ALMOST-ADDITIVITY OF ANALYTIC CAPACITY AND CAUCHY INDEPENDENT MEASURES

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Abstract. We show that, given a family of discs centered at a chord-arc curve, the analytic capacity of a union of arbitrary subsets of these discs (one subset in each disc) is comparable with the sum of their analytic capacities. However, we need that the discs in question would be separated, and it is not clear whether the separation condition is essential or not. We apply this result to find families $\{\mu_j\}$ of measures in $\mathbb{C}$ with the following property. If the Cauchy integral operators $\mathcal{C}_{\mu_j}$ from $L^2(\mu_j)$ to itself are bounded uniformly in j, then $\mathcal{C}_\mu$, $\mu = \sum \mu_j$, is also bounded from $L^2(\mu)$ to itself.

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1. Introduction

We consider two properties of families of sets and measures in the complex plane.

1.1. **Almost additivity of analytic capacity.** The *analytic capacity* $\gamma(F)$ of a compact set F in $\mathbb{C}$ is defined by the equality

$$\gamma(F) = \sup |f'(\infty)|,$$

where the supremum is taken over all analytic functions $f : \mathbb{C} \setminus F \to \mathbb{C}$ with $|f| \le 1$ on $\mathbb{C} \setminus F$. Here $f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty))$. For non-compact F we set

$$\gamma(F) = \sup\{\gamma(K) : K \text{ compact}, K \subset F\}$$

[G]. For a summary of equivalent definitions the reader can see [To] and [Vo].

In the celebrated paper [To1] Tolsa established the countable semiadditivity of the analytic capacity, i. e. that

$$\gamma\Big(\bigcup F_i\Big) \le C \sum \gamma(F_i)$$

with an absolute constant C. But the inverse inequality does not hold in general. To see that we consider the n-th generation $E_n^{1/4}$ of the corner 1/4-Cantor set constructed in the following way. Start with the unit square (0-th generation). The j-th generation consists of 4^j squares $E_{j,k}$ with side length 4^{-j}, each square $E_{j,k}$ contains four squares of $(j+1)$-th generation, located at the corners of $E_{j,k}$, and so on. It's known [MTV] that $\gamma(\bigcup_{k=1}^{4^n} E_{n,k}) = \gamma(E_n^{1/4}) \asymp 1/\sqrt{n}$ with absolute constants of comparison; here $P \asymp Q$ means that $cP \le Q \le CP$. Positive constants c, C, a, A (possibly with indices) are not necessarily the same at each appearance. On the other hand,

$$\sum_{k=1}^{4^n} \gamma(E_{n,k}) \asymp 4^n \cdot 4^{-n} = 1.$$

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Thus, "almost additivity" $\gamma\big(\bigcup F_i\big) \asymp \sum \gamma(F_i)$ of the analytic capacity does not take place in general. N. A. Shirokov posed the question on the validity of this property for the special class of sets described in the following Theorem 1.1.

We say that Γ is a chord-arc curve, if

$$|t-s| \le A_0|z(t)-z(s)|, \quad A_0 > 1,$$

where $z(t)$ is the arc-length parametrization of Γ.

Theorem 1.1. *Let D_j be discs with centers on a chord-arc curve Γ, such that $\lambda D_j \cap \lambda D_k = \emptyset$, $j \ne k$, for some $\lambda > 1$. Let $E_j \subset D_j$ be arbitrary compact sets. Then there exists a constant $c = c(\lambda, A_0)$, such that*

$$\gamma\big(\bigcup E_j\big) \ge c \sum_j \gamma(E_j). \tag{1.1}$$

Open question. It is not clear if the theorem is true or not when $\lambda = 1$.

1.2. **Cauchy independence of families of measures.** We use the results in the previous subsection to investigate the property of measures described below.

We call a finite Borel measure with compact support in the complex plane *a Cauchy operator measure* if the Cauchy operator $\mathcal{C}_\mu$ is bounded from $L^2(\mu)$ to itself wth norm at most 1.

The first natural question is how to interpret the "definition" of $\mathcal{C}_\mu$ as

$$\mathcal{C}_\mu f(z) = \int \frac{f(\xi)d\mu(\xi)}{\xi - z}.$$

One of the ways is to consider the so-called ε-truncations, defined by

$$\mathcal{C}_\mu^\varepsilon f(z) = \int\limits_{\varepsilon<|\xi-z|<\varepsilon^{-1}} \frac{f(\xi)d\mu(\xi)}{\xi - z}.$$

We now say that $\mathcal{C}_\mu$ is bounded as an operator from $L^2(\mu)$ to itself if the ε-truncations are bounded from $L^2(\mu)$ to itself uniformly in ε. Moreover, by the norm of $\mathcal{C}_\mu$ we understand the $\sup_\varepsilon \|\mathcal{C}_\mu^\varepsilon\|_\mu =: \|\mathcal{C}_\mu\|_\mu$, where $\|\mathcal{C}_\mu^\varepsilon\|_\mu$ is the norm of $\mathcal{C}_\mu^\varepsilon$ as an operator from $L^2(\mu)$ to itself. We encourage the reader to look for other interpretations in [NTrV1], [To] and [Vo].

