THE HAMBURGER MOMENT PROBLEM AND WEIGHTED POLYNOMIAL APPROXIMATION ON DISCRETE SUBSETS OF THE REAL LINE

Alexander Borichev, Mikhail Sodin

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0. Introduction

In this paper, we study a special case of the weighted polynomial approximation problem on the real line. In the general setting, this problem was investigated by S. Bernstein and M. Riesz, and later by N. Akhiezer, L. de Branges, L. Carleson, T. Hall, P. Koosis, B. Levin, P. Malliavin, S. Mandelbrojt, S. Mergelyan, H. Pollard and many others (for an extensive discussion see the survey papers [2, 28] and the book [21, Chapter VI]). In spite of significant efforts, the general problem is still far from being explicitly solved. In this paper we deal with a special case of the problem of density of the polynomials in $L^p(\mu)$ when the measure μ is supported by the zero set of an entire function of zero exponential type. This problem appears in the indeterminate case of the Hamburger moment problem [3, 5, 6, 7].

Given a (positive Borel) measure μ on the real line such that

$$\int_{\mathbb{R}} |t|^n \, d\mu(t) < \infty, \qquad n \geq 0,$$

we associate with this measure its moment sequence

$$s_n = \int_{\mathbb{R}} t^n \, d\mu(t), \qquad n = 0, 1, 2, \dots .$$

The Hamburger moment problem consists in finding, by a sequence of numbers $\{s_n\}_{n\geq 0}$, a positive Borel measure μ with moments s_n. If the solution is not unique, we say that the moment problem is *indeterminate*. Furthermore, measures μ solving such problems are

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called indeterminate ($\mu \in$ (indet)). In other words, for a measure μ to be indeterminate means that there exists another measure ν, $\nu \neq \mu$, with the same moments:

$$\int_{\mathbb{R}} t^n \, d\mu(t) = \int_{\mathbb{R}} t^n \, d\nu(t), \qquad n = 0, 1, 2, \dots .$$

Otherwise, the measure μ is said to be determinate ($\mu \in$ (det)).

R. Nevanlinna described in [29] (see also [3, Sections 2.4, 3.2]) the set of all solutions to an indeterminate moment problem. He parametrized this set using the class $(\mathcal{N})$ of functions φ holomorphic in the upper half-plane $\mathbb{C}_+$ and such that

$$\operatorname{Im} \varphi(z) \geq 0 \text{ for } \operatorname{Im} z > 0.$$

This class includes real constants, and we add formally the constant ∞ function. As a consequence of the Riesz–Herglotz formula, every function f in this class possesses an integral representation (see, for instance, [3, Section 3.1])

$$f(z) = az + b + \int_{\mathbb{R}} \frac{1 + uz}{u - z} d\sigma(u) = az + b + \int_{\mathbb{R}} \left(\frac{1}{u - z} - \frac{u}{1 + u^2} \right) (1 + u^2) \, d\sigma(u) \,, \quad (0.1)$$

for $z \in \mathbb{C}_+$, where a and b are real numbers, $a \geq 0$ and σ is a positive Borel measure of finite mass. If f is extended to the lower half-plane $\mathbb{C}_-$ by $f(z) = \overline{f(\overline{z})}$, $z \in \mathbb{C}_-$, then formula (0.1) holds for $z \in \mathbb{C} \setminus \mathbb{R}$. (Generally speaking, this is not an analytic continuation.)

For a fixed indeterminate moment problem there exists an entire matrix-function

$$\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}, \qquad AD - BC \equiv 1, \qquad (0.2)$$

whose elements A, B, C, and D are real entire functions (entire functions with real coefficients) such that for every $t \in \mathbb{R} \cup \{\infty\}$,

$$-\frac{C(z)t + D(z)}{A(z)t + B(z)} \in (\mathcal{N}).$$

The Nevanlinna formula

$$v(z, \nu) = -\frac{C(z)\varphi(z) + D(z)}{A(z)\varphi(z) + B(z)}, \qquad \varphi \in (\mathcal{N}), \qquad (0.3)$$

gives a bijection between the class $(\mathcal{N})$ and the set of the Stieltjes transforms

$$v(z, \nu) = \int_{\mathbb{R}} \frac{d\nu(t)}{t - z}$$

of all the solutions to the indeterminate moment problem.

A solution μ to an indeterminate moment problem is called *canonical* if it corresponds to $\varphi(z) \equiv t$, $t \in \mathbb{R}\cup\{\infty\}$, in formula (0.3). We shall also use the term *a canonical measure*. Canonical measures correspond to self-adjoint extensions (without extension of space) of symmetric operators with indices (1,1) associated with Jacobi matrices, see details in [3, Chapter 4]. These measures enjoy important extremal properties (see, for example, [3, Theorem 3.4.1]). Every canonical measure is a discrete measure with masses on the zero set of the corresponding entire function $A(z)t + B(z)$, $t \in \mathbb{R} \cup \{\infty\}$. (To prove that all the zeros are real we use that B/A is not a constant and $B/A \in (\mathcal{N})$. This last inclusion can be verified in the following way. If ν_0 and ν_∞ are the measures associated by (0.3) correspondingly with $\varphi = 0$, $\varphi = \infty$, then the measure $(\nu_0 + \nu_\infty)/2$ solving the same indeterminate moment problem is associated with B/A.)

Fix a canonical measure μ. Since the matrix-functions

$$\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

and

$$\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

correspond to the same indeterminate moment problem, without loss of generality we can assume that the support of μ coincides with the zero set of B. We denote this zero set by Λ_B, $\Lambda_B \subset \mathbb{R}$.

By a theorem of M. Riesz [3, Theorem 2.4.3], the elements of the matrix-function (0.2) describing the solutions of an indeterminate moment problem are entire functions of zero exponential type. Furthermore, we have

$$\sum_{\lambda\in\Lambda_B} \frac{|\lambda|^n}{|B'(\lambda)|} \le \sum_{\lambda\in\Lambda_B} |\lambda|^{n+1} \sqrt{\frac{D(\lambda)}{B'(\lambda)}} \sqrt{\frac{1}{D(\lambda)B'(\lambda)(1+\lambda^2)}}$$
$$\le \left[\sum_{\lambda\in\Lambda_B} \lambda^{2n+2}\left(\frac{D(\lambda)}{B'(\lambda)}\right)\right]^{1/2} \left[\sum_{\lambda\in\Lambda_B}\left(\frac{1}{D(\lambda)B'(\lambda)}\right)\frac{1}{1+\lambda^2}\right]^{1/2} < \infty, \qquad n \ge 0.$$

Here the sum of the series in the first square brackets is just the moment of order $2n+2$ of the measure ν_0 whose Stieltjes transform is equal to $-D/B$. The sum in the second square brackets converges because $A(\lambda)D(\lambda) = 1$, $\lambda \in \Lambda_B$, and $-A/B \in (\mathcal{N})$. We need only to use an immediate consequence of formula (0.1) which says that since the function $-A/B$ in the class $(\mathcal{N})$ is meromorphic in the plane and has poles only on the real line,

$$\sum_{\lambda\in\Lambda_B} \frac{A(\lambda)}{B'(\lambda)(1+\lambda^2)} < \infty. \tag{0.4}$$

Definition. *The Hamburger class* $\mathfrak{H}$ *consists of all transcendental real entire functions* B *of zero exponential type with only real (and simple) zeros* $\lambda \in \Lambda_B$ *such that*

$$\lim_{\substack{|\lambda|\to\infty \\ \lambda\in\Lambda_B}} \frac{|\lambda|^n}{|B'(\lambda)|} = 0, \qquad n \geq 0.$$

Without loss of generality, we always assume that the origin does not belong to the zero set Λ_B. A Hamburger class function is uniquely determined (up to a multiplicative constant) by its zero set.

Thus, entire functions involved in the Nevanlinna formula (0.3) belong to the Hamburger class. Furthermore, if

$$\mu = \sum_{\lambda\in\Lambda_B} \mu_\lambda \delta_\lambda$$

is a canonical measure, where δ_λ is the unit point mass measure at the point λ,

$$\mu_\lambda = \frac{D(\lambda)}{B'(\lambda)} = \frac{1}{A(\lambda)B'(\lambda)}, \qquad \lambda \in \Lambda_B,$$

then the functions A and D can be reconstructed by the formulas

$$\frac{A(z)}{B(z)} = \alpha z + \beta + \sum_{\lambda\in\Lambda_B} \frac{A(\lambda)}{B'(\lambda)}\Big[\frac{1}{\lambda - z} - \frac{1}{\lambda}\Big] = \alpha z + \beta + \sum_{\lambda\in\Lambda_B} \frac{1}{\mu_\lambda [B'(\lambda)]^2}\Big[\frac{1}{\lambda - z} - \frac{1}{\lambda}\Big],$$

$$\frac{D(z)}{B(z)} = \gamma z + \delta + \sum_{\lambda\in\Lambda_B} \frac{D(\lambda)}{B'(\lambda)} \frac{1}{\lambda - z} = \gamma z + \delta + \sum_{\lambda\in\Lambda_B} \frac{\mu_\lambda}{\lambda - z},$$

where $\alpha, \gamma \geq 0$, $\beta, \delta \in \mathbb{R}$. Estimate (0.4) ensures here that

$$\sum_{\lambda\in\Lambda_B} \frac{1}{\mu_\lambda [B'(\lambda)]^2 (1+\lambda^2)} < \infty.$$

In 1944 H. Hamburger claimed the following statement to be valid.

Statement (Hamburger [18], [3, Addenda and Problems to Chapter 4])**.** *A positive measure* μ *is a canonical solution to an indeterminate moment problem if and only if for some function* $B \in \mathfrak{H}$ *we have*

$$\text{(i)} \quad \mu = \sum_{\lambda\in\Lambda_B} \mu_\lambda \delta_\lambda; \qquad \sum_{\lambda\in\Lambda_B} |\lambda|^n \mu_\lambda < \infty, \quad n \geq 0,$$

$$\text{(ii)} \quad \sum_{\lambda\in\Lambda_B} \frac{1}{\mu_\lambda [B'(\lambda)]^2 (1+\lambda^2)} < \infty,$$

$$\text{(iii)} \quad \sum_{\lambda\in\Lambda_B} \frac{1}{\mu_\lambda [B'(\lambda)]^2} = +\infty.$$

In particular, for the masses $\mu_\lambda = [B'(\lambda)]^{-2}$, $\lambda \in \Lambda_B$, conditions (i)–(iii) are fulfilled, and as a result, the zero set Λ_B of an arbitrary entire function in $\mathfrak{H}$ should be the support of a canonical measure.

In 1989 a gap in the proof of Hamburger's Statement was found by C. Berg and H. Pedersen. Soon P. Koosis [23] constructed a counterexample to Hamburger's Statement.

What was the source of Hamburger's mistake? We have already pointed out that if μ is a canonical measure, then conditions (i) and (ii) should hold. On the other hand, if μ is a measure satisfying conditions (i) and (ii), then $\mu \in$ (indet), see [3, Addenda and Problems to Chapter 4, Lemma 2]. Furthermore, a theorem by M. Riesz, [3, Sections 2.3, 2.4], asserts that the following conditions are equivalent:

(a) The set of polynomials $\mathcal{P}$ is dense in $L^2(\mu)$,

$$\mathrm{Clos}_{L^2(\mu)}\mathcal{P} = L^2(\mu). \tag{0.5}$$

(b) Either $\mu \in$ (det) or μ is a canonical measure.

Thus, a measure μ is canonical if and only if conditions (i) and (ii) are fulfilled together with (0.5). Hamburger believed that when conditions (i) and (ii) are fulfilled, condition (iii) is necessary and sufficient for completeness of polynomials in $L^2(\mu)$. It is indeed necessary. Consider a function c defined by $c(\lambda) = [\mu_\lambda B'(\lambda)]^{-1}$, $\lambda \in \Lambda_B$. If

$$\sum_{\lambda\in\Lambda_B} \frac{1}{\mu_\lambda [B'(\lambda)]^2} < \infty,$$

then the function c is an element of $L^2(\mu)$, and it is orthogonal to $\mathcal{P}$,

$$\sum_{\lambda\in\Lambda_B} \frac{P(\lambda)}{B'(\lambda)} = 0, \qquad P \in \mathcal{P},$$

by Lemma 3 in Appendix 3.

However, condition (iii) is not sufficient for completeness of polynomials. In [23] an entire function $B \in \mathfrak{H}$ is constructed such that for the measure $\mu = \sum_{\lambda\in\Lambda_B}[B'(\lambda)]^{-2}\delta_\lambda$,

$$\mathrm{Clos}_{L^2(\mu)}\mathcal{P} \neq L^2(\mu),$$

and hence, μ is not canonical.

The above described situation was the reason for writing this paper. Here we consider the following problem.

Problem. *Let $B \in \mathfrak{H}$, $1 \le p < \infty$, and let $\mu = \sum_{\lambda\in\Lambda_B} \mu_\lambda \delta_\lambda$ be a (positive) measure such that $\mathcal{P} \subset L^p(\mu)$. When*

$$\mathrm{Clos}_{L^p(\mu)}\mathcal{P} = L^p(\mu)?$$

In Appendix 1, we prove that in the so called "singular case" of the weighted polynomial approximation in $L^p(\mu)$, the measure μ must be supported by the zero set of a

Hamburger class entire function. This gives another motivation for studying the above formulated problem. As a counterpart to this abstract result we show in Appendix 2 how our methods can be applied to yield the complete solution of the weighted approximation problem in a very concrete model case. Our main results are presented in Section 1. Making use of an approach suggested by de Branges [9], in Section 2 we give a solution to the above described problem (Theorem A) and derive a correct version of Hamburger's Statement (Corollary 1.2). In Theorems B and C, proved correspondingly in Sections 3 and 4, we give concrete sufficient conditions on a set in $\mathbb{R}$ to be the support of a canonical measure. Another sufficient condition, Theorem D, is formulated in Section 1. An example given in Section 5 shows that the conditions of Theorems B and C cannot be essentially weakened. Some results on Hamburger class functions we use in our paper and a lemma on divisors of entire functions constructed by regular subsequences of zeros are contained, correspondingly, in Appendices 3 and 4.

A part of the results proved in this paper and an intrinsic relation to de Branges' theory of Hilbert spaces of entire functions and to results by Akhiezer and Gurarii [1, 16] are described in [8].

