{D}_- : t \in \mathbb{R}) = E .$$

Therefore $\mathcal{T}$ maps the space E isometrically onto $L^2(\mathbb{R})$. It follows from $U_t \mathcal{D}_+ \subset \mathcal{D}_+$ for $t > 0$ that $e^{i\lambda t} \cdot S_a \cdot H^2_+ \subset \subset H^2_+$ for every $t > 0$. By P.Lax theorem [18] this means that S_a is an inner function $(|S_a(\lambda)| = 1, \ \lambda \in \mathbb{R})$. ●

REMARK. The function S_a being a quotient of entire functions, it is clear that

$$S_a = B \cdot \Theta .$$

Here B denotes a Blaschke product in $\mathbb{C}_+$ whose zeros have no limit points in $\mathbb{R}$ and $\Theta(z) = \exp(icz)$, $c > 0$.

The transformation $\mathcal{T}$ is called an *incoming spectral representation* for the unitary group $(U_t)_{t \in \mathbb{R}}$. The spectral property of $\mathcal{T}$ means that

$\mathcal{T}_-$ transforms the group $(U_t)_{t\in\mathbb{R}}$ onto the unitary group $(e^{i\lambda t})_{t\in\mathbb{R}}$ in $L^2(\mathbb{R})$.

Let now discuss the physical meaning of the reflection coefficient $\overline{S}_a$. It is clear that

$$\mathcal{T}_-^{-1} v = \mathcal{T}_-^{*} v = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Phi_a(x,\lambda)\, v(\lambda)\, d\lambda \qquad (6)$$

and that the evolution of the part of the "wave packet" $\mathcal{U}(x,t) = U_t \mathcal{T}_-^{-1} v$ in $\mathcal{D}_- \oplus \mathcal{D}_+$ is defined by

$$\{(\mathcal{P}_{\mathcal{D}_-} \oplus \mathcal{P}_{\mathcal{D}_+}) U_t \mathcal{T}_-^{-1} v\}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\lambda t} \begin{pmatrix} \frac{1}{i\lambda}\varphi_a(x,\lambda) \\ \varphi_a(x,\lambda) \end{pmatrix} v(\lambda)\, d\lambda, \; x > a.$$

Therefore for $x > a$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\lambda t} \begin{pmatrix} 1/i\lambda \\ 1 \end{pmatrix} e^{i\lambda(x-a)} v(\lambda)\, d\lambda +$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\lambda t} \begin{pmatrix} 1/i\lambda \\ 1 \end{pmatrix} e^{-i\lambda(x-a)} v(\lambda)\, \overline{S_a(\lambda)}\, d\lambda =$$

$$= \Phi_{in}(x+t) + \Phi_{out}(x-t) .$$

We see that the complex amplitudes $e^{-i\lambda a} \cdot v(\lambda) \cdot \begin{pmatrix} 1/i\lambda \\ 1 \end{pmatrix}$, $e^{-i\lambda a} \cdot v(-\lambda) S_a(\lambda) \begin{pmatrix} 1/-i\lambda \\ 1 \end{pmatrix}$ of the spectrum of the incoming and outgoing waves are connected with the help of reflexion coefficient.

2. The wave equation and the Regge problem.

A key to the connection between the Regge problem and the unitary group $(U_t)_{t\in\mathbb{R}}$ is given by an explicit description of the generator A of the contractive semigroup $(Z_t)_{t\geqslant 0}$.

THEOREM 2.1. The generator A of the semigroup $Z_t = e^{itA} = P_K e^{it\mathcal{L}} | K$ is a maximal completely dissipative operator in K . Its domain $D(A)$ is

$$\{\mathcal{U} \in K : u_0'' \in L^2(0,a);\; u_0'(0) = 0;\; u_1' \in L^2(0,a);\; u_0'(a) + u_1(a) = 0\}$$

and $A\mathcal{U} = \mathcal{L}\mathcal{U}$ for $\mathcal{U} \in K$.

PROOF. The operator A is a maximal dissipative operator, because $(Z_t)_{t\geqslant 0}$ is a contractive semigroup (see theorem X.48 [68]). Assuming that A has a non-trivial self-adjoint part, we see that there is a non-zero element f in K such that $Z_t f = U_t f$ for every $t>0$. Therefore $U_t f \perp \mathcal{D}_+$ for every $t>0$ and $f \perp U_{-t}\mathcal{D}_+$ for $t>0$. But $E=\mathrm{span}(U_{-t}\mathcal{D}_+ : t>0)$ and so $f=0$.