The following important fact (which we will repeatedly use) demonstrates the connection between the analytic capacity and boundedness of the Cauchy operator [To1, To2, To, Vo]: for every compact set F in $\mathbb{C}$,

$$\gamma(F) \asymp \sup\{\|\mu\| : \operatorname{supp}\mu \subset F,\ \mu \in \Sigma,\ \|\mathcal{C}_\mu\|_\mu \le 1\}, \tag{1.2}$$

where Σ is the class of nonnegative Borel measures μ such that $\mu(D(x,r)) \le r$ for every disc $D(x,r) := \{z \in \mathbb{C} : |z-x| < r\}$.

We call a collection $\{\mu_j\}$ of finite positive Borel measures with compact supports *C-Cauchy independent measures* if a) $\|\mathcal{C}_{\mu_j}\|_{\mu_j} \le 1$ (Cauchy operator measures) and b) $\|\mathcal{C}_\mu\|_\mu \le C < \infty$ for $\mu = \Sigma_j \mu_j$. We will call such collection *Cauchy independent* if it is C-Cauchy independent for some finite C.

The family $\{\mu_j\}$ can be finite or infinite. Two Cauchy operator measures are always Cauchy independent with an absolute constant C. A short proof of this nontrivial fact is given in [NToV, Proposition 3.1]. So, a finite family is always Cauchy independent for a sufficiently large constant C. But our main interest is in situations, when infinite families are independent (or when C is independent of the number of measures). The main result is the following

Theorem 1.2. *Suppose that $\lambda > 1$, and measures μ_j are supported on compact sets E_j lying in discs D_j such that λD_j are disjoint. We also assume that measures μ_j are extremal in the following sense: $\|\mathcal{C}_{\mu_j}\|_{\mu_j} \le 1$ and $\|\mu_j\| \asymp \gamma(E_j)$ with absolute comparison constants. Let $\mu = \sum_j \mu_j$ and $E = \cup E_j$. Then this family is Cauchy independent if and only if for any disc B,*

$$\mu(B) \le C_0 \gamma(B \cap E)\,. \tag{1.3}$$

Remark. In Section 3 we explain that the condition (1.3) with any disc B is necessary for the bound $\|\mathcal{C}_\mu\|_\mu \le C$ without any additional assumptions on the structure of μ. The example given in Section 5 shows that this condition alone is not sufficient even if μ consists of countably many pieces μ_j, and each of μ_j gives a bounded Cauchy operator with a uniform bound. Thus, additional conditions on the structure of μ are needed. The example of such assumptions on μ which seem reasonable is given in Theorem 1.2, where supports of μ_j are located in separated discs.

Remark. Theorem 1.2 is a bit astonishing. It is easier to explain, that under its conditions, there exists a piece of measure μ, namely $\mu' := \chi_{E'} \cdot \mu$, such that $\mu(E') \ge c\,\mu(E)$, and $\|\mathcal{C}_{\mu'}\|_{\mu'} \le C < \infty$, where $c > 0$ and C are constants depending only on parameters in Theorem 1.2. Even this fact is far from being trivial, it requires the full strength of non-homegeneous non-accretice Tb theorem of Nazarov–Treil–Volberg (see [NTrV2], [Vo], [To]), the reader can see this type of considerations on pp. 125–129, 135–146 of the paper of Tolsa [To1] in which Painlevé's conjecture is solved. In other words, to prove that a "good portion" of μ is a Cauchy operator measure is a non-trivial fact in itself. But it is even more remarkable that the whole measure μ is such.

As a corollary we derive the following independence theorem.

Theorem 1.3. *Let $\mu = \Sigma_j \mu_j$ be as above. Assume that measures μ_j are supported on compacts E_j lying in discs D_j such that λD_j are disjoint ($\lambda > 1$). We also assume that measures μ_j are extremal in the sense that $\|\mathcal{C}_{\mu_j}\|_{\mu_j} \le 1$ and $c_1 \mu_j(E_j) \le \gamma(E_j) \le c_2 \mu_j(E_j)$, $0 < c_1 < c_2 < \infty$. If for any disc B,*

$$\sum_j \gamma(B \cap E_j) \le C_1 \gamma(B \cap E), \quad E = \bigcup E_j, \tag{1.4}$$

then the norm $\|\mathcal{C}_\mu\|_\mu$ is bounded, and the bound depends only on comparison constants c_1, c_2, C_1.