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1. Main results

Fix $B \in \mathfrak{H}$. Let us consider a function w on the zero set of B such that $w(\lambda) > 0$,

$$\lim_{\substack{|\lambda|\to\infty \\ \lambda\in\Lambda_B}} |\lambda|^n w(\lambda) = 0, \qquad n \geq 0. \tag{1.1}$$

We introduce the Banach spaces $\ell^p(w, \Lambda_B)$, $1 \leq p \leq \infty$, of functions a on Λ_B, with norm

$$\|a\|^p_{\ell^p(w,\Lambda_B)} = \sum_{\lambda\in\Lambda_B} [w(\lambda)]^p |a(\lambda)|^p,$$
$$\|a\|_{\ell^\infty(w,\Lambda_B)} = \sup_{\lambda\in\Lambda_B} w(\lambda)|a(\lambda)|.$$

Since $\ell^\infty(w, \Lambda_B)$ is not separable, we consider, as the natural limit case of the scale $\ell^p(w, \Lambda_B)$ for $p = +\infty$, the space $c_0(w, \Lambda_B)$ of functions a on Λ_B such that

$$\lim_{\substack{|\lambda|\to\infty \\ \lambda\in\Lambda_B}} w(\lambda)|a(\lambda)| = 0,$$

with norm

$$\|a\|_{c_0(w,\Lambda_B)} = \sup_{\lambda\in\Lambda_B} w(\lambda)|a(\lambda)|.$$

Thus we obtain the scale of the spaces $\ell^p_*(w,\Lambda_B)$,

$$\begin{aligned} \ell^p_*(w,\Lambda_B) &= \ell^p(w,\Lambda_B), \qquad 1\le p<\infty,\\ \ell^\infty_*(w,\Lambda_B) &= c_0(w,\Lambda_B). \end{aligned}$$

Their dual spaces are

$$\big[\ell^p_*(w,\Lambda_B)\big]^* = \ell^q\Big(\frac{1}{w},\Lambda_B\Big), \qquad \frac{1}{p}+\frac{1}{q}=1,\ 1\le p\le\infty,$$

with the usual pairing: if $a\in\ell^p_*(w,\Lambda_B)$, $b\in\ell^q\big(1/w,\Lambda_B\big)$, then

$$\langle a,b\rangle = \sum_{\lambda\in\Lambda_B} a(\lambda)\overline{b(\lambda)}.$$

Since B is of zero exponential type, condition (1.1) implies that

$$\mathcal{P}\subset\ell^p_*(w,\Lambda_B), \qquad 1\le p\le\infty.$$

Theorem A. *The polynomials are dense in $\ell^p_*(w,\Lambda_B)$ if and only if for every function $F\in\mathfrak{H}$ such that $\Lambda_F\subset\Lambda_B$, we have, for $1<p\le\infty$,*

$$\sum_{\lambda\in\Lambda_F}\left|\frac{1}{w(\lambda)F'(\lambda)}\right|^{p/(p-1)} = +\infty, \tag{1.2}$$

and for $p=1$,

$$\liminf_{\substack{|\lambda|\to\infty\\ \lambda\in\Lambda_F}} w(\lambda)F'(\lambda)=0. \tag{1.3}$$

The case $p=\infty$ (with agreement $p/(p-1)=1$) in Theorem A is a special case of a remarkable theorem by de Branges [9, 21, Chapter VI] which gives one of the solutions of the Bernstein weighted polynomial approximation problem. The proof of (the general case of) de Branges' theorem uses extensively geometric properties of the dual space to $c_0(w)$. Another proof found recently [32] uses ideas that go back to P. Chebyshev and A. Markov. None of these proofs seems to work for the spaces $\ell^p(w,\Lambda_B)$. However, in the special case under consideration, when the weight is defined on a discrete set that does not accumulate too fast at infinity, the polynomial approximation problem in the spaces $\ell^p(w,\Lambda_B)$ can be reduced to that in the space $c_0(w,\Lambda_B)$, see Appendix 1.

The quasianalyticity theorems given in Chapter I of J. P. Kahane's work [20] may be interpreted as statements on weighted polynomial approximation on zero sets of entire functions of Hamburger class when the weight is log-concave. In the opposite direction, de Branges' theorem and its extension given above provide results on the kind of quasianalyticity problems considered in [20]. In the recent paper [4], J. M. Anderson, D. Khavinson and H. Shapiro study the following problem: what rate of decrease of the coefficients c_n of Dirichlet series with negative exponents $-\lambda_n$

$$\sum_{\lambda\in\Lambda} c_n e^{-\lambda_n z}$$

(absolutely convergent in the closed right half-plane) guarantees that such a series cannot represent a function which vanishes with all its derivatives at the boundary point 0 of the half-plane? This problem might be reformulated as the weighted polynomial approximation problem on the discrete set of exponents Λ. Their main results pertain to the case when $\Lambda = \{n^a\}$, $n \in \mathbb{N}$, $a > 2$. In this case, Λ is the zero set of a Hamburger class entire function. In Appendix 2, we show that in this special "model" situation the results obtained with the help of de Branges' theorem are stronger than results of the papers [4, 20].

Theorem A immediately yields a correct version of the Hamburger statement:

Corollary 1.1. *To make Hamburger's Statement correct, condition* (iii) *should be replaced by the following condition:*

(iii$'$) *for every $F \in \mathfrak{H}$ such that $\Lambda_F \subset \Lambda_B$, we have*

$$\sum_{\lambda\in\Lambda_F} \frac{1}{\mu_\lambda [F'(\lambda)]^2} = +\infty. \tag{1.4}$$

Remark 1.2. In the letter of October 5, 1997, A. Bakan informed us of his recent results concerning the Hamburger moment problem. In particular, he formulated

(a) *an analog of our Corollary* 1.1 *where the condition $F \in \mathfrak{H}$ is replaced by the condition that F is just an entire function of zero exponential type.*

Furthermore, he claimed that Hamburger's Statement is wrong in the following strong sense:

(b) *for every $B \in \mathfrak{H}$, there exists μ satisfying the conditions of Hamburger's Statement and such that μ is not a canonical measure.*

Let us describe how to derive these results from our Corollary 1.1.

(a) If A is a divisor of $B \in \mathfrak{H}$, such that $A \notin \mathfrak{H}$, and for a sequence $\{\mu_\lambda\}_{\lambda\in\Lambda_B}$

$$\sum_{\lambda\in\Lambda_A} \frac{1}{\mu_\lambda [A'(\lambda)]^2} < \infty, \tag{1.5}$$

we choose a rare subsequence of zeros of A and construct a divisor $F \in \mathfrak{H}$ of A by this sequence such that for some c, n

$$|A'(\lambda)| < c|\lambda|^n, \qquad \lambda \in \Lambda_F. \tag{1.6}$$

This is possible due to Lemma A3.4. Then

$$\sum_{\lambda\in\Lambda_F} \frac{1}{\mu_\lambda [F'(\lambda)]^2} < \infty, \tag{1.7}$$

since $|A'(\lambda)| = o(|F'(\lambda)|)$, $|\lambda| \to \infty$, $\lambda \in \Lambda_F$. Applying Corollary 1.1, we conclude that the polynomials are not dense in $L^2(\mu)$. □

(b) Given $B \in \mathfrak{H}$, take an arbitrary divisor F of B such that $F \in \mathfrak{H}$ and B/F is transcendental. Our problem now is to find a sequence $\{c_\lambda\}_{\lambda\in\Lambda_B}$ such that

$$\sum_{\lambda\in\Lambda_B} \frac{c_\lambda |\lambda|^n}{[B'(\lambda)]^2} < \infty, \qquad n \geq 0, \tag{1.8}$$

$$\sum_{\lambda\in\Lambda_B} \frac{1}{c_\lambda (1+\lambda^2)} < \infty, \tag{1.9}$$

$$\sum_{\lambda\in\Lambda_B} \frac{1}{c_\lambda} = \infty, \tag{1.10}$$

$$\sum_{\lambda\in\Lambda_F} \frac{1}{c_\lambda} \frac{[B'(\lambda)]^2}{[F'(\lambda)]^2} < \infty. \tag{1.11}$$

Put

$$c(\lambda) = \begin{cases} \max\Big(1, \dfrac{\lambda^2[B'(\lambda)]^2}{[F'(\lambda)]^2}\Big), & \lambda \in \Lambda_F, \\ 1, & \lambda \in \Lambda_B \setminus \Lambda_F. \end{cases}$$

Then (1.8) is satisfied because $B \in \mathfrak{H}$, $F \in \mathfrak{H}$, and the fact that (1.9)–(1.11) hold follows immediately. Now, setting $\mu_\lambda = c_\lambda [B'(\lambda)]^{-2}$, we obtain a finite measure supported by the zero set of B which according to Corllary 1.1 is not a canonical measure. □

The following Corollary explains the importance of the special case of Theorem A with $w(\lambda) = |B'(\lambda)|^{-1}$, $\lambda \in \Lambda_B$, $B \in \mathfrak{H}$:

Corollary 1.3. *If ν is a canonical solution to an indeterminate moment problem, $B \in \mathfrak{H}$, $\operatorname{supp}\nu = \Lambda_B$, then the measure $\mu = \sum_{\lambda\in\Lambda_B} [B'(\lambda)]^{-2}\delta_\lambda$ is also a canonical solution of an indeterminate moment problem.*

Proof. If μ is not a canonical solution to an indeterminate moment problem, then by Corollary 1.1 for some divisor $F \in \mathfrak{H}$ of B we have

$$\sum_{\lambda\in\Lambda_F} \left|\frac{B'(\lambda)}{F'(\lambda)}\right|^2 < \infty.$$

Clearly, B/F is not a constant function. Pick a zero w of B/F and consider $F_0(z) = F(z)(z-w)$. If $\nu = \sum_{\lambda\in\Lambda_B} \nu_\lambda \delta_\lambda$ is a canonical solution, then by condition (ii) of Hamburger's Statement, for some C we have

$$\frac{1}{\nu_\lambda} \leq C[B'(\lambda)]^2(1+\lambda^2).$$

Now,

$$\sum_{\lambda\in\Lambda_F} \frac{1}{\nu_\lambda [F_0'(\lambda)]^2} \le C \sum_{\lambda\in\Lambda_F} \Big|\frac{B'(\lambda)}{F'(\lambda)}\Big|^2 \frac{1+\lambda^2}{|\lambda-w|^2} < \infty,$$

and again by Corollary 1.1 we obtain that ν cannot be a canonical solution of an indeterminate moment problem. □

Thus, Koosis' example [23] shows that there are $B \in \mathfrak{H}$ for which no canonical measure μ exists with $\operatorname{supp}\mu = \Lambda_B$. This implies, in particular, that *not every function in* $\mathfrak{H}$ *can be an element of the matrix-function in* (0.2) *parametrizing the set of solutions for an indeterminate moment problem.* Our discussion in Introduction shows that the description of canonical solutions to the Hamburger moment problem and the description of the first row of Nevanlinna matrices parametrizing all solutions are basically equivalent problems. It is worth to mention that Krein [24] and de Branges [10, Chapter 2] described (in different terms) the first row of *an arbitrary* Nevanlinna matrix, see also [31].

Corollary 1.4. *A Hamburger class entire function $B(z)$ can be included into a Nevanlinna matrix parametrizing the solutions of an indeterminate moment problem if and only if the polynomials are dense in the space $L^2(\mu)$ with $\mu = \sum_{\lambda\in\Lambda_B}[B'(\lambda)]^{-2}\delta_\lambda$.*

This is just a reformulation of Corollary 1.3.

In what follows, we restrict ourselves by the special case $w(\lambda) = |B'(\lambda)|^{-1}$ and try to give some reasonable sufficient conditions for the completeness of the polynomials. It seems to be quite hard to find *explicit* necessary and sufficient conditions.

Generally speaking, in order to apply Theorem A one needs to verify condition (1.1) (or (1.2)) for a rather large family of "Hamburger divisors" F. Nevertheless, we show below that this theorem can be efficiently applied (compare with recent applications [30] of the original de Branges' theorem).

Let us introduce some notations. A set $\Lambda \subset \mathbb{R}$ is said to be M-separated, if for some $C < \infty$,

$$|\lambda - \lambda'| \ge C(1+|\lambda|)^{-M(\lambda)}, \qquad \lambda, \lambda' \in \Lambda,\ \lambda \ne \lambda'.$$

Frequently we deal with the case when M is just a constant function.

In a recent paper [13], A. Fryntov considered the situation when $\Lambda_B \subset \mathbb{R}_+$ is an (R)-set in the sense of Levin [26, Chapter II, Section 1]: for the counting function $n(t)$ of the set Λ_B there exists the limit

$$\lim_{t\to\infty} \frac{n(t)}{t^{\rho(t)}} = \Delta, \qquad 0 < \Delta < \infty\,, \tag{1.12}$$

where $\rho(t)$ is a Valiron proximate order (i.e. $\lim_{t\to\infty} \rho'(t) t \log t = 0$), $0 < \lim_{t\to\infty}\rho(t) = \rho < 1/2$, and the set Λ_B is $(\rho(t)-1)$-separated. Then the function B is of completely regular growth in the sense of Levin–Pfluger, and there exists the limit

$$\lim_{\substack{\lambda\to\infty \\ \lambda\in\Lambda_B}} \frac{\log|B'(\lambda)|}{\lambda^{\rho(\lambda)}} = \pi\Delta\,\cot\pi\rho > 0\,.$$

Therefore, in this case $B \in \mathfrak{H}$.

Theorem (Fryntov [13]). *For the entire function B satisfying the above listed conditions,*

$$\mathrm{Clos}_{\ell^2(\frac{1}{|B'|},\Lambda_B)}\mathcal{P} = \ell^2\Big(\frac{1}{|B'|},\Lambda_B\Big).$$

A similar situation was considered by H. Hamburger in [18], where he produced a false statement. A correct formulation (without proof) is contained in [3, Addenda and Problems to Chapter 4, Subsection 5] where the credit is given to B. Levin. However, the late Professor Levin told the second-named author that in his proof he had used the Hamburger statement (see above). Fryntov's proof of this theorem is rather ingenious and involved. We show here that a little bit more general result follows easily from Corollary 1.1.

Theorem B. *Let $B \in \mathfrak{H}$, $\Lambda_B \subset \mathbb{R}_+$. Suppose that B is of normal type with proximate order $\rho(t) \to \rho < 1/2$, $t \to \infty$, and the indicator function h_B. If*

$$\lim_{\substack{\lambda\to\infty\\ \lambda\in\Lambda_B}} \frac{\log|B'(\lambda)|}{\lambda^{\rho(\lambda)}} = h_B(0), \tag{1.13}$$

then

$$\mathrm{Clos}_{\ell^p_*(\frac{1}{|B'|},\Lambda_B)}\mathcal{P} = \ell^p_*\Big(\frac{1}{|B'|},\Lambda_B\Big), \qquad 1 < p \le \infty\,, \tag{1.14}$$

Remark 1.5. Note that as a consequence of Theorem A, for every Hamburger class function $B(z)$,

$$\mathrm{Clos}_{\ell^1(1/|B'|,\Lambda_B)}\mathcal{P} \ne \ell^1\Big(\frac{1}{|B'|},\Lambda_B\Big),$$

since condition (1.3) is violated already for $F = B$.

Remark 1.6. Note that since the Phragmén–Lindelöf indicator of the derivative does not exceed the indicator of a function, condition (1.13) means that the derivative of B grows maximally rapidly along Λ_B. The condition of the maximal growth of the derivative on the set of zeros occurs in the entire function theory, namely in interpolation theory (see [26, Chapter IV, Section 4; 15] and references therein) and in the theory of Dirichlet series with complex exponents (see [25, Theorem 8.3]). Conditions of Theorem B yield that $B(z)$ is a function of completely regular growth in the Levin-Pfluger sense (see e.g. [11]; this follows easily from the proof of Theorem B given in Section 3) and therefore the asymptotic relation (1.12) holds. K. Malyutin proved (see [15, Theorem 9]) that, for entire functions of completely regular growth, condition (1.13) is equivalent to the following separation condition:

$$\lim_{\delta\to 0}\sup_{z\in\mathbb{C}} \frac{1}{|z|^{\rho(|z|)}} \int_0^{\delta|z|} \frac{[n_z(t)-1]_+}{t}\,dt = 0 \tag{1.15}$$

where $n_z(t)$ is a number of zeros lying in the closed disc of radius t with center z.