The computation of the domain for A is a more subtle problem. Let $\mathcal{D}_0$ be the set of smooth data in K supported on compact subsets of $(0,a)$. Let $\mathcal{L}_0=\mathcal{L}\,|\,\mathcal{D}_0$. Clearly, $\mathcal{L}_0$ is symmetric in K . Using Theorem 1.2 , one can easily prove that

$$\mathrm{D}(\mathcal{L}_0^*)=\{\mathcal{U}\in K : u_0''\in L^2(0,a),\ \ u_0'(0)=0,\ u_1'\in L^2(0,a)\}$$

and that $\mathcal{L}_0^*= \mathcal{L}\,|\,\mathrm{D}(\mathcal{L}_0^*)$. Standard arguments lead to the conclusion that the deficiency indices of $\mathcal{L}_0$ are (1,1). Indeed, if, for example,

$$i\begin{pmatrix}u_0\\ u_1\end{pmatrix}=i\begin{pmatrix}\mathbb{O} & -I\\ L & 0\end{pmatrix}\begin{pmatrix}u_0\\ u_1\end{pmatrix}$$

for $\mathcal{U}$ in $\mathrm{D}(\mathcal{L}_0^*)$, then $u_1=-u_0$ and $u_0=Lu_0$. It follows that $\dim\operatorname{Ker}(i-\mathcal{L}_0^*)=1$.

By theorem 1.3 for any $\mathcal{U}$ in $\mathcal{D}_0$ we have $U_t\,\mathcal{U}\in K$ if t is small enough. Therefore $\mathcal{L}_0\subset A$ and also $\mathcal{L}_0\subset A^*$. But then $A^*\subset\mathcal{L}_0^*$, $A=A^{**}\subset\mathcal{L}_0^*$ and therefore the domain of A is contained in $\mathrm{D}(\mathcal{L}_0^*)$.

Let, for the time being, B denote the restriction of $\mathcal{L}$ onto the subset of data in $\mathrm{D}(\mathcal{L}_0^*)$ satisfying the boundary condition $u_0'(a)+u_1(a)=0$. Clearly, B is a closed operator. Moreover B is a dissipative operator in K , i.e. $\operatorname{Im}\langle B\mathcal{U},\mathcal{U}\rangle_E \geqslant 0$. Indeed, for every $\mathcal{U}\in\mathcal{D}(\mathcal{L}_0^*)$

$$\langle\mathcal{L}\mathcal{U},\mathcal{U}\rangle_E=\frac{1}{2}\int_0^a -iu_1'\bar{u}_0'\,dx+\frac{1}{2}\int_0^a -iu_0''\bar{u}_1\,dx=$$

$$=\frac{1}{2}\int_0^a \overline{iu_0'}\,du_1+\frac{1}{2}\int_0^a \overline{iu_1}\,du_0'=\overline{iu_0'(a)}u_1(a)+\overline{iu_1(a)}\,u_0'(a)-$$

$$-\frac{1}{2}\int_0^a \left[u_1\cdot\overline{iu_0''}+u_0'\cdot\overline{iu_1'}\right]dx = 2i|u_1(a)|^2+\langle \mathcal{U},\mathcal{L}\mathcal{U}\rangle_E .$$

Therefore $\operatorname{Im}\langle \mathcal{L}\mathcal{U},\mathcal{U}\rangle_E = |u_1(a)|^2 \geqslant 0$.

The operator B is a one-dimensional perturbation of $\mathcal{L}_0$. So to prove $B = A$ it is sufficient to check that $Z_t D(B) \subset \subset D(B)$ (see theorem X.49 [68]), for $t>0$.
Let $\mathcal{U}\in D(B)$. Then $\mathcal{P}_{\mathcal{D}_-} U_t \mathcal{U} = 0$ for $t>0$ because $U_{-t}\mathcal{D}_- \subset \mathcal{D}_-$ and $\mathcal{U}\perp \mathcal{D}_+ \oplus \mathcal{D}_-$. This means that outside of the interval $(0,a)$ the solution $U_t\mathcal{U}(x)$ is outgoing and therefore $(U_t\mathcal{U})_0'(x)+(U_t\mathcal{U})_1(x) \equiv 0$ for $x>a$. But $(U_t\mathcal{U})_0\in W_2^2(0,a+t)$ and $(U_t\mathcal{U})_1\in W_1^1(0,a+t)$ and in particular, these functions are continuous. Therefore $(U_t\mathcal{U})_0'(a)+(U_t\mathcal{U})_1(a)=0$. To finish the proof it is sufficient only to remark that the projection of $\mathcal{U}\in E$ onto K is a pair in K $\begin{pmatrix} v_0 \\ v_1\end{pmatrix}$, $v_0(x)\equiv const$, $v_1(x)\equiv 0$ for $x>a$ which coincides with $\mathcal{U}$ on $(0,a)$. ●