Unlike Theorem 1.1, Theorems 1.2, 1.3 do not have any assumptions on the location of discs D_j.

However, if we have λ-separated discs D_j (meaning λD_j are disjoint) and almost additivity (1.4) then it turns ut that we automatically must have a very special geometry of such discs. Let us show this right away (modulo the proof of Theorem 1.3).

Let us recall that the curve Γ on the plane is called Ahlfors regular (with constant A) if for any disc B we have

$$\mathcal{H}^1(\Gamma \cap B) \le A \operatorname{diam} B.$$

Corollary 1.4. *Under the conditions of* (1.4) *for some* $E_j \subset D_j$ *with* $\gamma(E_j) > 0, j = 1, 2, \dots,$ *and* λ*-separation of the discs* D_j, $\lambda > 1$, *there exists an Ahlfors regular curve* Γ, *whose Ahlfors constant depends only on* λ *and on* C_1 *of* (1.4), *such that all discs* D_j *intersect* Γ.

Proof. Let E be a compact set of finite 1-Hausdorff measure: $\mathcal{H}^1(E) < \infty$. Let $c^2(E)$ mean Melnikov–Menger's curvature of measure $\mathcal{H}^1|E$, see [M], [To], [Le]. While proving Theorems 1.2, 1.3 we will construct sets L_j inside $L_j \cap D_j \neq \emptyset$, formed by finitely many straight segments, and such that measure $\mathcal{H}^1|L, L := \bigcup_j L_j$, satisfies

$$c^2(L \cap B) \leq C \operatorname{diam} B, \ \forall \text{ discs } B,$$

where C depends on C_1, λ only. This property turns out to be equivalent to the fact that L is contained in a single Ahlfors regular curve of the plane by a theorem of G. David and S. Semmes, see [DS], [Le].

□

Remark. Now we see that almost additivity of analytic capacity in the form of inequality (1.4) and a "small" geometric condition of λ-separation of ambient discs (with $\lambda > 1$) means that our discs have to "line-up" along a good (Ahlfors regular) curve. Theorem 1.1 claims a sort of a converse statement.

To prove Theorems 1.2, 1.3, we will need only the special case of Theorem 1.1, when Γ is a subset of the real axis or of a circle. In this case there is a short proof based only on some classical facts in complex analysis. We give this proof in the next Section 2. Theorem 1.2 is proved in the Section 3, and Theorem 1.3 in Section 4. In Section 5 we give the example mentioned above. Section 6 contains the proof of Theorem 1.1 in the full generality, which is completely different from the proof in Section 2. The main tool of this proof is Melnikov–Menger's curvature of a measure. All necessary definitions are given in Section 6. In the last Section 7 we formulate an open question.

2. Almost-additivity of analytic capacity: string of beads attached to the real line

A result close to the theorem below for some special sets $\{E_j\}_{j=1}^{\infty}$ was proved (but not stated) in [NV]. Here we use the approach via the Marcinkiewicz function, the approach in [NV] was a bit more complicated. Unlike [NV], we do not need any special size properties of these sets.

Theorem 2.1. *Let* D_j *be discs, each of which has a non-empty intersection with the real line* $\mathbb{R}$, *such that* $\lambda D_j \cap \lambda D_k = \emptyset$, $j \neq k$, *for some* $\lambda > 1$. *Let* $E_j \subset D_j$ *be arbitrary compact sets. Then there exists a constant* $c = c(\lambda) > 0$, *such that*

$$\gamma\big(\bigcup E_j\big) \geq c \sum_j \gamma(E_j).$$

Proof. It is enough to prove the result for finite families of indices j. We first notice that $\gamma_j := \gamma(E_j) \leq \operatorname{diam}(E_j) \leq 2r_j$, where r_j is the radius of D_j. Let y_j be the center of the chord $\mathbb{R} \cap D_j$. Denote $\lambda' := \frac{1+\lambda}{2}$. For each j we draw a horizontal line segment $L_j \subset \lambda' D_j$ with center at y_j and with capacity $b(\lambda)\gamma_j$. We choose and fix $b(\lambda)$: $0 < b(\lambda) \leq \min(\frac{1}{100}, \frac{r_j\sqrt{\lambda'^2-1}}{2}$. Thus, the length ℓ_j of L_j satisfies $\ell_j = \frac{1}{25}\gamma_j < \frac{r_j}{10}$, $\ell_j \leq 2r_j\sqrt{\lambda'^2 - 1}$. In particular, the whole segment L_j lies in $\lambda' D_j$.