For Hamburger class functions of completely regular growth, condition (1.13) is not a *necessary* one for the density of the polynomials (1.14). For every γ, $0 \le \gamma < 1$, there exists a Hamburger class function $B(z)$, $\Lambda_B \subset \mathbb{R}_+$, of mean type and completely regular growth with respect to order $\rho < 1/2$ such that

$$\liminf_{\substack{|\lambda|\to\infty \\ \lambda\in\Lambda_B}} \frac{\log|B'(\lambda)|}{\lambda^\rho} = \gamma h_B(0)\,. \tag{1.16}$$

and the polynomials are still dense. On the other hand, a simple modification of Koosis' example [23] produces Hamburger class functions B with positive zeros and of completely regular growth satisfying (1.16) with arbitrary γ not exceeding 1/2 and such that the polynomials are not dense in all spaces $\ell^p_*\big(|B'(\lambda)|^{-1}, \Lambda_B\big)$, $1 \le p \le \infty$. A gap remains here: we do not know any example with these properties for $\gamma > 1/2$.

Another possible approach to the problem under consideration is to use the method by Fryntov [13] of constructing supporting polynomials for the Riesz–Hall–Mergelyan majorant. In this direction we obtain the following result.

Theorem C. *Let $B \in \mathfrak{H}$ be such that*

$$\sum_{\lambda\in\Lambda_B} \frac{1}{|\lambda|} < \infty. \tag{1.17}$$

Denote

$$B^{\#}(r) = \prod_{\lambda\in\Lambda_B} \Big(1 + \frac{r}{|\lambda|}\Big).$$

Suppose that, for some constant $M < \infty$,

$$\Lambda_B \quad \textit{is} \quad M\textit{-separated}, \tag{1.18}$$

and

$$\liminf_{\substack{|\lambda|\to\infty \\ \lambda\in\Lambda_B}} \frac{\log|B'(\lambda)|}{\log B^{\#}(|\lambda|)} > 0. \tag{1.19}$$

Then equalities (1.14) *are fulfilled.*

Comparatively to Theorem B, we impose here a much weaker condition on the growth of $|B'(\lambda)|$. In particular, we do not insist anymore on the asymptotic relation (1.12) which, as was explained above, follows from the assumptions of Theorem B. However, we must add an additional condition (1.18) which, locally, is much stronger than (1.15). An analysis of the example constructed by Koosis in [23] shows that condition (1.18) in Theorem C cannot be omitted. Furthermore, using Theorem A, we construct in Section 5 an entire function $B \in \mathfrak{H}$ of convergence class (1.17), with M-separated Λ_B, such that condition (1.19) holds with the upper limit instead of the lower one and nevertheless the polynomials are not dense in $\ell^p_*\big(1/|B'|, \Lambda_B\big)$.

Remark 1.7. If the zero set Λ_B is a subset of $\mathbb{R}_+$, then $B^{\#}(r) = B(-r) = M(r, B)$. Here, as usual,

$$M(r, B) = \max_{|z|=r} |B(z)| .$$

In the general case (we assume for simplicity that $B(0) = 1$), a standard estimate of the canonical product of genus zero gives

$$\log M(r, B) \le \log B^{\#}(r) \le r \int_r^{\infty} \frac{\log M(t, B)}{t^2}\, dt . \tag{1.20}$$

In general, we do not know whether $B^{\#}$ may be replaced by $M(\cdot, B)$ in the conditions of Theorem C. Of course, if B has only finite number of zeros on one of the semi-axes, or if, for some $\rho < 1$, the function $r \mapsto r^{-\rho} M(r, B)$ decreases for big r, then $B^{\#}$ is equivalent to $M(\cdot, B)$. The last statement is a consequence of (1.20).

Remark 1.8. Lemma A3.5 shows that, under conditions (1.17) and (1.18), the property (1.19) is equivalent to a lower bound for $|B(z)|$ which holds outside exceptional discs around the zero set: for some $\eta > 0$, $M < \infty$,

$$\log |B(z)| \ge \eta \log B^{\#}(|z|), \qquad z \notin \bigcup_{\lambda \in \Lambda} D\big(\lambda, |\lambda|^{-M}\big) .$$

In particular, conditions (1.17)–(1.19) imply that $B^{\#}(r) \le M^{1/\eta}(r, B)$.

By no means, we try here to squeeze from Fryntov's approach everything it can give. Our goal was rather to demonstrate that his approach works far beyond his original assumptions. This fact was not evident at all since, from the first look, his proof in [13] is rather rigid.

Here is another sufficient condition for the density given just in terms of Λ. Consider a function $V(r) = r^{\rho(r)}$, where $\rho(r)$ is a proximate order, $\rho(r) \to \rho$, $0 \le \rho \le 1/2$, $r^{\rho}(r) = o(r^{1/2})$ as $r \to +\infty$, and without loss of generality assume that $V(r)$ increases for $r \ge 0$. Denote by Φ the inverse function to V.

Theorem D. *Let $\Lambda = \{\lambda_n\} \subset \mathbb{R}_+$ and $\lambda_n/\Phi(n)$ increase. Then Λ is the zero set of a Hamburger class function B, and*

$$\operatorname{Clos}_{\ell^p_*(\frac{1}{|B'|}, \Lambda)} \mathcal{P} = \ell^p_*\Big(\frac{1}{|B'|}, \Lambda\Big), \qquad 1 < p \le \infty.$$

Conditions on Λ of such type appeared in similar problems in Kahane's thesis [20, Chapter I].

2. Proof of Theorem A

We start with a proposition which goes back to Koosis, he considered in [23] the case $p = 2$, $w(\lambda) = |B'(\lambda)|^{-1}$:

Proposition 2.1. *The polynomials are not dense in* $\ell^p_*(w,\Lambda_B)$, $1\le p\le\infty$, *if and only if there exists an entire function* $f\neq 0$ *of zero exponential type such that*

$$f\in \ell^q\Big(\frac{1}{w|B'|},\Lambda_B\Big),\qquad \frac{1}{p}+\frac{1}{q}=1, \tag{2.1}$$

and

$$\lim_{|y|\to\infty}\Big[|y|^n\Big|\frac{f(iy)}{B(iy)}\Big|\Big]=0,\qquad n\ge 0. \tag{2.2}$$

Proof. If such an entire function f does exist, put $c(\lambda)=f(\lambda)/B'(\lambda)$. Then $c\in \ell^q\big(1/w,\Lambda_B\big)$. Let us verify that the functional on $\ell^p_*(w,\Lambda_B)$ defined by c, is not identically 0, and vanishes on all polynomials. By (2.1) and the Hölder inequality,

$$\sum_{\lambda\in\Lambda_B}\Big|\frac{f(\lambda)}{B'(\lambda)}\Big||\lambda|^n=\sum_{\lambda\in\Lambda_B}\Big|\frac{f(\lambda)}{w(\lambda)B'(\lambda)}\Big||\lambda|^n w(\lambda)<\infty,\qquad n\ge 0.$$

This implies that

$$\frac{z^{n+1}f(z)}{B(z)}=\sum_{\lambda\in\Lambda_B}\lambda^n c(\lambda)\frac{\lambda}{z-\lambda},\qquad n\ge 0,\ z\notin\Lambda_B. \tag{2.3}$$

The reason is that the difference of the left-hand side and the right hand-side is an entire function of zero exponential type tending to zero along the imaginary axis (compare to Lemma A3.2). The Phragmén–Lindelöf principle yields that this difference is 0.

Equality (2.3) shows that $c\neq 0$. Furthermore, setting $z=0$ in (2.3), we obtain

$$\sum_{\lambda\in\Lambda_B}\lambda^n c(\lambda)=0,$$

and the polynomials are not dense.

Arguing in the opposite direction, assume that there exists a non-zero functional $c\in\ell^q\big(1/w,\Lambda_B\big)$ which vanishes on the polynomials. We define an entire function f by the Lagrange interpolation series

$$\frac{f(z)}{B(z)}=\sum_{\lambda\in\Lambda_B}\frac{c(\lambda)}{z-\lambda}.$$

The series in the right hand-side converges absolutely:

$$\sum_{\lambda\in\Lambda_B}\Big|\frac{c(\lambda)}{z-\lambda}\Big|=\sum_{\lambda\in\Lambda_B}\frac{|c(\lambda)|}{w(\lambda)}\frac{w(\lambda)}{|z-\lambda|}<\infty,\qquad z\notin\Lambda_B.$$

Therefore, we easily obtain that f is of zero exponential type and satisfies conditions (2.1) and (2.2). □

Applying this Proposition with $f(z)=B(z)/F(z)$, we obtain

Corollary 2.2. *Let $1 \le p \le \infty$, $1/p + 1/q = 1$. If $F \in \mathfrak{H}$ is a divisor of B such that*

$$\mathbf{1} \in \ell^q\Big(\frac{1}{w|F'|}, \Lambda_F\Big),$$

then

$$\mathrm{Clos}\,_{\ell^p_*(w,\Lambda_B)}\mathcal{P} \ne \ell^p_*(w, \Lambda_B).$$

This gives necessity in Theorem A.

For the case $p = \infty$, this corollary as well as the much more delicate converse theorem is proved by de Branges. Here we formulate a special case of his theorem:

De Branges' Theorem (see [9, 21, Section VI F, 32])**.** *If*

$$\mathrm{Clos}\,_{c_0(w,\Lambda_B)}\mathcal{P} \ne c_0(w, \Lambda_B),$$

then for some $F \in \mathfrak{H}$ which is a divisor of B,

$$\mathbf{1} \in \ell^1\Big(\frac{1}{w|F'|}, \Lambda_F\Big).$$

Now, our aim is to extend this converse theorem to all p.

Definition. *An exponent p, $1 \le p \le \infty$, is normal (for the pair B, w) if*

$$\mathbf{1} \notin \ell^q\Big(\frac{1}{w|B'|}, \Lambda_B\Big), \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

If p is a normal exponent, then all r, $p < r \le \infty$, are normal exponents as well. For normal exponents, the entire function f in Proposition 2.1 is automatically transcendental (since there is an infinite subsequence $\{\lambda_n\} \subset \Lambda_B$ such that $f(\lambda_n) \to 0$ as $n \to \infty$), and as a consequence has infinitely many zeros. Dividing it by an arbitrary polynomial divisor, we get another function satisfying the conditions of Proposition 2.1.

Corollary 2.3. *Let p be a normal exponent. The polynomials are not dense in the space $\ell^p_*(w, \Lambda_B)$ if and only if for every $n < \infty$ there exists an entire function $f \ne 0$ of zero exponential type satisfying condition (2.2) and such that*

$$|f(\lambda)| \le \frac{w(\lambda)|B'(\lambda)|}{1 + |\lambda|^n}, \qquad \lambda \in \Lambda_B.$$

In particular, we obtain

Corollary 2.4. *The equality*

$$\mathrm{Clos}\,_{\ell^p_*(w,\Lambda_B)}\mathcal{P} = \ell^p_*(w, \Lambda_B)$$

holds simultaneously for all normal exponents p.

Combining this result with Proposition 2.1 and de Branges' theorem, we obtain sufficiency in Theorem A for normal exponents p. It remains to note that for exponents p that are not normal, the polynomials are not dense as a consequence of Corollary 2.2 (with $F = B$). This completes the proof of Theorem A. □

We finish this Section with a proposition needed in Section 4. For the sake of simplicity assume now that $\Lambda_B \cap (-1,1) = \emptyset$. Consider the Riesz–Hall–Mergelyan majorant

$$M_N(z) = \sup\bigl\{|P(z)| : P \in \mathcal{P},\ |P(\lambda)|w(\lambda) \le |\lambda|^N\bigr\}, \qquad N \ge 1.$$

For normal exponents we can improve the Mergelyan theorem [17, 28] a little bit.

Proposition 2.5. (compare to [13]) *For the polynomials to be complete in $\ell^p_*(w, \Lambda_B)$ with normal p it is sufficient that for some $N < \infty$, $z \in \mathbb{C} \setminus \Lambda_B$,*

$$M_N(z) = +\infty, \tag{2.4}$$

and it is necessary that (2.4) *holds for all N, $z \in \mathbb{C} \setminus \Lambda_B$.*

Proof. To prove sufficiency we write the Lagrange interpolation formula for Pf, where P is a polynomial and f satisfies the conditions of Corollary 2.3 with $n = N + 1$,

$$\frac{P(z)f(z)}{B(z)} = \sum_{\lambda\in\Lambda_B} \frac{P(\lambda)f(\lambda)}{B'(\lambda)(z-\lambda)}.$$

This equality is verified like formula (2.3). Furthermore, it implies that M_N is finite outside Λ_B:

$$\left|\frac{P(z)f(z)}{B(z)}\right| = \left|\sum_{\lambda\in\Lambda_B} \frac{P(\lambda)w(\lambda)}{\lambda^N}\,\frac{f(\lambda)\lambda^{N+1}}{B'(\lambda)w(\lambda)}\,\frac{1}{\lambda(z-\lambda)}\right| \le M_N(z) \sum_{\lambda\in\Lambda_B} \frac{1}{|\lambda(z-\lambda)|}\,.$$

The necessity follows from the usual L^p-version of the theorem (see [6, 27, 28]). □

3. Proof of Theorem B

If the polynomials are not dense for some p, then by Theorem A,

$$\sum_{\lambda\in\Lambda_F} \left|\frac{B'(\lambda)}{F'(\lambda)}\right|^{p/(p-1)} < +\infty\,, \tag{3.1}$$

for some Hamburger divisor F of B. Then, for some positive constant c,

$$|F'(\lambda)| \ge c|B'(\lambda)|, \tag{3.2}$$

and for every $\varepsilon > 0$ and sufficiently large λ,

$$|F'(\lambda)| \ge \exp\left[(h_B(0) - \varepsilon)\lambda^{\rho(\lambda)}\right], \qquad \lambda \in \Lambda_F.$$

Since F and B are canonical products of genus zero with positive zeros, on the circle $|z| = r$ they achieve their maximal and minimal values on the negative and positive rays correspondingly. In particular, F also has a mean type with respect to the proximate order $\rho(r)$: for some positive constant C_1,

$$\log|F(z)| \le \log|F(-|z|)| \le \log|B(-|z|)| \le C_1 |z|^{\rho(|z|)}.$$

As a result, we obtain a lower bound for F on a sequence of circles: there exist sequences $r_n \to +\infty$, $\varepsilon_n \to 0$, $n \to \infty$, such that

$$|F(r_n e^{i\theta})| \ge |F(r_n)| \ge \exp\left[(h_B(0) - \varepsilon_n) r_n^{\rho(r_n)}\right]. \tag{3.3}$$

Indeed, if

$$\log|F(\lambda + t)| \le (h_B(0) - \varepsilon)\lambda^{\rho(\lambda)}, \qquad t \in [-1, 1], \quad \lambda \in \Lambda_F,$$

then the theorem on two constants applied in the domain

$$\left\{z \in \mathbb{C} : |z - \lambda| < 1\right\} \setminus \left([\lambda - 1, \lambda - 1/\lambda] \cup [\lambda + 1/\lambda, \lambda + 1]\right)$$

to the subharmonic function $z \mapsto \log|F(z)| - \log|z - \lambda|$, implies that, for big $\lambda \in \Lambda_F$,

$$\log|F'(\lambda)| \le (h_B(0) - \varepsilon/2)\lambda^{\rho(\lambda)},$$

that is impossible. Thus there are sequences r_n and ε_n such that (3.3) holds.