REMARK. It is easy to see that for the generator $A' = -A^*$ of the conjugate semigroup the following formula holds

$$D(A') = \{\mathcal{U}\in K : u_0\in W_2^2(0,a),\ u_1\in W_2^1(0,a),\ u_0'(0)=0,\ u_0'(a)-u_1(a)=0\}.$$

Now we are in a position to describe spectral properties of the operator A . Let $\sigma_d(B)$ denote a point spectrum of an operator B , i.e. the set of all eigen-values. Remind the reader, see lemma 1.4 , that a vector-function $\mathcal{U}$ in K is completely determined by its restriction on the interval $(0,a)$ and that $y_a(x,\lambda)$ denotes the Jost solution corresponding to a point a :

$$L y_a = \lambda^2 y_a;\quad y_a(a,\lambda)=1,\ y_a'(a,\lambda) = -i\lambda .$$

THEOREM 2.2. The spectrum $\sigma(A)$ of the dissipative operator A is equal to $\sigma_d(A)\cup\{\infty\}$, and $\sigma_d(A) = \{k\in\mathbb{C}_+ : S_a(k)=0\}$. The resolvent $(A-\lambda I)^{-1}$ is compact. For $k\in\sigma_d(A)$ the eigen-function $\mathcal{U}_k$ corresponding to the eigen-value k is defined by

$$\mathcal{U}_k(x) = \begin{pmatrix} \frac{1}{ik}\ y_a(x,k) \\ y_a(x,k)\end{pmatrix},\qquad x\in[0,a].$$

The spectrum $\sigma_d(A)$ is symmetric with respect to the imaginary axis: $\sigma_d(A) = -\overline{\sigma_d(A)}$.

PROOF. The first statement of the theorem is implied by theorem 1.5. To prove the resolvent of A is compact it is, obviously, sufficient to check that the operator $Tf = P_{K_{S_a}} \frac{1}{z+i} f$ is compact in K_{S_a} . A simple, but important formula, connecting Hankel and model operators, (see [18] , p.237) implies

$$S_a H_{\overline{S}_a \cdot \frac{1}{x+i}} = T \oplus \mathbb{O} .$$

The function S_a being holomorphic on $\mathbb{R}$, it is clear that $\overline{S}_a \cdot (x+i)^{-1} \in C_0(\mathbb{R})$ and therefore by the Hartman-Sarason theorem (see, for example, [18]) the operator $H_{\overline{S}_a \cdot (x+i)^{-1}}$ is compact.

We have by the definition of the reflection coefficient $S_a(k) = -y_a'(0,k)\cdot(y_a'(0,-k))^{-1}$ and therefore $k \in \sigma_d(A)$ iff $y_a'(0,k) = 0$ *). It follows from the definition of the Jost solution that $\mathcal{U}_k \in D(A)$. Now the proof of the equality $A\mathcal{U}_k = k\mathcal{U}_k$ is reduced to a calculation. The last statement of the theorem is an obvious consequence of (5). ●

A completely analogous result holds for the adjoint operator A^* . Clearly, $\sigma_d(A^*) = \overline{\sigma_d(A)}$. Here is a formula for the eigen-function $\mathcal{U}_k^*$, $A^*\mathcal{U}_k^* = \bar{k}\,\mathcal{U}_k^*$:

$$\mathcal{U}_k^* = \begin{pmatrix} \frac{1}{i\bar{k}}\, y_a(x,-\bar{k}) \\ y_a(x,-\bar{k}) \end{pmatrix}, \quad x \in [0,a] .$$

The following formulae will be useful in what follows:

$$\mathcal{U}_k + \overline{\mathcal{U}}_k^* = 2\begin{pmatrix} 0 \\ y_a(x,k) \end{pmatrix}, \quad \mathcal{U}_k - \overline{\mathcal{U}}_k^* = \frac{2}{ik}\begin{pmatrix} y_a(x,k) \\ 0 \end{pmatrix}. \tag{7}$$

It should be remarked that

$$A^*\overline{\mathcal{U}}_k^* = -k\cdot\overline{\mathcal{U}}_k^* , \qquad k \in \sigma_d(A) .$$

THEOREM 2.3. The following are equivalent:

a) the family $\{\mathcal{U}_k, \mathcal{U}_k^* : k \in \sigma_d(A)\}$ is complete in K ;

b) the family $\{y_a(x,k) : k \in \sigma_d(A)\}$ of the eigen-func-

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*) The operator A being dissipative, it follows $\operatorname{Im} k > 0$, otherwise we would get an eigen-value for A in $\mathbb{C}_-$.

tions for problem (2) is complete in $L^2_\rho(0,a)$.