Next, let f_j be the function that gives the capacity of E_j. Also let φ_j be the function that gives the capacity of L_j in the following sense:

$$\varphi_j(z) = \int\limits_{L_j} \frac{\varphi_j(x)}{x-z}dx, \qquad \int \varphi_j(x)dx = b(\lambda)\gamma_j.$$

Positive functions $\varphi_j(x)$ have a uniform bound $\|\varphi_j\|_\infty \le A$ with an absolute constant A. In particular, if $\mathcal{F}$ is any subset of indices j we have

$$\left|\operatorname{Im} \sum_{j\in\mathcal{F}} \varphi_j(z)\right| \le A \int_{\cup_{j\in\mathcal{F}} L_j} \frac{|\operatorname{Im} z|}{|t-z|^2}\, dt \le \pi A\,, \quad \forall z \in \mathbb{C}\,. \tag{2.1}$$

Remark. It is important here that the intervals L_j are situated on the real line (or at least are not far away from $\mathbb{R}$). For any $M > 0$ one can easily construct a chord-arc curve and discs centered on it such that the left hand side in (2.1) exceeds M. This is the obstacle for extension of these arguments to chord-arc curves.

Our next goal is to find a family $\mathcal{F}$ of indices and absolute positive constants a_1, a_2, such that the following two assertions hold:

$$\sum_{j\in\mathcal{F}} \gamma_j \ge a_1 \sum_j \gamma_j\,, \tag{2.2}$$

$$\sum_{j\in\mathcal{F}} |f_j(z) - b(\lambda)^{-1}\varphi_j(z)| \le a_2\,, \quad \forall z \in \mathbb{C} \setminus \Big(\bigcup_{j\in\mathcal{F}} (E_j \cup L_j)\Big)\,. \tag{2.3}$$

Let us finish the proof of the theorem, taken these assertions as granted (for a short while). Let $F := \sum_{j\in\mathcal{F}} f_j$. Combining (2.1) and (2.3) we get $|\operatorname{Im} F(z)| \le C_1(\lambda)$, $z \in \mathbb{C} \setminus (\cup_{j\in\mathcal{F}} E_j)$. Hence, the function $F_1 := e^{iF} - 1$ is bounded in $\mathbb{C} \setminus (\cup_{j\in\mathcal{F}} E_j)$ by constant $C(\lambda)$. Since $F(\infty) = 0$, we have $|F_1'(\infty)| = |F'(\infty)| = \sum_{j\in\mathcal{F}} \gamma_j$. Thus,

$$\gamma\Big(\bigcup_{j\in\mathcal{F}} E_j\Big) \ge \frac{a_1}{C(\lambda)} \sum_{j\in\mathcal{F}} \gamma_j\,.$$

Combine this with (2.2). We obtain, that

$$\gamma\Big(\bigcup_j E_j\Big) \ge \gamma\Big(\bigcup_{j\in\mathcal{F}} E_j\Big) \ge a_3 \sum_j \gamma_j\,,$$

and Theorem 1.1 would be proved. So we are left to chose the family $\mathcal{F}$ such that (2.2), (2.3) hold.

By the Schwartz lemma in the form we borrow from [G, p. 12–13], we have

$$|f_j(z) - b(\lambda)^{-1}\varphi_j(z)| \le \frac{A r_j \gamma_j}{\operatorname{dist}(z, E_j \cup L_j)^2}\,, \quad z \notin E_j \cup L_j\,. \tag{2.4}$$

Denote $\lambda_0 = \sqrt{\lambda'^2 - 1}$, $h := (\frac{\lambda}{\lambda'} - 1)$,

$$Q_i := [-hr_i, hr_i] \times (\mathbb{R} \cap D(y_i, \lambda_0 r_i)) \quad g_i := \sum_{j:\, j\ne i} \frac{r_j\gamma_j}{D(Q_j, Q_i)^2}\,,$$

where $D(Q_i, Q_j) := \operatorname{dist}(Q_i, Q_j) + r_i + r_j$. Notice that rectangles Q_i lies entirely in D_i, and moreover, augmented rectangles $(1 + h)Q_i$ are disjoint (it is a simple excercise in trigonometry).

Remark. We do not need this, but for the sake of explanation, let us define a function $g = \sum g_j \chi_{Q_j \cap \mathbb{R}}$. This function is often called a Marcinkiewicz function. The main trick with Marcinkiewicz functions is to integrate them with respect to a suitable measure. What in fact happens next is that we integrate it with respect to Lebesgue measure on $\mathbb{R}$.