The lower bound (3.3) extends to the whole complex plane outside the union of small exceptional discs around Λ_F. Let H be an arbitrary entire function of completely regular growth in the sense of Levin–Pfluger (see [26, Chapter III]) (with respect to the same proximate order $\rho(r)$) with the constant Phragmén–Lindelöf indicator function $h_H(\theta) \equiv h_0 < h_B(0)$. Then the integrals

$$\frac{1}{2\pi i} \int_{|\zeta| = r_n} \frac{H(\zeta)}{F(\zeta)} \frac{d\zeta}{z - \zeta}$$

tend to 0 as $n \to \infty$ for every fixed z, and the residue theorem yields the Lagrange interpolation formula

$$\frac{H(z)}{F(z)} = \sum_{\lambda \in \Lambda_F} \frac{H(\lambda)}{F'(\lambda)(z - \lambda)}, \qquad z \in \mathbb{C} \setminus \Lambda_F.$$

Our estimates on $F'\big|\Lambda_F$ and H imply that the series in the right-hand side of this equality converges absolutely and for some constant c,

$$|F(z)| \ge c|H(z)|, \qquad \operatorname{dist}(z, \Lambda_F) \ge 1.$$

The entire function $G = B/F$ satisfies the estimate

$$c|H(z)G(z)| \le |B(z)|$$

on the set $\{z \in \mathbb{C} : \operatorname{dist}(z, \Lambda_F) \ge 1\}$. Since H has completely regular growth, $h_{HG} = h_H + h_G$, and since every ray $\{\arg z = \alpha > 0\}$ eventually does not intersect exceptional discs, $h_{HG}(\alpha) \le h_B(\alpha)$. We conclude that

$$h_G(\alpha) = h_{HG}(\alpha) - h_H(\alpha) \le h_B(\alpha) - h_0\,,$$

and, by the continuity of the indicator functions and the choice of h_0, we obtain $h_G(0) = 0$. Since G is of order less than $1/2$, we conclude that G is of minimal type with respect to the proximate order $\rho(r)$.

Let us recall that, by (3.2), G is bounded on Λ_B. Therefore, we can use an argument due to Ganapathy Iyer (see [14] or [22]). The Lagrange interpolation formula applies to G^n and B for every integer $n \ge 1$:

$$\frac{G^n(z)}{B(z)} = \sum_{\lambda \in \Lambda_B} \frac{G^n(\lambda)}{B'(\lambda)(z-\lambda)}\,, \qquad z \in \mathbb{C} \setminus \Lambda_B.$$

We obtain that G is bounded on the whole complex plane: if $|G(\lambda)| \le M$, $\lambda \in \Lambda_B$, and if

$$X(z) = \sum_{\lambda \in \Lambda_B} \frac{1}{|B'(\lambda)(z-\lambda)|}\,, \qquad z \in \mathbb{C} \setminus \Lambda_B\,,$$

then

$$|G(z)| \le M\,|B(z)X(z)|^{1/n} \le M\,.$$

As a consequence, G is a constant function that contradicts (3.1). $\square$

4. Proof of Theorem C

The proof of Theorem C uses the following factorization lemma. A similar lemma was also used by Fryntov [13, Lemma 2]. Since the proof is rather standard, we give it in Appendix 4.

Lemma 4.1. *Let $\Gamma = \{\gamma_j\}$ be a real M-separated sequence (possibly, finite) such that $\Gamma \cap (-2, 2) = \emptyset$ and*

$$\sum_{\gamma \in \Gamma} \frac{1}{\gamma} = L < \infty. \tag{4.1}$$

Put

$$f(z) = \prod_{\gamma \in \Gamma} \Big(1 - \frac{z}{\gamma}\Big).$$

Fix $m \ge 2$, take an arbitrary sequence of entire numbers $\{r_s\}$, $0 \le r_s \le m-1$, and consider

$$f_m(z) = \prod_s \Big(1 - \frac{z}{\gamma_{sm+r_s}}\Big).$$

There exist numbers $C = C(L) > 0$, $K = K(M) > 0$, which do not depend on Γ, m and $\{r_s\}$, such that

$$\frac{1}{C(1+|z|^K)} \le \frac{|f_m(z)|}{|f(z)|^{1/m}} \le C(1+|z|^K), \qquad z \notin \bigcup_{\gamma\in\Gamma} D\big(\gamma, |\gamma|^{-M-1}\big), \tag{4.2}$$

$$\frac{1}{C|\gamma|^K} \le \frac{|f_m(\gamma)|}{|f'(\gamma)|^{1/m}} \le C|\gamma|^K, \qquad \gamma \in \Gamma \setminus \big\{\gamma_{sm+r_s}\big\}, \tag{4.3}$$

and

$$\frac{1}{C|\gamma|^K} \le \frac{|f'_m(\gamma)|}{|f'(\gamma)|^{1/m}} \le C|\gamma|^K, \qquad \gamma \in \big\{\gamma_{sm+r_s}\big\}. \tag{4.4}$$

Proof of Theorem C. Since exponents $p > 1$ are normal for the weight $w = 1/|B'|$, to prove Theorem 4.1 it suffices to verify that the Riesz–Hall–Mergelyan majorant $M_N(0)$ is infinite for some N depending on M and η.

We work with the zero sequence $\Lambda = \Lambda_B$. Without loss of generality, we assume that $\Lambda_B \cap (-2, 2) = \emptyset$. There are two steps in the proof: "Thickening the zero sequence" and "Rarefying the zero sequence". As a result, for every sufficiently big m, we obtain a finite set $\Lambda_m \subset \Lambda$ such that

$$\lim_{m\to\infty} \inf \{|\lambda| : \lambda \in \Lambda \setminus \Lambda_m\} = \infty, \tag{4.5}$$

and "supporting polynomials"

$$P_m(z) = \prod_{\lambda\in\Lambda_m} \Big(1 - \frac{z}{\lambda}\Big),$$

satisfying the property

$$|P_m(\lambda)| \le |\lambda|^N |B'(\lambda)|, \qquad \lambda \in \Lambda \setminus \Lambda_m, \tag{4.6}$$

with $N = N(\eta, M)$ which does not depend on m and λ.

Furthermore, property (4.5) and estimate (4.6) imply that there exists a sequence c_m of positive numbers, $\lim_{m\to\infty} c_m = \infty$, such that

$$c_m |P_m(\lambda)| \le |\lambda|^{N+1} |B'(\lambda)|, \qquad \lambda \in \Lambda.$$

Since $P_m(0) = 1$, the definition of the polynomial majorant M_N (see Section 2) gives that

$$M_{N+1}(0) \ge \sup_m c_m = +\infty,$$

and finally, by Proposition 2.5,

$$\mathrm{Clos}_{\ell^p_*(\frac{1}{|B'|},\Lambda_B)}\mathcal{P} = \ell^p_*\Big(\frac{1}{|B'|},\Lambda_B\Big), \qquad 1 < p \le \infty\,.$$

Step 1 ("Thickening the zero sequence"). We choose inductively two sequences of numbers $\{N_k\}$ and $\{N'_k\}$,

$$1 = N_0 < N'_0 < N_1 < N'_1 < \ldots < N_k < N'_k < \ldots\,,$$

such that the following set of properties holds for $k \ge 1$.

(a) $N'_k \ge 2N_k$, $N_{k+1} \ge 2N'_k$;
(b) For every $\lambda \in \Lambda$, $k \ge 0$,

$$\mathrm{dist}\,(N_k, |\lambda|) \ge \frac{1}{|\lambda|^M},$$
$$\mathrm{dist}\,(N'_k, |\lambda|) \ge \frac{1}{|\lambda|^M};$$

(c) The function $\tilde{n}$, defined by $d\tilde{n}(t) = kdn(t)$, $N_{k-1} \le t \le N_k$, $k \ge 1$, $\tilde{n}(t) = 0$, $t \le 1$, belongs to the convergence class

$$\int_0^\infty \frac{\tilde{n}(t)}{t^2}dt < \infty;$$

(d) $N'_k \displaystyle\int_{N_{k+1}}^\infty \frac{\tilde{n}(t)}{t^2}dt \le \frac{1}{2}$;
(e) $2kn(N_k)\log r \le \eta \log B^{\#}(r)$, for $r \ge N'_k$, where η is the constant from the condition of Theorem, and $K = K(M+1)$ is the constant from the conclusion of Lemma 4.1.

Let us introduce a "thickened" sequence $\widehat{\Lambda} \subset \Lambda$. On both intervals $[-N_{k+1}, -N_k]$, $[N_k, N_{k+1}]$ we add by k new points between every two consecutive points λ and λ' of Λ. The same thing is done for consecutive λ and λ' such that $N_k < \lambda < N_{k+1} < \lambda'$ or $\lambda' < -N_{k+1} < \lambda < -N_k$. Roughly speaking, the density of $\widehat{\Lambda}$ is $k+1$ times bigger than that of the original set Λ on $[-N_{k+1}, -N_k] \cup [N_k, N_{k+1}]$. Moreover, we add the new points in such a way that

(f) they lie not far away from the old ones, namely for every $\mu \in \widehat{\Lambda} \setminus \Lambda$ there exists $\lambda \in \Lambda$ such that

$$|\mu| \ge |\lambda| \ \text{ and } \ |\mu - \lambda| \le \frac{1}{4|\lambda|^M};$$

(g) the set $\widehat{\Lambda}$ is $(M+1)$-separated.

Since $N_k \geq 4^k$ by (a), conditions (f) and (g) are compatible (without loss of generality, $M \geq 1$). Furthermore, we can choose new points in such a way that

(b′) For every $\lambda \in \widehat{\Lambda}$, $k \geq 1$,

$$\operatorname{dist}(N_k, |\lambda|) \geq \frac{1}{|\lambda|^{M+1}},$$
$$\operatorname{dist}(N'_k, |\lambda|) \geq \frac{1}{|\lambda|^{M+1}}.$$

As a consequence of property (c),

$$\sum_{\lambda \in \widehat{\Lambda}} \frac{1}{|\lambda|} < \infty.$$

If $\hat{n} = n_{\widehat{\Lambda}}$, then property (d) implies

(d′) $N'_k \displaystyle\int_{N_{k+1}}^{\infty} \frac{\hat{n}(t)}{t^2} dt \leq 1.$

Using the "thickened" sequence $\widehat{\Lambda}$ we define an entire function $\widehat{B}$ of zero exponential type,

$$\widehat{B}(z) = \prod_{\lambda \in \widehat{\Lambda}} \Big(1 - \frac{z}{\lambda}\Big).$$

The main result we obtain on this step is the estimate

$$|\widehat{B}'(\lambda)| \geq c\big[B^{\#}(|\lambda|)\big]^{\eta}, \qquad \lambda \in \widehat{\Lambda}, \tag{4.7}$$

for some $c > 0$, which implies, in particular, that $\widehat{B} \in \mathfrak{H}$.

First, however, we need to state two auxiliary estimates.

Lemma 4.2.

$$W = \exp\Big(\sum_{\lambda \in \Lambda} \frac{1}{|\lambda|}\Big) \geq \frac{B^{\#}(r+1)}{B^{\#}(r)}.$$

Proof.

$$\frac{B^{\#}(r+1)}{B^{\#}(r)} = \prod_{\lambda \in \Lambda} \frac{1 + \frac{r+1}{|\lambda|}}{1 + \frac{r}{|\lambda|}} = \prod_{\lambda \in \Lambda} \Big(1 + \frac{1}{|\lambda| + r}\Big) \leq \prod_{\lambda \in \Lambda} \Big(1 + \frac{1}{|\lambda|}\Big) \leq \exp\Big(\sum_{\lambda \in \Lambda} \frac{1}{|\lambda|}\Big). \quad \square$$

Lemma 4.3. *If B satisfies the conditions of Theorem* C, $\lambda \in \Lambda$, *and z is such that*

$$|z-\lambda| \le \frac{1}{4|\lambda|^M},$$

then, for sufficiently big $|\lambda|$, we have

$$|B'(\lambda)|^{1/3} \le \left|\frac{B(z)}{z-\lambda}\right| \le |B'(\lambda)|^3. \tag{4.8}$$

Proof. Consider the function h,

$$h(z) = \log\left|\frac{B(z)}{z-\lambda}\right|,$$

harmonic in the disc $D = D(\lambda, 1/(2|\lambda|^M))$. Since $B \in \mathfrak{H}$, Lemma A3.2 implies that

$$\frac{|z|^n}{|B(z)|} \le \sum_{\lambda\in\Lambda} \frac{|\lambda|^n}{|z-\lambda||B'(\lambda)|}, \qquad n \ge 0\,.$$

Therefore,

$$|B(z)| \ge |z|^n \min_{\lambda\in\Lambda} |z-\lambda| \cdot \Big[\sum_{\lambda\in\Lambda} \frac{|\lambda|^n}{|B'(\lambda)|}\Big]^{-1},$$

and, as a result, for $z \in \partial D$ and some positive c,

$$\left|\frac{B(z)}{z-\lambda}\right| \ge c|z|\,.$$

Hence, for sufficiently big $|\lambda|$, the function h is positive and harmonic in D. Finally, in the twice smaller disc $D(\lambda, 1/(4|\lambda|^M))$ we have

$$\frac{1}{3}\log|B'(\lambda)| = \frac{1}{3}h(\lambda) \le h(z) \le 3h(\lambda) = 3\log|B'(\lambda)|$$

which is equivalent to (4.8). $\square$

Now, fix $\mu \in \widehat{\Lambda}$, $N'_{k-1} < |\mu| < N'_k$. For sufficiently big k, we are going to get a lower estimate for

$$|\widehat{B}'(\mu)| = \frac{1}{|\mu|}\prod_{\substack{\lambda\in\widehat{\Lambda}\\ \lambda\ne\mu}}\left|1-\frac{\mu}{\lambda}\right| = \frac{1}{|\mu|}\cdot\prod_{\substack{\lambda\in\widehat{\Lambda}\\ \lambda<N_{k-1}}}\left|1-\frac{\mu}{\lambda}\right|\cdot\prod_{\substack{\lambda\in\widehat{\Lambda},\,\lambda\ne\mu\\ N_{k-1}<|\lambda|<N_{k+1}}}\left|1-\frac{\mu}{\lambda}\right|\cdot\prod_{\substack{\lambda\in\widehat{\Lambda}\\ \lambda>N_{k+1}}}\left|1-\frac{\mu}{\lambda}\right|. \tag{4.9}$$

The first and the third products can be dealt with easily:

$$\prod_{\substack{\lambda\in\widehat{\Lambda}\\ \lambda<N_{k-1}}} \Bigl|1-\frac{\mu}{\lambda}\Bigr| \ge 1 \quad \text{since} \quad N'_{k-1}\ge 2N_{k-1}, \tag{4.10}$$

$$\prod_{\substack{\lambda\in\widehat{\Lambda}\\ \lambda>N_{k+1}}} \Bigl|1-\frac{\mu}{\lambda}\Bigr| \ge \frac{1}{e^2} \quad \text{because of property (d}'\text{)}. \tag{4.11}$$

The difficult part here is to estimate the middle product. There are two cases: $N'_{k-1}<|\mu|<N_k$ and $N_k<|\mu|<N'_{k+1}$. We consider only the first one; the second one does not require essential modifications. Without loss of generality assume that $\mu>0$. Put

$$\Lambda_k=\bigl\{\lambda\in\Lambda : N_k<|\lambda|<N_{k+1}\bigr\}, \qquad k\ge 0,$$
$$\widehat{\Lambda}_k=\bigl\{\lambda\in\widehat{\Lambda} : N_k<|\lambda|<N_{k+1}\bigr\}, \qquad k\ge 0,$$

and

$$\Pi_k(z)=\prod_{\lambda\in\Lambda_k}\Bigl(1-\frac{z}{\lambda}\Bigr),$$
$$Q_k(z)=\prod_{\lambda\in\widehat{\Lambda}_{k-1}\cup(\widehat{\Lambda}_k\setminus\Lambda_k)}\Bigl(1-\frac{z}{\lambda}\Bigr).$$

Then

$$\frac{1}{|\mu|}\cdot\prod_{\substack{\lambda\in\widehat{\Lambda},\,\lambda\ne\mu\\ N_{k-1}<|\lambda|<N_{k+1}}}\Bigl|1-\frac{\mu}{\lambda}\Bigr| = |Q'_k(\mu)|\cdot|\Pi_k(\mu)|. \tag{4.12}$$

It remains to estimate two terms in the right-hand side. Denote by λ_μ the point of Λ closest to μ (possibly, $\lambda_\mu=\mu$).