PROOF. a) $\Rightarrow$ b) is obvious in view of (7). b) $\Rightarrow$ a). It is sufficient to check that the completeness of the family $\{y_a(x,k): k\in\sigma_d(A)\}$ in $L^2_\rho(0,a)$ implies its completeness in $W^1_2(0,a)$. We have

$$<f,u>_1=\int_0^a f'\bar{u}'dx=-\int_0^a[f-f(a)]\,\bar{u}''dx=\bar{\lambda}^2\int_0^a[f(x)-f(a)]\,\rho^2\bar{u}\,dx=$$

$$=\bar{\lambda}^2<f-f(a),\,u>_{L^2_\rho}$$

for any u satisfying $Lu=\lambda^2u$. ●

Henceforth we shall often assume the following technical condition is satisfied:

$$\varlimsup_{y\to+\infty,\,S(iy)=0} 2y\,|S'(iy)|<1. \tag{*}$$

It should be remarked that trivial estimates using the Cauchy formula imply $2y\cdot|S'(iy)|\leq 1$.

THEOREM 2.4. Suppose the family $\{u_k, u_k^*: k\in\sigma_d(A)\}$ forms an unconditional basis in K . Then the family of the eigen--functions for the Regge problem (2) forms an unconditional basis in $L^2_\rho(0,a)$ and in $W^1_2(0,a)$ simultaneously. The converse is true if $S_a\in(*)$.

PROOF. The family of 2-dimensional subspaces spanned by the vectors u_k , $\bar{u}_k^*$ forms, clearly, an unconditional basis in K . Therefore the first statement of the theorem is a consequence of (7).

To prove the second one we remark the functions u_k and $\bar{u}_k^*$ are othogonal for $k\neq-\bar{k}$. Indeed,

$$k<u_k,\bar{u}_k^*>=<Au_k,\bar{u}_k^*>=<u_k,A^*\bar{u}_k^*>=-\bar{k}<u_k,\bar{u}_k^*>.$$

It remains to discuss the case $k=iy$, $y>0$. It follows from (*) that the angles between the vectors u_{iy} and $\bar{u}_{iy}^*$ are bounded away from zero. To see this we use theorem 1.5. Then the angle between $S\cdot\sqrt{y}\cdot(z-iy)^{-1}$, $(z+iy)^{-1}\sqrt{y}$ coincides with arccos $(2y|S'(iy)|)$. ●

Let $S_a=\theta^d\cdot B$, where $\theta^d=exp(idz)$, $d>0$, and B is a Blaschke product with simple zeroes in a half-plane $\mathbb{C}_\sigma$ for some $\sigma>0$.

THEOREM 2.5. Suppose the family $(e^{ikx})_{B(k)=0}$ forms an unconditional basis in $L^2(0,d)$. Then the family of the eigen-func -

tions for the Regge problem (2) forms an unconditional basis in $L^2_\rho(0,a)$ and in $W^1_2(0,a)$. The converse is true if $S_a \in (*)$.

PROOF. The first statement of the theorem results from theorem 2, Part I, theorem 1.5 and theorem 2.4. To prove the second statement one should simply inverse the order of theorems cited above. ●

3. Asymptotic properties of the reflexion coefficient and an example to the Regge problem.

It is assumed in this section that $\rho \in C^2[0,a_\rho]$, $\inf_{0<x<a_\rho} \rho(x) > 0$ and that $\lim_{x\to a_\rho - 0} \rho(x) \neq \rho(a_\rho + 0) = 1$. It follows from the formula

$$S_a(\lambda) = -e^{2i(a-a_\rho)\lambda} \cdot \frac{y'(0,\lambda)}{y'(0,-\lambda)}, \quad \Im m\,\lambda \geqslant 0,$$

that all needed information about S_a can be extracted from the Jost solution $y(x,\lambda)$ corresponding to the point a_ρ.