The important point is that we can estimate $\sum_i g_i \gamma_i$. In fact,

$$\sum_i g_i\gamma_i = \sum_i \gamma_i \sum_{j\colon j\neq i} \frac{r_j\gamma_j}{D(Q_j,Q_i)^2} = \sum_j r_j\gamma_j \sum_{i\colon i\neq j} \frac{\gamma_i}{D(Q_i,Q_j)^2}$$
$$\leq 2\sum_j r_j\gamma_j \sum_{i\colon i\neq j} \frac{r_i}{D(Q_i,Q_j)^2} \leq A_0 \sum_j r_j\gamma_j r_j^{-1} = A_0(\sum_j \gamma_j).$$

In the last estimate we used that

$$\sum_{i\colon i\neq j} \frac{r_i}{D(Q_i,Q_j)^2} \leq A_0 \int_{t:|t-y_j|\geq r_j} \frac{1}{r_j^2+|t-y_j|^2}\,dt \leq \frac{\pi A_0}{r_j}\,.$$

Now we use the Tchebysheff inequality. Denote $I^* := \{i : g_i > 10A_0\}$, $I_* := \{i : g_i \leq 10A_0\}$. We immediately see that

$$\sum_{j\in I_*} \gamma_j \geq \frac{9}{10}\sum_j \gamma_j\,. \tag{2.5}$$

Obviously, by (2.4) for every index i we have

$$\sum_{j\colon j\neq i} |f_j(z) - b(\lambda)^{-1}\varphi_j(z)| \leq C(\lambda) g_i\,,\ \ z\in\mathbb{C}\setminus \cup_{j\in I_*: j\neq i} Q_j.$$

This estimate and the choice of I_* imply that

$$\sum_{j\colon j\neq i\,,\, j\in I_*} |f_j(z) - b(\lambda)^{-1}\varphi_j(z)| \leq C(\lambda)g_i \leq 10A_0C(\lambda)\,,\quad \forall i\in I_*\,,\ \forall z\in\mathbb{C}\setminus\cup_{j\in I_*: j\neq i}Q_j.\,.$$

But all functions $|f_i|, |\varphi_i|$ are bounded by 1 in $\mathbb{C}\setminus(E_i\cup L_i)$. Therefore, the last inequality implies the estimate

$$\sum_{j\colon j\in I_*} |f_j(z) - b(\lambda)^{-1}\varphi_j(z)| \leq 10A_0C(\lambda) + b(\lambda)^{-1} =: a_2\,,\ \forall z\in\mathbb{C}\setminus\cup_{j\in I_*}(E_j\cup L_j)\,. \tag{2.6}$$

The function $\sum_{j\in I_*}(f_j - b(\lambda)^{-1}\varphi_j)$ is analytic in $\mathbb{C}\setminus\left(\bigcup_{i\in I_*}(E_i\cup L_i)\right)$ and vanishes at infinity. Therefore, (2.6) implies (2.3) if we put $\mathcal{F} := I_*$. Assertion (2.2) is proved in (2.5), and the proof of the theorem is completed. □

Using the conformal map of the half-plane onto the unit disc and an obvious observation on dilations, we have the following corollary.

Corollary 2.2. *The statement of Theorem 2.1 remains true if centers are on a circle instead of being on the real line.*

3. Proof of Theorem 1.2

3.1. **Necessity of the condition** (1.3)**.** Suppose that $\|\mathcal{C}_\mu\|_\mu \le C < \infty$ and $\operatorname{supp}\mu \subset E$. One can easily see that $\|\mathcal{C}_{\mu|B}\|_{\mu|B} \le C < \infty$ for any disc B. Moreover, boundedness of $\mathcal{C}_\mu$ implies that $\alpha\mu \in \Sigma$ with α depending only on C, see for example [Da]. Thus, the measure $c\mu|B$, $c = c(C, \alpha) > 0$, participates in the right hand side of (1.2) with $F = B \cap E$, and we get (1.3).

3.2. **Sufficiency of the condition** (1.3)**.** The following result was proved in [NToV], although was not formulated explicitly (see the last three pages of Section 3 in [NToV]).

Theorem 3.1. *Suppose that $\{D_j\}$ are discs on the plane and the dilated discs λD_j, $\lambda > 1$, are disjoint. Let ν, σ be two positive measures supported in $\cup_j D_j$ such that $c_1\nu(D_j) \le \sigma(D_j) \le c_2\nu(D_j)$, $0 < c_1 < c_2 < \infty$. Suppose also that the Cauchy operators $\mathcal{C}_{\sigma_j}$, $\sigma_j = \sigma|_{D_j}$, are uniformly bounded. Then if ν is a Cauchy operator measure, then σ is also a Cauchy operator measure, and its constant depends only on c_1, c_2, λ and and the constant for measure ν.*

We need some preliminary constructions and notations. First, we define new L_j.