First, we verify that

$$|\Pi_k(z)|=B(z)\Bigl[\prod_{\substack{\lambda\in\Lambda\\ |\lambda|<N_k}}\bigl|1-\frac{z}{\lambda}\bigr|\cdot\prod_{\substack{\lambda\in\Lambda\\ |\lambda|>N_{k+1}}}\bigl|1-\frac{z}{\lambda}\bigr|\Bigr]^{-1}\ge\frac{1}{eW}|\lambda_\mu|^{-M-1}\frac{|B'(\lambda_\mu)|^{1/3}}{B^{\#}(|\lambda_\mu|)}, \tag{4.13}$$

for $z=\mu$. Indeed, if $\mu\in\widehat{\Lambda}\setminus\Lambda$, then Lemma 4.3 and properties (f) and (g) imply that

$$|B(\mu)|\ge|\lambda_\mu|^{-M-1}|B'(\lambda_\mu)|^{1/3}.$$

Furthermore, as a consequence of property (d),

$$\prod_{\substack{\lambda\in\Lambda\\ |\lambda|>N_{k+1}}}\Bigl|1-\frac{\mu}{\lambda}\Bigr|\le e, \tag{4.14}$$

and by Lemma 4.2 ($|\mu-\lambda_\mu|\le 1$),

$$\prod_{\substack{\lambda\in\Lambda\\|\lambda|<N_k}} \left|1-\frac{\mu}{\lambda}\right| \le \prod_{\substack{\lambda\in\Lambda\\|\lambda|<N_k}} \left(1+\left|\frac{\mu}{\lambda}\right|\right) \le W B^{\#}(|\lambda_\mu|).$$

If $\mu\in\Lambda$, then

$$\left|\frac{B(z)}{1-\frac{z}{\mu}}\right|\Big|_{z=\mu} = |\lambda_\mu||B'(\lambda_\mu)|,$$
$$\prod_{\substack{\lambda\in\Lambda\\|\lambda|<N_k}} \left(1+\left|\frac{\mu}{\lambda}\right|\right) \le B^{\#}(|\lambda_\mu|),$$

and we get even better estimates. This proves (4.13).

To estimate $|Q_k'(\mu)|$ we use Lemma 4.1. The set of zeros of Q_k, $\widehat{\Lambda}_{k-1}\cup(\widehat{\Lambda}_k\setminus\Lambda_k)$, is a k times "thickened" set $\Lambda_{k-1}\cup\Lambda_k$. Therefore, if $\mu\in\Lambda_{k-1}$, then

$$|Q_k'(\mu)| \ge \left[\frac{|\Pi_{k-1}'(\mu)||\Pi_k(\mu)|}{C|\mu|^K}\right]^k = \left[\frac{|B'(\mu)|}{C|\mu|^K}\cdot\Big[\prod_{\substack{\lambda\in\Lambda\\|\lambda|<N_{k-1}}}\left|1-\frac{\mu}{\lambda}\right|\cdot\prod_{\substack{\lambda\in\Lambda\\|\lambda|>N_{k+1}}}\left|1-\frac{\mu}{\lambda}\right|\Big]^{-1}\right]^k.$$

By property (e) we have

$$\prod_{\substack{\lambda\in\Lambda\\|\lambda|<N_{k-1}}} \left|1-\frac{\mu}{\lambda}\right| \le \left(1+\frac{\mu}{N_{k-1}}\right)^{n(N_{k-1})} \le \mu^{n(N_{k-1})} \le \left[B^{\#}(\mu)\right]^{\eta/k},$$

and using estimate (4.14) we obtain, for sufficiently big k, that

$$|Q_k'(\mu)| \ge \left[\frac{|B'(\mu)|}{eC|\mu|^K\left[B^{\#}(|\mu|)\right]^{\eta/k}}\right]^k$$
$$= \left[\frac{|B'(\mu)|^{1/k}}{eC|\mu|^K}\right]^k\cdot\frac{|B'(\mu)|}{\left[B^{\#}(|\mu|)\right]^{\eta}}\cdot|B'(\mu)|^{k-2} \ge |B'(\mu)|^{k-2}. \quad (4.15)$$

The last inequality is a consequence of condition (4.1) and the inequality

$$\frac{|B'(\mu)|^{1/k}}{eC|\mu|^K} \ge \exp\Big[\frac{\eta}{k}\log B^{\#}(|\mu|) - \log(eC) - K\log|\mu|\Big] \ge 1, \qquad |\mu|\ge N_k',$$

which follows from property (e).

If, on the opposite, $\mu \in \widehat{\Lambda}_{k-1} \setminus \Lambda_{k-1}$, then, again by Lemma 4.1,

$$|Q_k'(\mu)| \geq \left[\frac{|\Pi_{k-1}(\mu)||\Pi_k(\mu)|}{C|\mu|^K}\right]^k = \left[\frac{|B(\mu)|}{C|\mu|^K} \cdot \Big[\prod_{\substack{\lambda\in\Lambda \\ |\lambda|\leq N_{k-1}}} \Big|1-\frac{\mu}{\lambda}\Big| \cdot \prod_{\substack{\lambda\in\Lambda \\ |\lambda|\geq N_{k+1}}} \Big|1-\frac{\mu}{\lambda}\Big|\Big]^{-1}\right]^k,$$

and by Lemma 4.3 and property (e) we get like before that

$$|Q_k'(\mu)| \geq \left[\frac{|B'(\lambda_\mu)|^{1/3}}{eC|\lambda_\mu|^{K+M+1}} \frac{1}{[B\#(|\lambda_\mu|)]^{\eta/k}}\right]^k \geq |B'(\lambda_\mu)|^{k/4} \tag{4.16}$$

for sufficiently big k.

Gathering together (4.9)–(4.13), (4.15) and (4.16), we obtain that

$$|\widehat{B}'(\mu)| \geq \frac{|B'(\lambda_\mu)|^{k/4}}{B\#(|\lambda_\mu|)} \geq |B'(\lambda_\mu)|^{k/5} \geq |B^{\#}(\lambda_\mu)|^{\eta k/5}$$

for sufficiently big k. Lemma 4.1 implies now estimate (4.7).

Remark. Actually, we have proven that the function $\widehat{B}$ satisfies all the conditions of Theorem (with $M = M+1$ and different η).

Step 2 ("Rarefying the sequence"). Here we follow [13].

Let us enumerate the elements of $\widehat{\Lambda} = \{\lambda_j\}_{j\in\mathbb{Z}}$ in such a way that $|\lambda_0| \leq |\lambda|$, $\lambda \in \widehat{\Lambda}$, and

$$\ldots < \lambda_{j-1} < \lambda_j < \lambda_{j+1} < \ldots .$$

For every positive integer m, choose a sequence

$$\widehat{\Lambda}(m) = \big\{\lambda_{km+r_k(m)}\big\}, \qquad 0 \leq r_k(m) \leq m-1, \quad k \in \mathbb{Z},$$

in such a way that

(u) $\widehat{\Lambda}(m) \cap [-N_{m-1}, N_{m-1}] \subset \Lambda \cap [-N_{m-1}, N_{m-1}]$;
(v) $\widehat{\Lambda}(m) \cap \big([-N_m, -N_{m-1}] \cup [N_{m-1}, N_m]\big) = \Lambda \cap \big([-N_m, -N_{m-1}] \cup [N_{m-1}, N_m]\big)$;
(w) $\widehat{\Lambda}(m) \cap \big((-\infty, -N_m] \cup [N_m, \infty)\big) \supset \Lambda \cap \big((-\infty, -N_m] \cup [N_m, \infty)\big)$.

The fact that we are able to make such a choice follows from the construction of the set $\widehat{\Lambda}$. Furthermore, we require additionally that the numbers $r_k(m)$ satisfy the property

$$\lim_{m\to\infty} \inf\big\{|\lambda| : \lambda \in \Lambda \cap \widehat{\Lambda}(m)\big\} = \infty.$$

Put

$$\Lambda_m = \Lambda \setminus \widehat{\Lambda}(m).$$

To prove Theorem we need only to verify that the "supporting" polynomials

$$P_m(z) = \prod_{\lambda\in\Lambda_m}\Big(1-\frac{z}{\lambda}\Big),$$

satisfy estimate (4.6). Put

$$F_m(z) = \frac{B(z)}{P_m(z)} = \prod_{\lambda\in\Lambda\cap\widehat{\Lambda}(m)}\Big(1-\frac{z}{\lambda}\Big).$$

Using Lemma 4.1, we can estimate $|F_m'(\mu)|$ from below at points $\mu\in\Lambda\cap\widehat{\Lambda}(m)$.

Case 1, $|\mu| < N_{m-1}'$**.** By properties (d′), (u) and (v), we have

$$|F_m'(\mu)| = \frac{1}{|\mu|}\prod_{\substack{\lambda\in\Lambda\cap\widehat{\Lambda}(m)\\ \lambda\neq\mu}}\Big|1-\frac{\mu}{\lambda}\Big| \geq \frac{1}{e|\mu|}\prod_{\substack{\lambda\in\widehat{\Lambda}(m)\\ \lambda\neq\mu}}\Big|1-\frac{\mu}{\lambda}\Big|.$$

Applying Lemma 4.1 and using (4.7) we get

$$|F_m'(\mu)| \geq \frac{1}{Ce|\mu|^K}|\widehat{B}'(\mu)|^{1/m} \geq \frac{c}{Ce|\mu|^K}\big[B^{\#}(|\mu|)\big]^{\eta/m}. \tag{4.17}$$

Case 2, $|\mu| > N_{m-1}'$**.** Our first inequality

$$|F_m'(\mu)| = \prod_{\substack{\lambda\in\Lambda\cap\widehat{\Lambda}(m)\\ |\lambda|<N_{m-1}}}\Big|1-\frac{\mu}{\lambda}\Big| \cdot \frac{1}{|\mu|}\prod_{\substack{\lambda\in\Lambda\cap\widehat{\Lambda}(m)\\ \lambda\neq\mu,\,|\lambda|>N_{m-1}}}\Big|1-\frac{\mu}{\lambda}\Big| \geq \frac{1}{|\mu|}\cdot\prod_{\substack{\lambda\in\Lambda\\ \lambda\neq\mu,\,|\lambda|>N_{m-1}}}\Big|1-\frac{\mu}{\lambda}\Big|,$$

holds because all factors in the first product are bigger than 1; to rewrite the index set in the second product we use properties (v) and (w). Therefore, by property (e) and condition (4.1), we have for big m:

$$|F_m'(\mu)| \geq |B'(\mu)|\Big[\prod_{\substack{\lambda\in\Lambda\\ |\lambda|<N_{m-1}}}\big|1-\frac{\mu}{\lambda}\big|\Big]^{-1} \geq \frac{|B'(\mu)|}{\big[B^{\#}(|\mu|)\big]^{\eta/(m-1)}} \geq \big[B^{\#}(|\mu|)\big]^{\eta/m}. \tag{4.18}$$

By (4.17) and (4.18) we obtain that in both cases

$$|F_m'(\mu)| \geq \frac{c}{Ce|\mu|^K}\big[B^{\#}(|\mu|)\big]^{\eta/m}.$$

Thus, for big N,

$$|P_m(\mu)| \leq \frac{|B'(\mu)|}{|F_m'(\mu)|} \leq |\mu|^N|B'(\mu)|, \qquad \mu\in\Lambda\setminus\Lambda_m.$$

Now, (4.6) is proved and the theorem follows. $\square$

An analysis of the proof of Theorem C shows that it runs under the following condition imposed on the M-separable zero set Λ of a Hamburger class function:

there is a sequence $0 = R_0 < R_1 < \ldots < R_k \to \infty$, $R_k > C(R_{k-1}, \Lambda)$, such that every $(M+1)$-separable sequence $\widehat{\Lambda}$, obtained by adding k points between every consecutive $\lambda, \lambda' \in \Lambda$ with $R_k < |\lambda| < R_{k+1}$, $|\lambda| < |\lambda'|$, is the zero set of a Hamburger class function.

In particular, if an increasing sequence of positive numbers $\{\lambda_n\}$ satisfies these conditions and $\{\mu_n\}$ is an increasing sequence of positive numbers, then $\{\mu_n\lambda_n\}$ also satisfies these conditions. This leads to Theorem D. We leave the details to the interested reader.

5. An Example

Here we produce an example which shows that condition (1.19) in Theorem C cannot be relaxed.