We begin with an analysis of a "standard" equation

$$-y'' + \rho^{1/2}(\rho^{-1/2})'' y = \lambda^2 \rho^2 y$$

which, obviously, can be solved explicity:

$$y(x) = \rho^{-1/2}(x) \cdot e^{\pm i\lambda \int_x^{a_\rho} \rho(s)\,ds}.$$

One can easily prove that the Green function $G(x,t,\lambda)$ of the "standard" equation is defined by

$$G(x,t,\lambda) = \begin{cases} \rho^{-1/2}(x)\rho(t)^{-1/2} \cdot \dfrac{\sin \lambda \int_t^x \rho(s)\,ds}{\lambda}, & \text{if } x \leqslant t \\ 0, & \text{if } x > t. \end{cases}$$

Remind that by definition the Green function satisfies the equation:

$$-G'' + \rho^{1/2}(\rho^{-1/2})'' G - \lambda^2 \rho^2 G = \delta(x-t).$$

Therefore for any solution $y_0(x,\lambda)$ of the "standard" equation the solution $y(x,\lambda)$ of the integral equation

$$y(x,\lambda) = y_0(x,\lambda) + \int_x^{a_\rho} G(x,t,\lambda)\rho^{1/2}(t)(\rho^{-1/2}(t))'' y(t,\lambda)\,dt$$

satisfies (2) with the boundary conditions

$$y(a_\rho,\lambda)=y_0(a_\rho,\lambda)=1,\quad y'(a_\rho,\lambda)=y_0'(a_\rho,\lambda)=-i\lambda\,.$$

The following formula defines the function $y_0(x,\lambda)$:

$$y_0(x,\lambda)=\frac{\rho(a_\rho)^{\frac{1}{2}}}{\rho(x)^{\frac{1}{2}}}\cdot\cos\left(\lambda\int_x^{a_\rho}\rho(s)ds\right)+\frac{i\lambda-\frac{\rho'(a_\rho)}{2\rho(a_\rho)}}{[\rho(x)\cdot\rho(a_\rho)]^{\frac{1}{2}}}\cdot\frac{\sin(\lambda\int_x^{a_\rho}\rho(s)ds)}{\lambda}\,.$$

A well-known method of iteration can be applied now to investigate the asymptotic behavior of $y'(0,\lambda)$.

Let $\Gamma(x,t,\lambda)\overset{def}{=\!=}G(x,t,\lambda)\,\rho^{1/2}(t)\,\rho^{-1/2}(t)''$. Then

$$|y_0(x,\lambda)|\leq const\,\exp\{|Im\lambda|\cdot\int_x^{a_\rho}\rho(s)ds\};$$

$$|\Gamma(x,t,\lambda)|\leq const\,\frac{1}{|\lambda|}\cdot\exp\{|Im\lambda|\int_x^{a_\rho}\rho(s)\,ds\}.$$

Let $g_0(t,\lambda)\overset{def}{=\!=}y_0(t,\lambda)$ and let

$$g_{n+1}(x,\lambda)=\int_x^{a_\rho}\Gamma(x,t,\lambda)\,g_n(t,\lambda)\,dt,\quad n\in\mathbb{Z}_+\,.$$

The induction arguments imply

$$|g_n(t,\lambda)|\leq c^{n+1}\cdot\frac{(a-t)^n}{n!\cdot|\lambda|^n}\cdot\exp\{|Im\lambda|\cdot\int_t^{a_\rho}\rho(s)\,ds\}$$

and therefore the series

$$y(t,\lambda)=\sum_{n=0}^{\infty}g_n(t,\lambda)$$

converges to an entire function $\lambda\mapsto y(t,\lambda)$ of the exponential type:

$$y(t,\lambda)-y_0(t,\lambda)=O(1)\cdot\frac{1}{|\lambda|}\cdot\exp\{|Im\lambda|\int_t^{a_\rho}\rho(s)ds\}.$$

Let now $d\overset{def}{=\!=}\int_0^{a_\rho}\rho(s)\,ds$. A formal differentiation of the asymptotic formula gives

$$y'(0,\lambda)-y_0'(0,\lambda)=O(1)\,e^{d\cdot|Im\lambda|}\quad.$$

The proof is given by the iteration method. A simple computation leads to the following formula:

$$y_0'(0,\lambda)=\left[\rho(a_\rho)\cdot\rho(0)\right]^{1/2}\{\lambda\sin\lambda d-i\rho(a_\rho)^{-1}\lambda\cos\lambda d\}+O(1)e^{d|\operatorname{Im}\lambda|}. \quad (9)$$