Definition. *We call by cross two perpendicular line segments of equal length intersecting in their centers, one of them being horizontal.*

By $\mathcal{H}^1$ we denote the 1-dimensional Hausdorff measure. Here is an easy lemma.

Lemma 3.2. *For any cross and any disc B,*

$$\gamma(\text{cross} \cap B) \asymp \mathcal{H}^1(\text{cross} \cap B)$$

with absolute constants of comparison.

We now need an number $N = N(\lambda)$, which is defined as follows. Recall that $\lambda > 1$ and $\lambda' = \frac{1+\lambda}{2}$. Let a disc D with radius r be given. We place a cross of length (that is of $\mathcal{H}^1$-measure) less than $\frac{r}{1000}$ in the center of D, and N crosses that touch $\partial(\lambda' D_j)$ on the inside, of the same length as the first cross, and on equal distance from each other. We also require that crosses do not intersect. By L we denote the union of all crosses. Let N be a minimal integer such that the following holds.

If a disc B intersects D and $\mathbb{C} \setminus (\lambda D)$, then at least one cross from L lies inside B.

Clearly, such $N = N(\lambda) < \infty$ exists. The following lemma is almost obvious.

Lemma 3.3. *For the set L defined above it is true that $\gamma(L) \asymp \mathcal{H}^1(L)$, where the comparison constants can depend only on N.*

Proof. Since crosses do not intersect, we have

$$\gamma(L) \asymp \gamma(\text{cross}) \asymp \gamma(\text{horizontal part of the cross}) \asymp \mathcal{H}^1(\text{cross}) = \tfrac{1}{N+1}\mathcal{H}^1(L).$$

□

Let L_j be the union of crosses associated with D_j. We have chosen the number of crosses in each L_j, but we have a freedom to choose their size. We define the size so that

$$\mathcal{H}^1(L_j) = \tfrac{N+1}{1000}\gamma(E_j). \tag{3.1}$$

Then, in particular, $\mathcal{H}^1(\text{one cross}) = \frac{1}{N+1}\mathcal{H}^1(L_j) = \frac{1}{1000}\gamma(E_j) \le \frac{r_j}{1000}$, since $\gamma(E_j) \le \gamma(D_j) = r_j$.

We need the following lemma.

Lemma 3.4. *Fix an index j. Let B be a disc such that at least one cross from L_j lies inside B. Then $\gamma(L_j) \asymp \gamma(L_j \cap B)$ with constants depending only on λ. In particular this is true if $D_j \subset B$, or if B intersects D_j and $\mathbb{C} \setminus \lambda D_j$.*

Proof. Indeed, by semiadditivity of γ we have $\gamma(L_j) \le A\cdot(N+1)\cdot\gamma(\text{central cross}) \le A(N+1)\gamma(L_j \cap B)$. □

Lemma 3.5. *For any disc B the following relation holds with constants depending only on λ:*

$$\gamma\big(\bigcup_{j:D_j\subset B} L_j\big) \asymp \gamma\big(\bigcup_{j:D_j\subset B} L_j \cap B\big).$$

Proof. By semiadditivity of γ,

$$\gamma\big(\bigcup_{D_j\subset B} L_j\big) \le A\Big(\gamma\big(\bigcup_{\lambda' D_j\subset B} L_j\big) + \gamma\big(\bigcup_{D_j\subset B, \lambda' D_j\not\subset B} L_j\big)\Big).$$

The first term is the same as $\gamma(\bigcup_{\lambda' D_j\subset B} L_j \cap B)$. For the second, we use that $L_j \cap B \subset \lambda' D_j$, and thus we can apply Theorem 2.1, or rather Corollary 2.2 prepared in the previous section as $\frac{\lambda}{\lambda'}\lambda' D_j$ are pairwise disjoint, and we can use Corollary 2.2 with just a new dilation constant $\lambda_{new} := \frac{\lambda}{\lambda'}$. Thus

$$\gamma\big(\bigcup_{D_j\subset B, \lambda' D_j\not\subset B} L_j \cap B\big) \ge c \sum_{D_j\subset B, \lambda' D_j\not\subset B} \gamma(L_j \cap B) \ge$$

$$c_1 \sum_{D_j\subset B, \lambda' D_j\not\subset B} \gamma(L_j) \ge c_2\gamma\big(\bigcup_{D_j\subset B, \lambda' D_j\not\subset B} L_j\big),$$

which finishes the proof. The second inequality uses Lemma 3.4, as obviously all L_j for $D_j \subset B$ satisfy this lemma (the central cross definitely lies in B for such discs). □

For a given disc B denote by $\mathcal{J} = \mathcal{J}(B)$ the set of indices $\mathcal{J} := \{j : D_j \cap B \ne \emptyset \text{ and } D_j \not\subset B\}$.