Example 5.1. *For every ρ, $0 \le \rho < 1/2$, there exists a function $B \in \mathfrak{H}$ of order ρ and mean type, with zero set $\{\lambda_k\}_{k\ge 1}$ such that*

$$\mathrm{Clos}_{\ell^p_*(\frac{1}{|B'|},\Lambda_B)}\mathcal{P} \neq \ell^p_*\Big(\frac{1}{|B'|},\Lambda_B\Big), \qquad 1 \le p \le \infty,$$

and, for $\rho > 0$, there exists $\delta > 0$ such that

$$0 < \lambda_{k-1} < \lambda_k - \delta\lambda_k^{1-\rho}, \qquad k > 1, \tag{5.1}$$

$$\limsup_{\substack{\lambda\to\infty\\ \lambda\in\Lambda_B}} \frac{\log|B'(\lambda)|}{\lambda^\rho} > 0\,; \tag{5.2}$$

for $\rho = 0$ for every $\delta > 0$ and for sufficiently big k

$$0 < \lambda_{k-1} < \lambda_k - \lambda_k^{1-\delta}.$$

Proof. Let us consider only (the more difficult) case $\rho > 0$. First, take the sequence $\Lambda_0 = \{\lambda_{0,k}\}_{k\ge1}$, $\lambda_{0,k} = k^{1/\rho}$. We are going to add some points to this sequence to get the sequnce $\{\lambda_k\}_{k\ge1}$. By induction, we choose sequences $\{r_n\}_{n\ge1}$, $\{t_n\}_{n\ge1}$, $\{x_n\}_{n\ge1}$ of points on the positive axis and finite sets of points $\Lambda_n = \{\lambda_{n,k}\}_{1\le k\le m_n}$, such that

(a) $r_n < t_n < r_{n+1}/2$, $\quad n \ge 1$;
(b) $m_n < C \cdot t_n^\rho$, $\quad n \ge 1$;
(c) $t_n < \lambda_{n,k} < (3/2)t_n$, $\quad 1 \le k \le m_n$;
(d) the set $\Lambda = \Lambda_0 \cup \Lambda_1 \cup \ldots \cup \Lambda_n$ rearranged in the order of increasing, satisfies condition (5.1);

(e) if $B_n(z) = \prod_{z\in\Lambda_n}(1 - z/\lambda)$, $\widetilde{B}_n = \prod_{0\le s\le n} B_s$, $F_0(z) = 1$, $F_n(z) = \prod_{1\le s\le n}(1 - z/x_s)$, then for some N independent of n (and to be fixed later)

$$|B_n(x)| > 1 - 2^{-n}, \qquad x < 2r_n, \tag{5.3}$$

$$\Big|\prod_{0<s\le n-1} B_s(x)\Big| > 1, \qquad x > r_n, \tag{5.4}$$

$$|\widetilde{B}_n'(\lambda)| > \frac{|F_{n-1}(\lambda)|^{1/3}}{\lambda^N}, \qquad \lambda \in \Lambda_0 \cup \Lambda_n,\ r_n < \lambda < r_{n+1}, \tag{5.5}$$

and for some $x_n \in \Lambda_0 \cup \Lambda_n$,

$$|\widetilde{B}_n'(x_n)| < |F_{n-1}(x_n)|^{2/3}. \tag{5.6}$$

Condition (5.4) evidently holds for sufficiently big r_n; properties (b), (c) imply that (5.3) holds for sufficiently big t_n. If Λ_n is empty, then (5.5) follows from (5.4) and the estimates on the original product B_0 given in [26, Chapter II, Section 2],

$$\left.\begin{aligned} \log|B_0'(\lambda)| &\sim C_\rho \lambda^\rho, \qquad && \lambda \in \Lambda_0, \\ \log|B_0(\lambda)| &\sim C_\rho \lambda^\rho, \qquad && \operatorname{dist}(\lambda, \Lambda_0) > 1. \end{aligned}\right\} \tag{5.7}$$

Let us add to Λ_n groups of N new equidistant points between consecutive points $\lambda_{0,k}$, $\lambda_{0,k+1}$, beginning from $\lambda_{0,k}$ closest from above to t_n. We stop at the first moment when condition (5.6) starts to be valid for some point $x \in \Lambda_0 \cup \Lambda_n$ and put $x_n = x$. Since on the previous step condition (5.6) did not hold, condition (5.5) is still fulfilled.

It remains to verify that if we add by N points between all the points $\lambda_{0,k}$ in the interval $[t_n, (3/2)t_n]$, then condition (5.6) is satisfied for the point $x_n = \lambda_{0,k}$ closest from above to t_n. (Note that in this case $m_n \asymp Ct_n^\rho$). Indeed,

$$|B_n(\lambda_{0,k})| \le \exp\big(-N\big(2^\rho - 1\big)\lambda_{0,k}^\rho\big).$$

Fixing $N > C_\rho/\big(2^\rho - 1\big)$ we get $|\widetilde{B}_n'(\lambda_{0,k})| < 1$ for big n, and (5.6) holds.

Now, $B = \prod_{s\ge 0} B_s$ is of order ρ and mean type because of properties (a) and (b), is in $\mathfrak{H}$ because of property (e), satisfies condition (5.1) because of property (d). To verify (5.2), we note that on the intervals $[r_n, 2r_n]$ the quotient $|B/B_0|$ is uniformly bounded from above because of (5.3)–(5.4) and then use estimates (5.7).

Put $F(z) = \prod_{s\ge 1}(1 - z/x_s)$. Then the function F is of zero order and belongs to $\mathfrak{H}$ by Lemma A.4. Finally, $|F'(\lambda)/B'(\lambda)|$ tends to 0 more rapidly than any polynomial for $\lambda \to \infty$, $F(\lambda) = 0$, because of (5.6). It remains to apply Theorem A. $\square$

Here we give some general information related to the so called singular case of the weighted polynomial approximation problem in the spaces $\ell^p(w)$ and $L^p(\mu)$. The facts stated below provide additional motivation for the results discussed in our paper. Probably, most of these facts are known to the specialists.

1. Let $w:\mathbb{R}\mapsto[0,c]$, $c<+\infty$, be a function continuous on $S_w=\{x\in\mathbb{R}: w(x)>0\}$ and such that

$$\lim_{|x|\to\infty}|x|^n w(x)=0, \qquad n\geq 0. \tag{A1.1}$$

We consider the space $C_0(w)$ consisting of all functions continuous on S_w and such that

$$\lim_{|x|\to\infty}|f(x)|w(x)=0,$$
$$\|f\|_{C_0(w)}=\sup_{x\in\mathbb{R}}|f(x)|w(x).$$

Condition (A1.1) guarantees that all polynomials belong to the space $C_0(w)$.

We are going to use de Branges' theorem [9, 21, Section VI F, 32] in full generality:

De Branges' Theorem. *The polynomials are not dense in $C_0(w)$ if and only if there exists an entire function F of zero exponential type, with simple real zeros $\Lambda_F\subset S_w$ such that*

$$\sum_{\lambda\in\Lambda_F}\frac{1}{w(\lambda)|F'(\lambda)|}<\infty. \tag{A1.2}$$

Conditions (A1.1) and (A1.2) imply that F is in the Hamburger class $\mathfrak{H}$ (and consequently, by a theorem of M. Krein [26, Chapter V, Section 6], in the Cartwright class).

For every $s\in\mathbb{R}$, define a weight $w_s(x)=(1+|x|)^s w(x)$. We call a weight w *singular* if the polynomials are dense in $C_0(w_s)$ and are not dense in $C_0(w_t)$ for some $s<t$. Otherwise, w is called *regular*.

The next result improves somewhat a theorem by Mergelyan [28, Subsection 24].

Proposition A1.1. *If w is a singular weight, then S_w coincides with the zero set of a Hamburger class function.*

Proof. Since the set S_{w_s} does not depend on s, without loss of generality we may assume that the polynomials are dense in $C_0(w)$ and are not dense in $C_0(w_1)$. Then, by de Branges' theorem, there exists $F\in\mathfrak{H}$ such that $\Lambda_F\subset S_w$ and

$$\sum_{\lambda\in\Lambda_F}\frac{1}{(1+|\lambda|)w(\lambda)|F'(\lambda)|}<\infty.$$

If $\Lambda_F\neq S_w$, then we can choose a point $\lambda_0\in S_w\setminus\Lambda_F$ and set $F_0(z)=(z-\lambda_0)F(z)$. Then $F_0\in\mathfrak{H}$, $\Lambda_{F_0}\subset S_w$ and

$$\sum_{\lambda\in\Lambda_{F_0}}\frac{1}{w(\lambda)|F_0'(\lambda)|}<\infty.$$

One more application of de Branges' theorem gives us that the polynomials are not dense in $C_0(w)$, and we arrive at a contradiction. Thus, $\Lambda_F = S_w$, and the proof is completed. $\square$

For singular weights w, the set S_w is discrete and has no finite limit points, therefore we interpret the space $C_0(w)$ as a space of sequences and we for it our previous notation $c_0(w)$.

2. Given a positive measure μ on $\mathbb{R}$ having finite moments of all orders we consider the spaces $L^p(\mu)$,

$$L^p(\mu) = \bigl\{f : \|f\|^p_{L^p(\mu)} = \int_{\mathbb{R}} |f(x)|^p \, d\mu(x) < \infty\bigr\}, \qquad 1 \le p < \infty \,.$$

Set $d\mu_s(x) = (1+|x|)^{ps} d\mu(x)$, $s \in \mathbb{R}$. A measure μ is called *p-singular* if the polynomials are dense in $L^p(\mu_s)$ and are not dense in $L^p(\mu_t)$ for some $s < t$. An analog of Proposition A1.1 holds in this situation.

Proposition A1.2. *If μ is a p-singular measure, then* $\operatorname{supp} \mu$ *coincides with the zero set of a Hamburger class function.*

In the case $p = 2$, this assertion follows from classical results related to the Hamburger moment problem [3, Chapters 3 and 4].

Proof. First, using Mergelyan's argument [28, Subsection 24], we prove that $\operatorname{supp} \mu$ is a discrete set and

$$\operatorname{card} (\operatorname{supp} \mu \cap [-r, r]) = o(r), \qquad r \to \infty. \tag{A1.3}$$

As in the previous proof, assume that the polynomials are dense in $L^p(\mu)$ and are not dense in $L^p(\mu_1)$. Pick an arbitrary point $x_0 \in \operatorname{supp} \mu$, and choose a function $\psi \in C(\mathbb{R})$ with compact support such that $\psi(x_0) = 1$. Then there exists a sequence of polynomials P_n such that $\|\psi - P_n\|_{L^p(\mu)} \to 0$, $n \to \infty$. Therefore,

$$\sup_n \|P_n\|_{L^p(\mu)} \le C.$$

On the other hand, since the polynomials are not dense in $L^p(\mu_1)$, the Riesz–Hall–Mergelyan majorant

$$M_{p,\mu}(z) = \sup\bigl\{|P(z)| : P \in \mathcal{P},\, \|P\|_{L^p(\mu)} \le 1\bigr\}$$

is finite and, moreover, [21, 27, 28]

$$\log M_{p,\mu}(z) = o(|z|), \qquad |z| \to \infty.$$

This implies that $\{P_n\}$ is a normal sequence in $\mathbb{C}$; a subsequence $\{P_{n_k}\}$ converges locally uniformly to an entire function Ψ of zero exponential type. Furthermore, $\Psi = \psi$ μ-a.e.

Varying x_0 and ψ we obtain that $\Lambda = \operatorname{supp}\mu$ is discrete and satisfies estimate (A1.3). Thus,

$$\mu = \sum_{\lambda\in\Lambda} \mu_\lambda \delta_\lambda,$$

where

$$\sum_{\lambda\in\Lambda} |\lambda|^n \mu_\lambda < \infty, \qquad n \ge 0,$$
$$\sum_{\lambda\in\Lambda} \frac{1}{(1+|\lambda|)^2} < \infty.$$

We introduce an auxiliary weight w,

$$w(x) = \begin{cases} \mu_\lambda^{1/p}, & x = \lambda \in \Lambda, \\ +\infty, & x \notin \Lambda, \end{cases}$$

satisfying the conditions given at the beginning of Subsection 1. Then for every f with compact support,

$$\|f\|^p_{c_0(w)} = \max_{\lambda\in\Lambda} |f(\lambda)|^p \mu_\lambda \le \sum_{\lambda\in\Lambda} |f(\lambda)|^p \mu_\lambda = \|f\|^p_{L^p(\mu)}$$
$$\le \max_{\lambda\in\Lambda}\Big[|f(\lambda)|^p (1+|\lambda|)^2 \mu_\lambda\Big] \cdot \sum_{\lambda\in\Lambda} \frac{1}{(1+|\lambda|)^2} \le \|f\|^p_{c_0(w_{2/p})} \cdot \sum_{\lambda\in\Lambda} \frac{1}{(1+|\lambda|)^2}.$$

As a result, we get continuous embeddings

$$c_0(w_{2/p}) \hookrightarrow L^p(\mu) \hookrightarrow c_0(w). \tag{A1.4}$$

Therefore, μ is p-singular if and only if w is singular. An application of Proposition A1.1 completes the proof. $\square$

3. Now, we are able to characterize the supports of singular weights and p-singular measures. Let us call $B \in \mathfrak{H}$ a *good* Hamburger class function if it has no divisors G of zero exponential type bounded on Λ_B such that B/G is transcendental. Observe, that if such a divisor G exists, then $F = B/G$ automatically belongs to the Hamburger class.

Proposition A1.3. *The following conditions on a subset Λ of $\mathbb{R}$ are equivalent*:

(i) $\Lambda = \operatorname{supp}\mu$ *for a p-singular measure μ*;
(ii) $\Lambda = S_w$ *for a singular weight w*;
(iii) $\Lambda = \Lambda_B$ *for a good Hamburger class function B*;
(iv) $\Lambda = \Lambda_B$ *for a Hamburger class function B satisfying the following property*: *every entire function S of zero exponential type such that*

$$\lim_{|y|\to\infty} \left|\frac{y^n S(iy)}{B(iy)}\right| = 0\,, \qquad n \ge 0, \tag{A1.5}$$

is equal to a constant provided it is bounded on the set Λ_B.

The equivalence of the properties (i)–(iii) follows from Propositions A1.1, A1.2 and Theorem A. The equivalence of the properties (i) and (iv) follows from Koosis' argument [23] (which was already used in Section 2). Observe that condition (iii) looks much weaker than (iv): if $B \in \mathfrak{H}$ and G is a divisor of B of zero exponential type bounded on Λ_B, then $S = G$ satisfies condition (A1.5) which yields that G is a constant function. However, we cannot see how to prove directly the equivalence of these two conditions.

Theorems B, C and D give sufficient conditions for a Hamburger class function to be a good function. We think that the class of good Hamburger functions deserves a much better understanding than that we have achieved in this work.

4. Let X be one of the Banach spaces $L^p(\mu)$, $1 \le p < \infty$, or $C_0(w)$, and let

$$\delta_X(\mathcal{P}) = \dim \mathcal{P}^\perp,$$

where

$$\mathcal{P}^\perp = \{x^* \in X^* : x^*(P) = 0, \quad \forall P \in \mathcal{P}\}$$

be the annihilator of the polynomials.