Hence,

$$\frac{y'(0,\lambda)}{y'(0,-\lambda)}=\frac{\sin\lambda d-i\rho(a_\rho)^{-1}\cos\lambda d+O\left(\frac{1}{|\lambda|}\right)e^{d|\operatorname{Im}\lambda|}}{\sin\lambda d+i\rho(a_\rho)^{-1}\cos\lambda d+O\left(\frac{1}{|\lambda|}\right)e^{d|\operatorname{Im}\lambda|}}. \quad (\mathrm{10})$$

Clearly,

$$\sin\lambda d-i\rho(a_\rho)^{-1}\cos\lambda d=-\frac{i}{2}e^{-i\lambda d}(1+\rho^{-1}(a_\rho))\left\{e^{2i\lambda d}-\frac{\rho(a_\rho)-1}{\rho(a_\rho)+1}\right\}$$

and therefore $\sin\lambda d-i\rho(a_\rho)^{-1}\cos\lambda d$ is a sine-type function with the sequence of zeroes

$$\lambda_n^0=\frac{\pi n}{d}+\Delta+\frac{i}{2d}\log\left|\frac{1+\rho(a_\rho)}{1-\rho(a_\rho)}\right|,\quad n\in\mathbb{Z},$$

where $\Delta=0$ if $\rho(a_\rho)>1$ and $\Delta=\frac{\pi}{2d}$ if $\rho(a_\rho)<1$. One can easily check now that the sequence $(\lambda_n)_{n\in\mathbb{Z}}$ of the zeroes of $y'(0,-\lambda)$ satisfies

$$\lambda_n=\lambda_n^0+O\left(\frac{1}{n}\right).$$

This implies the sequence $(\lambda_n)_{n\in\mathbb{Z}}$ is an interpolating one for H_+^∞. It follows also from (10) that

$$\lim_{t\to+\infty}-\frac{y'(0,it)}{y'(0,-it)}=\frac{1-\rho(a_\rho)}{1+\rho(a_\rho)}$$

and therefore

$$S_a(\lambda)=e^{2i(a-a_\rho)\lambda}\cdot B(\lambda), \quad (\mathrm{11})$$

where B is a Blaschke product.

THEOREM 3.1. 1) If $a_\rho\leq a\leq a_\rho+d$ then the family of eigen-functions for the Regge problem (2) is complete in $L_\rho^2(0,a)$.

2) If $a=a_\rho+d$ then the family of eigen-functions forms a Riesz basis in $L_\rho^2(0,a)$.

PROOF. The function $y'(0,\lambda)$ being equal to zero at $\lambda=0$, we see that $\lambda^{-1}y'(0,\lambda)$ is an entire function. It follows from the asymptotic formula (9) that $\lambda^{-1}y'(0,\lambda)\in STF$. The width of the indicator diagram of this function is equal to $2d$.

So we see that

$$\frac{1}{\lambda} y'(0,\lambda) = e^{-i\lambda d} \cdot B \cdot h ,$$

where h is an outer function for $\mathbb{C}_+$. On the other hand, we have by (II)

$$\frac{1}{\lambda} y'(0,\lambda) = - B \frac{1}{\lambda} y'(0,-\lambda)$$

and therefore $h = -\frac{1}{\lambda} y'(0,-\lambda) e^{i\lambda d}$. This implies

$$\frac{\bar{h}}{h}(\lambda) = \frac{y'(0,\lambda)}{y'(0,-\lambda)} e^{-2i\lambda d} = - B e^{-2i\lambda d} , \quad \lambda \in \mathbb{R} .$$

It follows from $\lambda^{-1} y'(0,\lambda) \in STF$ that $|h|^2 \in (HS)$ and therefore

$$\operatorname{dist}(\bar{\theta}^{2d} B, H_+^\infty) < 1 ,$$
$$\operatorname{dist}(\theta^{2d} \bar{B}, H_+^\infty) < 1 .$$

We see that statement 2) of the theorem is now a simple corollary of theorem 3, Part I, theorems 2.3-2.5.

To prove statement one we should use lemma 3-bis instead of theorem 3, Part I. ●

The number $d = \int_0^{a_\rho} \rho(x)\, dx$ has a nice physical interpretation. Namely, it coincides with the time needed for the point perturbation of the end $x = 0$ of the string (2) to reach the point $x = a_\rho$ (see theorem 1.3.).

An example discussed in this section is closely related with an interesting paper [69] .

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