Lemma 3.6. *Suppose that a disc B intersects more than one D_j. Then with absolute constants,*

$$\gamma\big(\bigcup_{\mathcal{J}} L_j\big) \asymp \gamma\big(\bigcup_{\mathcal{J}} L_j \cap B\big).$$

Proof. Here again we will use Corollary 2.2 of Theorem 2.1. Since B intersects more than one D_j, it cannot be contained in λD_j, $j \in \mathcal{J}$. Thus, it contains at least one cross from L_j for each $j \in \mathcal{J}$ (it follows from the choice of N). Call this cross C_j. We apply Corollary 2.2 to D_j with dilation constant λ to get the estimate

$$\gamma\big(\bigcup_{\mathcal{J}} L_j \cap B\big) \ge c\sum_{\mathcal{J}} \gamma(L_j \cap B) \ge \sum_{\mathcal{J}} \gamma(C_j) \ge c_1 \sum_{\mathcal{J}} \gamma(L_j) \ge c_2\gamma\big(\bigcup_{\mathcal{J}} L_j\big),$$

which finishes the proof. □

Finally, we need the following notation. Fix a disc B. Denote

$$F_j = \begin{cases} E_j, & D_j \subset B \\ \emptyset, & D_j \not\subset B. \end{cases}, \quad F = \bigcup F_j.$$

Remark. A disc B will be free to change in what follows. The constants in further inequalities will never depend on B.

Our next goal is to prove that under assumptions of Theorem 1.2, the inequality

$$\gamma\big(\bigcup L_j \cap B\big) \geq c \sum \gamma(L_j \cap B)$$

holds with a universal constant c (universality means that c will not depend on the disc B). We need the following two lemmas.

We fix a small positive absolute constant ε. The choice of smallness will be clear from what follows.

Lemma 3.7 (The first case)**.** *Suppose that $\gamma(F) \leq \varepsilon\gamma(E \cap B)$. Then there exists a constant c, that can depend only on N, ε and other universal constants, such that*

$$\gamma\big(\bigcup L_j \cap B\big) \geq c \sum \gamma(L_j \cap B).$$

Proof. Suppose that B intersects only one λD_j. Then the $\bigcup$ and the $\sum$ have only one term, and there is nothing to prove. So, we can assume that B intersects at least two of λD_j's. Notice also that by this assumption, by the fact that λD_i are pairwise disjoint, and by the choice of N, if B intersects D_j then at least one cross from L_j lies inside B. Let $\mathcal{J}$ be as in Lemma 3.6. Using Lemma 3.4 and Corollary 2.2 we get

$$\sum_{\mathcal{J}} \gamma(L_j) \leq A_1 \sum_{\mathcal{J}} \gamma(L_j \cap B) \leq A_2 \gamma\big(\bigcup_{\mathcal{J}} L_j \cap B\big)\,. \tag{3.2}$$

On the other hand, by the assumption of Theorem 1.2,

$$\sum_{D_j \subset B} \gamma(L_j) \leq C \sum_{D_j \subset B} \gamma(E_j) \leq C' \sum_{D_j \subset B} \mu_j(D_j) \leq C'\mu(B) \leq C'C_0\gamma(E \cap B). \tag{3.3}$$

Also with an absolute constant A,

$$\gamma(E \cap B) \leq A\Big(\gamma(F) + \gamma\big(\bigcup_{\mathcal{J}} E_j \cap B\big)\Big) \leq \varepsilon A\gamma(E \cap B) + A\gamma\big(\bigcup_{\mathcal{J}} E_j \cap B)\big).$$

Thus, if ε is small enough (notice that the smallness depends only on A), we have

$$\gamma(E \cap B) \leq C\gamma\big(\bigcup_{\mathcal{J}} E_j \cap B)\big). \tag{3.4}$$

Therefore, combining (3.3), (3.4), and (3.2), we obtain

$$\begin{aligned} \sum_{D_j \subset B} \gamma(L_j) &\leq C\gamma\big(\bigcup_{\mathcal{J}} E_j \cap B)\big) \leq C_1 \sum_{\mathcal{J}} \gamma(E_j \cap B) \\ &\leq C_2 \sum_{\mathcal{J}} \gamma(L_j) \leq C_3 \gamma\big(\bigcup_{\mathcal{J}} L_j \cap B\big). \end{aligned} \tag{3.5}$$