Proposition A1.4. *For $1 \le p < \infty$ and $d \in \mathbb{N}$, the following conditions are equivalent*:

(i) $\delta_{L^p(\mu)}\mathcal{P} = d$;

(ii) *for every integer r, $0 \le r < d$, the polynomials are not dense in $L^p(\mu_{-r})$, and are dense in $L^p(\mu_{-d})$*;

(iii) *the polynomials are dense in $L^p(\mu_{-d})$, and for every $a \in \mathbb{R}$ such that $\mu(\{a\}) = 0$ and every $c > 0$, the polynomials are not dense in $L^p(\mu_{-d} + c\delta_a)$*;

(iv) A. $\operatorname{supp}(\mu) = \Lambda_B$, $B \in \mathfrak{H}$;

B. *for every $F \in \mathfrak{H}$ such that $\Lambda_F \subseteq \Lambda_B$, $\operatorname{card}(\Lambda_B \setminus \Lambda_F) < d$, we have*

$$\sum_{\lambda \in \Lambda_F} \frac{1}{\mu_\lambda |F'(\lambda)|^{p/(p-1)}} < +\infty\,, \qquad 1 < p < \infty,$$

and

$$\liminf_{\substack{\lambda \to \infty \\ \lambda \in \Lambda_F}} \mu_\lambda |F'(\lambda)| > 0\,, \qquad p = 1;$$

C. *for every $F \in \mathfrak{H}$ such that $\Lambda_F \subset \Lambda_B$, $\operatorname{card}(\Lambda_B \setminus \Lambda_F) \ge d$, we have*

$$\sum_{\lambda \in \Lambda_F} \frac{1}{\mu_\lambda |F'(\lambda)|^{p/(p-1)}} = +\infty\,, \qquad 1 < p < \infty,$$

and

$$\liminf_{\substack{\lambda \to \infty \\ \lambda \in \Lambda_F}} \mu_\lambda |F'(\lambda)| = 0\,, \qquad p = 1.$$

A similar statement is valid for $C_0(w)$. In the L^2-setting, the equivalence of conditions (i)–(iii) is known, see [6] for references. Canonical measures are 2-singular. Moreover, a measure μ is canonical if and only if $\delta_{L^p(\mu_1)}\mathcal{P} = 1$, and a special case of Proposition A1.4 with $p = 2$, $d = 1$ (and with μ_1 instead of μ) coincides with the corrected version of Hamburger's statement given in Section 1 (Corollary 1.1).

Proof. The equivalence of conditions (i) and (ii) follows from the following simple argument from linear algebra. We use the pairing $\langle f, g\rangle = \int fg\,d\mu$ for $f \in L^p(\mu_r)$ and $g \in L^q(\mu_{-r})$, $1/p + 1/q = 1$. Assume first that (i) holds so there are d linearly independent vectors in $L^q(\mu)$ annihilating the polynomials. Then a linear combination v of them annihilates $d-1$ functions $(i+x)^{-r}$, $0 < r < d$, or, what is the same, $(i+x)^r v$ belongs to $L^q(\mu_r)$ and annihilates the polynomials. Thus the polynomials are not dense in $L^p(\mu_{-r})$, $r < d$. Furthermore, the polynomials are dense in $L^p(\mu_{-d})$, otherwise, there is a vector $w \in L^q(\mu_d)$ annihilating the polynomials and therefore $d+1$ vectors $w, xw, \dots, x^d w$ from $L^q(\mu)$ annihilate the polynomials which contradicts (i). Arguing in the same way, we obtain the opposite implication.

The equivalence of conditions (ii) and (iii) was proved in the C_0-setting by Mergelyan [28, Subsection 28]; his proof works in the L^p spaces as well. Assume first that (ii) holds. Then there is a vector $v_1 \in L^q(\mu_1)$ annihilating the polynomials. By Proposition A1.2, the support of μ is discrete. Choose a point $a \in \mathbb{R}$ such that $\mu(\{a\}) = 0$. Then the function $v(x) = v_1(x)/(x-a)$ belongs to $L^q(\mu)$ and, for every polynomial P,

$$P(a)\int v\,d\mu = P(a)\int \frac{v_1(x)}{x-a}\,d\mu(x) = \int \frac{v_1(x)P(x)}{x-a}\,d\mu(x) = \int vP\,d\mu\,. \tag{A1.6}$$

Since the polynomials are dense in $L^p(\mu)$, the function v_1 cannot be orthogonal to $1/(x-a)$, thus

$$\int v\,d\mu = \int \frac{v_1(x)}{x-a}\,d\mu \neq 0\,,$$

and we can normalize v in such a way that $\int v\,d\mu = -c$. Relation (A1.6) says that the polynomials are not dense in $L^p(\mu + c\delta_a)$, and (iii) is done. Reversing the argument, we obtain the opposite implication.

The equivalence of conditions (ii) and (iv) follows from Theorem A. □

5. The weighted approximations by polynomials and by linear combinations of exponents are rather similar [21, 27]. However, there are some differences. One of them appears in the singular case [22, 23]. In weighted approximation by linear combinations of exponents, the Hamburger class is replaced by the Krein class consisting of the real entire functions C with real zeros, such that

$$\frac{1}{C(z)} = \sum_{\lambda\in\Lambda_C} \frac{1}{C'(\lambda)(z-\lambda)}\,,$$

where the series in the right-hand side converges absolutely outside exceptional discs around the points of Λ_C. By Krein's theorem [21, 26], the Krein class is a subset of the Cartwright class. In particular, the Krein class functions have exponential type.

De Branges' theorem claims that *the linear combinations of exponents* $\{e^{ilx}\}_{-\sigma<l<\sigma}$ *are not dense in the space* $C_0(w)$ *if and only if there exists a Krein class entire function* C *of exponential type* σ *such that* $\Lambda_C \subset S_w$ *and*

$$\sum_{\lambda\in\Lambda_C} \frac{1}{w(\lambda)|C'(\lambda)|} < +\infty\,.$$

The counterparts of Theorem A and Proposition A1.4 hold in this setting with Krein class entire functions of exponential type σ instead of Hamburger class entire functions. Furthermore, an observation by Koosis [22] shows that in this case the notion of a good function is not meaningful: *every Krein class entire function of mean type is good.* Thus the problem of description of supports of singular measures has a simple answer: a set $\Lambda \subset \mathbb{R}$ is the support of a singular measure (or of a singular weight) if and only if Λ is the zero set of an entire function of the Krein class having exponential type σ.

6. Let us consider the situation where the set $\Lambda = S_w$ is discrete and does not accumulate too fast at infinity:

$$\operatorname{card}\big(\Lambda\cap[-r,r]\big) = O(r^N), \qquad r\to\infty,$$

for some $N<\infty$.

We introduce the scale of spaces $\ell^p_*(w) = \ell^p_*(w,\Lambda)$, and assume, as before, that w satisfies condition (A1.1) and hence, the polynomials belong to these spaces. The weight w is called *singular* if the polynomials are dense in $\ell^p_*(w_s)$ and are not dense in $\ell^p_*(w_t)$ for some $t>s$. Otherwise, w is called *regular*. Analogously to (A1.4), for $s>N/p$ we have continuous embeddings

$$c_0(w_s) \hookrightarrow \ell^p_*(w) \hookrightarrow c_0(w).$$

Therefore, the notion of regularity does not depend on p, and in the regular case the polynomials are dense or not dense simultaneously in all $\ell^p_*(w)$, $1\le p\le+\infty$. Now we apply Proposition A1.1 and de Branges' criterion of density of the polynomials in $C_0(w)$ to obtain such a result.

Proposition A1.5. *If* Λ *is not a zero set of a Hamburger class function, then the weight* w *is regular, and the following two conditions are equivalent*:

(i) *the polynomials are dense in all* $\ell^p_*(w)$, $1\le p\le+\infty$;

(ii) *there exist* $s<+\infty$, $c>0$, *and a Hamburger class function* F, $\Lambda_F\subset\Lambda$, *such that*

$$w(\lambda)|F'(\lambda)| \ge c(1+|\lambda|)^{-s}, \qquad \lambda\in\Lambda_F.$$

Correspondingly, in the singular case, Λ should be the zero set of a Hamburger class function. This case is the main subject of our paper.

Finally, we note that the situation with general $L^p(\mu)$ spaces is quite different unless some *apriori* conditions are imposed on the support of μ: given $1<r<\infty$, A. Kesarev constructed recently a smooth measure μ on $\mathbb{R}$ (which is automatically p-regular for every p) such that for $s<r$ the polynomials are dense in $L^s(\mu)$ and for $s>r$ they are not dense in $L^s(\mu)$.

The Bernstein approximation problem on discrete subsets Λ of the real line can be analyzed rather completely in the case when both Λ and the weight $w(\lambda)$ behave fairly regular. In this appendix we show how de Branges' criterion and the techniques developed in the paper can be applied in the following concrete situation:

$$\Lambda = \{n^{1/\rho}\}_{n\in\mathbb{N}}, \qquad \rho > 0,$$
$$w(\lambda) = \exp(-c\lambda^m), \qquad w_s(\lambda) = \lambda^s \exp(-c\lambda^m), \qquad c > 0,\ m > 0,\ s \in \mathbb{R}.$$

Most of the results presented below are not new. Some of them could be extracted, for example, from [20, Chapter 1] or from [4, Theorems 2.1, 3.1 and 4.1]. However, the methods we use give the most precise results in this direction.

Following the previous appendix, we say that the weight w is regular if the polynomials are dense or not dense simultaneously in all the spaces $\ell^p_*(w_s)$, $-\infty < s < \infty$. Otherwise, the weight is called singular. The discussion in the previous appendix shows that this notion does not depend on the exponent p, $1 \le p \le \infty$. The results for regular weights given below are presented on Figure 1.

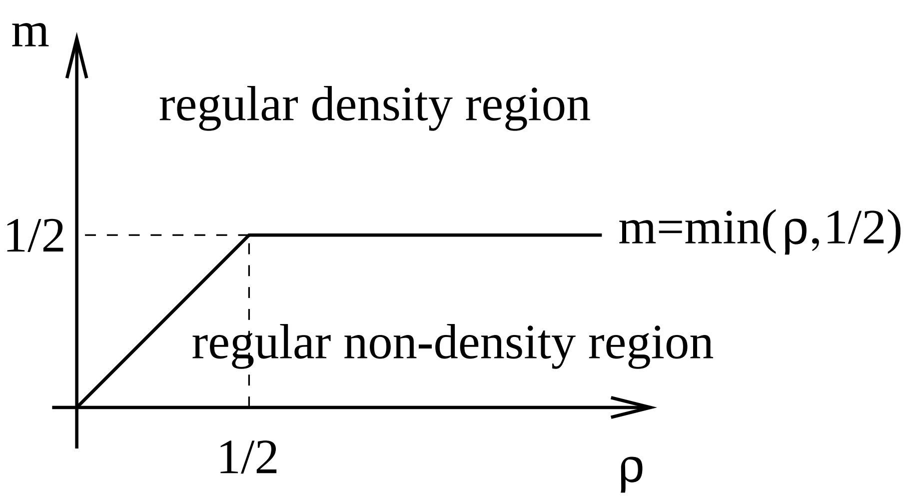

FIGURE 1.

1. For $m \ge 1/2$ the polynomials are regularly dense in $\ell^p_*(w)$. To verify this fact, we associate to every functional a on $\ell^p_*(w_s)$, $a \in \ell^q_*(1/w_s)$, $1/p+1/q = 1$, its cosine-transform

$$F_a(z) = \sum_{\lambda\in\Lambda} a(\lambda)\cos(\lambda^{1/2} z)$$

holomorphic in a neighborhood of 0. If a functional a vanishes on all polynomials, then F_a vanishes at the point 0 together with all derivatives, and correspondingly, $F_a = 0$, $a = 0$. This argument is given in [4, Theorems 2.1].

2. For $\rho < m < 1/2$ the polynomials are regularly dense in $\ell^p_*(w)$. Otherwise, by Theorem A, there exists a Hamburger class function F such that

$$\Lambda_F \subset \Lambda \tag{A2.1}$$

and

$$|F'(\lambda)| \geq c_1 \lambda^{-s} \exp(c\lambda^m), \qquad \lambda \in \Lambda_F, \tag{A2.2}$$

with $c_1 > 0$, $s < \infty$. Let ρ_F be the order of growth of F. The inclusion (A2.1) implies that $\rho_F \leq \rho$. Therefore, $\rho_{F'} = \rho_F \leq \rho < m$ that contradicts (A2.2) because Λ_F is unbounded.

3. For $m < \min(\rho, 1/2)$ the polynomials are regularly non-dense in $\ell^p_*(w)$. Indeed, pick m_1, $m < m_1 < \min(\rho, 1/2)$, and choose a subset $\Lambda_1 \subset \Lambda$ which is an R-set with respect to the order m_1 (see Section 1 for the definition). In this situation the canonical product

$$F(z) = \prod_{\lambda \in \Lambda_1} \left(1 - \frac{z}{\lambda}\right)$$

satisfies the asymptotic relation (like in Section 1)

$$\lim_{\substack{\lambda \to \infty \\ \lambda \in \Lambda_F}} \frac{\log |F'(\lambda)|}{\lambda^{m_1}} = c \cot \pi m_1, \tag{A2.3}$$

for some $c > 0$. As a result, F is in the Hamburger class, and by Theorem A, the polynomials are regularly non-dense.

4. Let $m = \rho \in (0, 1/2)$, $c > \pi \cot \pi\rho$. Like in Subsection 3 we use that the canonical product

$$B_\rho(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^{1/\rho}}\right)$$

satisfies the estimate

$$\lim_{\substack{\lambda \to \infty \\ \lambda \in \Lambda}} \frac{\log |B'_\rho(\lambda)|}{\lambda^\rho} = \pi \cot \pi\rho. \tag{A2.4}$$

Now, the proof of Theorem B may be easily modified to prove that the polynomials are regularly dense in $\ell^p_*(w)$. Alternatively, this result may be derived directly from Fryntov's theorem.

If $1/\rho$ is an integer (≥ 3), then this result follows from [4, Theorem 4.1] where much simpler methods are used. Apparently, these methods are insufficient to study the case of arbitrary $\rho > 2$ which is left open in [4].

5. Let $m = \rho \in (0, 1/2)$, $c < \pi \cot \pi\rho$. In this case the polynomials are regularly non-dense in $\ell^p_*(w)$. Indeed, estimate (A2.4) implies that for some $\eta > 0$ and every s there exists $c_1 > 0$ such that

$$w_s(\lambda)|B'_\rho(\lambda)| \geq c_1 \exp(\eta\lambda^\rho), \qquad \lambda \in \Lambda_B.$$

This gives immediately that all the exponents p are not normal for the weights w_s, and the polynomials are not dense (see Section 2).

This case is basically covered by Theorem 3.1 in [4]. A somewhat less precise result could also be extracted from [20, Chapter 1].

6. The remaining case $m = \rho \in (0, 1/2)$, $c = \pi \cot \pi\rho$ is the most delicate one because the weights w_s are singular. Our results are presented on Figure 2.

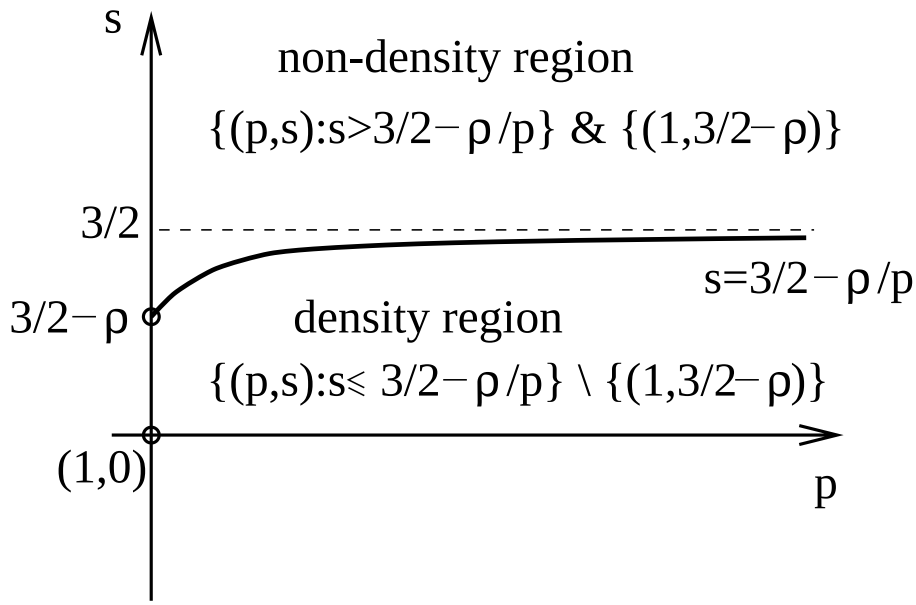

FIGURE 2.

To apply our technique we need a more precise asymptotics for $|B'_\rho|$. It follows from the results of G. H. Hardy in [19] (see also [12, Section 3.3, Theorem 2]) that there exists a finite limit

$$\lim_{|z|\to\infty} \frac{B_\rho(z)}{z^{-1/2}\sin(\pi z^\rho)\exp((\pi\cot\pi\rho)z^\rho)}, \qquad |\arg z| \le \pi/2. \tag{A2.5}$$

Put $f(z) = \sin(\pi z^\rho)$, $g = B_\rho/f$. Since $B'_\rho(\lambda) = g(\lambda)f'(\lambda)$, $\lambda \in \Lambda$, estimate (A2.5) implies that there exists a finite limit

$$\lim_{\substack{\lambda\to\infty \\ \lambda\in\Lambda}} \frac{|B'_\rho(\lambda)|}{\lambda^{\rho-3/2}\exp((\pi\cot\pi\rho)\lambda^\rho)}.$$

Therefore, for $p > 1$ and for every Hamburger class divisor F of B_ρ we have

$$\sum_{\lambda\in\Lambda_F}\left[\frac{1}{w_s(\lambda)|F'(\lambda)|}\right]^q = \sum_{\lambda\in\Lambda_F}\lambda^{-q(s+\rho-3/2)}\left|\frac{B'_\rho(\lambda)}{F'(\lambda)}\right|^q$$
$$= \sum_{k=1}^{\infty} n_k^{-q(s+\rho-3/2)/\rho}\left|\frac{B'_\rho(n_k^{1/\rho})}{F'(n_k^{1/\rho})}\right|^q, \qquad \frac{1}{p}+\frac{1}{q}=1, \tag{A2.6}$$

where $\Lambda_F = \{n_k^{1/\rho}\}_{k=1}^\infty$. Hence, if $q(s+\rho-3/2)/\rho > 1$, that is $s > 3/2 - \rho/p$, then the exponent p is not normal for the weight w_s, and the polynomials are not dense in $\ell^p_*(w_s)$. If $p=1$, and F is a Hamburger class divisor of B_ρ, then we have

$$w_s(\lambda)|F'(\lambda)| = \lambda^{s+\rho-3/2}\Big|\frac{F'(\lambda)}{B'_\rho(\lambda)}\Big|.$$

Therefore, if $s \ge 3/2-\rho$, then $p=1$ is not a normal exponent for w_s, and the polynomials are not dense in $\ell^1(w_s)$.

On the orther hand, if $p>1$ and $s \le 3/2-\rho/p$ or $p=1$ and $s < 3/2-\rho/p$, then the polynomials are dense in $\ell^p_*(w_s)$. To prove this we apply the criterion in Theorem A. Consider the case $p>1$ and $\delta = q(s+\rho-3/2)/\rho \le 1$. If the polynomials are not dense, then for some Hamburger divisor F of B the series (A2.6) converges. Put $G = B/F$. Then we have

$$\sum_{k=1}^\infty n_k^{-\delta}|G(n_k^{1/\rho})|^q < \infty. \tag{A2.7}$$

Now, an argument like in the proof of Theorem B (Section 3) shows that $\Lambda\setminus\Lambda_F$ is finite. Otherwise, we may add to F some zeros of G in such a way that (3.1) holds, and as a consequence, $B=F$, and we get a contradiction. Hence, G is a polynomial, and (A2.7) cannot hold. The same argument works for $p=1$ and $s<3/2-\rho/p$.

Appendix 3. The Hamburger class

The properties of functions in the Hamburger class $\mathfrak{H}$ we need in our paper are given here. For some of these and other properties see also [3, Chapter 4], [5].

Lemma A3.1. *If B is a transcendental entire function of zero exponential type with real zeros, then*

$$\lim_{|y|\to\infty}\frac{|y|^n}{|B(iy)|} = 0, \qquad n \ge 0.$$

Proof. To prove this statement note that for all such functions B we have $|B(iy)| > |B(0)|$, $y\in\mathbb{R}\setminus\{0\}$. Therefore, it is enough to divide B by $(n+1)$ terms $(z-z_k)$, where z_k, $1\le k\le n+1$, are (different) zeros of B. □

Lemma A3.2. *If $B\in\mathfrak{H}$, then for every polynomial P,*

$$P(z) = \sum_{\lambda\in\Lambda_B}\frac{B(z)P(\lambda)}{B'(\lambda)(z-\lambda)}.$$

Proof. Denote the difference between the right-hand side and the left-hand side by $R(z)$. Then R and, as a consequence, R/B are entire functions of zero exponential type, and by Lemma A3.1 the function R/B tends to 0 along the imaginary axis. The Phragmén–Lindelöf principle yields now that $R/B=0$, $R=0$. □

Lemma A3.3. *If $B \in \mathfrak{H}$, then for every polynomial P,*

$$\sum_{\lambda \in \Lambda_B} \frac{P(\lambda)}{B'(\lambda)} = 0.$$

Proof. Without loss of generality assume that $B(0) \neq 0$. Then we apply Lemma A3.2 to the polynomial $zP(z)$ and put $z = 0$. $\square$

Lemma A3.4. *If $0 < 2\lambda_k < \lambda_{k+1}$, $1 \le k < \infty$, then $F(z) = \prod_{k \ge 1}(1 - z/\lambda_k)$ is an entire function of order 0 in the Hamburger class.*

Proof. First,

$$|F'(\lambda_k)| = \frac{1}{\lambda_k} \cdot \prod_{s<k} \Bigl(\frac{\lambda_k}{\lambda_s} - 1\Bigr) \cdot \prod_{s>k} \Bigl(1 - \frac{\lambda_k}{\lambda_s}\Bigr),$$

$$\prod_{s>k} \Bigl(1 - \frac{\lambda_k}{\lambda_s}\Bigr) \ge \prod_{l>0} \bigl(1 - 2^{-l}\bigr) > 0.$$

Furthermore, for every n,

$$\lim_{k\to\infty} \frac{1}{\lambda_k^n} \cdot \prod_{s<k} \Bigl(\frac{\lambda_k}{\lambda_s} - 1\Bigr) > \lim_{k\to\infty} \prod_{s<n} \Bigl(\frac{1}{\lambda_s} - \frac{1}{\lambda_k}\Bigr) = \prod_{s<n} \frac{1}{\lambda_s} > 0. \quad \square$$

Lemma A3.5. *Let B be an entire function in the Hamburger class $\mathfrak{H}$ satisfying condition* (1.17)*. Suppose that the zeros of B are M-separated for some $M < \infty$. Then the following conditions on B are equivalent:*

(i) *for some $\eta > 0$,*

$$\log |B'(\lambda)| \ge \eta \log B^{\#}(|\lambda|), \qquad \lambda \in \Lambda.$$

(ii) *for some $\eta > 0$,*

$$\log |B(z)| \ge \eta \log B^{\#}(|z|), \qquad z \notin \bigcup_{\lambda \in \Lambda} D\bigl(\lambda, |\lambda|^{-M}\bigr).$$

These conditions imply that, for some $R < \infty$,

$$B^{\#}(r) \le M(r, B)^{1/\eta}, \qquad R < r < \infty.$$

Proof. Without loss of generality assume that $\Lambda \cap (-2, 2) = \emptyset$. We show that (i) implies (ii); the opposite implication and the last assertion are obvious. According to Lemma 4.3,

$\log|B(z)| \ge \eta \log B^{\#}(|z|)$ on the circles $C_\lambda = \{|z-\lambda| = |\lambda|^{-M-2}\}$, $\lambda \in \Lambda$. Consider the function $F = \log|B| - \eta \log B^{\#}$. Then, for $x \in \mathbb{R}_+ \setminus \Lambda$,

$$-F''(x) = \sum_{\lambda\in\Lambda} \frac{1}{(x-\lambda)^2} - \eta \sum_{\lambda\in\Lambda} \frac{1}{(x+|\lambda|)^2} > 0\,.$$

Hence, F is concave between the zeros of B and, consequently, is positive on $\mathbb{R}_+ \setminus \cup_{\lambda\in\Lambda} C_\lambda$. We argue similarly for $x \in \mathbb{R}_- \setminus \Lambda$. If B has no zeros on a half-axis, we just use that F tends to $+\infty$ along this half-axis.

Therefore, $\log|B| \ge \eta \log B^{\#}$ on the boundary $\partial\Omega$ of the domain

$$\Omega = \mathbb{C}_+ \setminus \bigcup_\lambda D\big(\lambda, |\lambda|^{-M}\big).$$

Since $\log|B|$ is harmonic and bounded from below on Ω, the function $\eta \log B^{\#} - \log|B|$ is subharmonic and has zero exponential type on Ω. Applying the Phragmén–Lindelöf principle, we conclude that $\log|B| \ge \eta \log B^{\#}$ on Ω. The same argument works in the lower half-plane as well. $\Box$

Appendix 4. Proof of Lemma 4.1

Estimate (4.3) is an immediate consequence of estimate (4.2) written in the form

$$\frac{1}{C(1+|z|^K)} \le \frac{|f_m(z)|}{|f(z)|^{1/m}} |z-\gamma|^{1/m} \le C(1+|z|^K), \qquad |z-\gamma| = \frac{1}{|\gamma|^{M+1}}\,.$$

We need only to use that

$$f'(\gamma) = \frac{f(z)}{z-\gamma}\Big|_{z=\gamma}\,,$$

and that the function

$$z \mapsto \log|f_m(z)| - \frac{1}{m}\log|f(z)| + \frac{1}{m}\log|z-\gamma|$$

is harmonic in the disc $D\big(\gamma, |\gamma|^{-M}\big)$. Estimate (4.4) is proved in the same way.

To prove estimate (4.2), we consider the family of functions

$$f_{m,r}(z) = \prod_s \Big(1 - \frac{z}{\gamma_{sm+r}}\Big), \qquad 0 \le r \le m-1.$$

The zeros of functions $f_{m,r}$ and $f_{m,l}$, $r \ne l$, are interlaced, and by a theorem of M. Krein (see [26, Chapter VII, Section 1]) the function

$$\operatorname{Im} \frac{f_{m,r}(z)}{f_{m,l}(z)} \cdot \operatorname{Im} z$$

is of a constant sign for $\operatorname{Im} z \neq 0$. By the Carathéodory inequality for the half-plane [26, Chapter I, Section 6], every function g enjoying this property, $\operatorname{Im} g(z) \cdot \operatorname{Im} z \neq 0$, $\operatorname{Im} z \neq 0$, satisfies the estimate

$$\frac{1}{5}|g(i)|\frac{|\operatorname{Im} z|}{|z|^2} \leq |g(z)| \leq 5|g(i)|\frac{|z|^2}{|\operatorname{Im} z|}, \qquad |z| \geq 1.$$

Applying this estimate to $g = f_{m,r}/f_{m,l}$ and using that $|g(i)| \leq C(L)$, where L is the constant in condition (4.1), we obtain

$$\frac{|\operatorname{Im} z|}{C|z|^2} \leq \left|\frac{f_{m,r}(z)}{f_{m,l}(z)}\right| \leq \frac{C|z|^2}{|\operatorname{Im} z|}, \qquad |z| \geq 1. \tag{A4.1}$$

Multiplying these inequalities for $l = 0, 1, \dots, m-1$ and taking the root of m-th degree, we obtain

$$\frac{|\operatorname{Im} z|}{C|z|^2} \leq \frac{|f_{m,r}(z)|}{|f(z)|^{1/m}} \leq \frac{C|z|^2}{|\operatorname{Im} z|}, \qquad |z| \geq 1.$$

Since the zeros of the functions f_m and $f_{m,0}$ are interlaced, we have an estimate like (A4.1) for the quotient $f_m/f_{m,0}$. Therefore,

$$\frac{|\operatorname{Im} z|^2}{C|z|^4} \leq \frac{|f_m(z)|}{|f(z)|^{1/m}} \leq \frac{C|z|^4}{|\operatorname{Im} z|^2}, \qquad |z| \geq 1. \tag{A4.2}$$

This implies immediately that

$$\frac{1}{C(1+|z|^K)} \leq \frac{|f_m(z)|}{|f(z)|^{1/m}} \leq C(1+|z|^K), \qquad |z| \geq 1,\ |\operatorname{Im} z| \geq \frac{1}{1+|z|^{M+2}}. \tag{A4.3}$$

The function $h = \log|f_m| - (1/m)\log|f|$ is harmonic in $\mathbb{C} \setminus \Gamma$. For every $\gamma \in \Gamma$, there exists c_γ, $|c_\gamma| \leq 1$, such that $h - c_\gamma \log|\cdot - \gamma|$ is harmonic in $D(\gamma, |\gamma|^{-m})$. Making use of estimate (A4.2) on the boundary of this disc and the Poisson-Jensen formula, we obtain that

$$|h(z)| \leq K \log\frac{1}{|\gamma|} \qquad |z-\gamma| = |\gamma|^{-m-1},$$

and henceforth

$$\frac{1}{C(1+|z|^K)} \leq \frac{|f_m(z)|}{|f(z)|^{1/m}} \leq C(1+|z|^K), \qquad |z-\gamma| = |\gamma|^{-m-1}. \tag{A4.4}$$

Consider the open set

$$\mathcal{O} = \left[\left\{z = x+iy : |y| < \frac{1}{(1+|x|^{M+2})}\right\} \cup \{z : |z| < 2\}\right] \setminus \bigcup_{\gamma\in\Gamma} D\big(\gamma, |\gamma|^{-M-1}\big).$$

The function h is harmonic on each connectivity component of $\mathcal{O}$, and, by the maximum principle, the estimate (A4.4) extends to these components. Taking into account (A4.3), we get the statement of the lemma. □

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A. A. Borichev, Laboratoire de Mathématiques Pures de Bordeaux,
UPRESA 5467 CNRS, Université Bordeaux I, 351, cours de la Libération,
33405 Talence, FRANCE
E-mail: borichev@math.u-bordeaux.fr

M. L. Sodin, Institute of Mathematical Sciences, Tel Aviv University,
Ramat Aviv 69978, ISRAEL
E-mail: sodin@math.tau.ac.il