Now combine (3.2) and (3.5) to get

$$\gamma\big(\bigcup L_j \cap B\big) \geq \gamma\big(\bigcup_{\mathcal{J}} L_j \cap B\big) \geq c \sum_{D_j \subset B} \gamma(L_j) + c \sum_{\mathcal{J}} \gamma(L_j) = c \sum_{D_j \cap B \neq \emptyset} \gamma(L_j)\,. \tag{3.6}$$

Obviously,

$$\gamma\big(\bigcup L_j \cap B\big) \geq \gamma\big(\bigcup_{\mathcal{J}_1} L_j \cap B\big), \quad \mathcal{J}_1 := \{j : D_j \cap B = \emptyset,\ L_j \cap B \neq \emptyset\}. \tag{3.7}$$

For $j \in \mathcal{J}_1$ we again consider the new dilation constants $\lambda_{new} := \frac{\lambda}{\lambda'}$, discs $D'_j := \lambda' D_j$. The discs $\lambda_{new} D'_j$, $j \in \mathcal{J}_1$, are disjoint, and D'_j intersects B for $j \in \mathcal{J}_1$. By Corollary 2.2 applied to L_j, $j \in \mathcal{J}_1$, playing the roles of E_j, we get

$$\gamma\big(\bigcup_{\mathcal{J}_1} L_j \cap B\big) \geq c \sum_{\mathcal{J}_1} \gamma(L_j \cap B)\,. \tag{3.8}$$

The combination of (3.6)–(3.8) finishes the proof. □

Lemma 3.8 (The second case)**.** *Suppose that $\gamma(F) \geq \varepsilon\gamma(E \cap B)$ with ε from the previous lemma. Then there exists a universal constant c such that*

$$\gamma\big(\bigcup L_j \cap B\big) \geq c \sum \gamma(L_j \cap B).$$

Proof. By Theorem 2.1 or rather Corollary 2.2 we need only to prove the inequality

$$\gamma\big(\bigcup_{D_j \subset B} L_j \cap B\big) \geq c \sum_{D_j \subset B} \gamma(L_j \cap B). \tag{3.9}$$

Using the assumption of our lemma as well as the conditions (1.3) and $\|\mu_j\| \asymp \gamma(E_j)$ of Theorem 1.2, we get

$$\gamma(F) \geq \varepsilon\gamma(E \cap B) \geq \varepsilon c \mu(B) \geq \varepsilon c \sum_{D_j \subset B} \mu_j(B) \geq \varepsilon c' \sum_j \gamma(F_j). \tag{3.10}$$

By ν we denote the measure on F participating in (1.2) for which $\|\nu\| \asymp \gamma(F)$. Denote $d\nu_j = \chi_{F_j} d\nu$. Then $\mathcal{C}_{\nu_j}$ is bounded on $L^2(\nu_j)$ (with norm at most 1), and (1.2) yields the estimate

$$\|\nu_j\| \leq C\gamma(F_j) \leq C_1\gamma(L_j) \leq C_2\mathcal{H}^1(L_j) =: C_2\ell_j.$$

We call j **good** if $D_j \subset B$ **and** $\|\nu_j\| \geq \tau\ell_j$. The choice of τ will be clear from the next steps. However, we want to emphasize now that this choice will be universal. By (3.10) we have:

$$\begin{aligned}\varepsilon c' A^{-1} \sum \gamma(F_j) &\leq A^{-1}\gamma(F) \leq \|\nu\| = \sum \|\nu_j\| \\ &\leq C_2 \sum_{j \text{ is good}} \ell_j + \tau \sum_{D_j \subset B} \ell_j \leq C_2 \sum_{j \text{ is good}} \ell_j + C_3\tau \sum \gamma(F_j)\end{aligned}$$

(in the last inequality we use (3.1)). Therefore,

$$\sum_{j \text{ is good},\ D_j \subset B} \ell_j \geq c \sum \gamma(F_j) \geq c_1 \sum_{D_j \subset B} \ell_j. \tag{3.11}$$

Actually, τ is chosen exactly here. We see that it indeed depends only on universal constants such as A and ε. Recall that C_j denotes the central cross of each L_j. We set

$$d\sigma_g := \sum_{j \text{ is good}, \, D_j \subset B} \chi_{C_j} d\mathcal{H}^1, \quad d\nu_g := \sum_{j \text{ is good}, \, D_j \subset B} d\nu_j \, .$$

Then for good j, $\sigma_g(D_j) = \mathcal{H}^1(C_j) \asymp \mathcal{H}^1(L_j) = \ell_j \asymp \nu_g(D_j)$. In the last inequalities the comparison constants can depend on previous universal constants and τ. Operators $\mathcal{C}_{\sigma_g|D_j}$ are uniformly bounded (since the Cauchy operator for each of two intervals in a cross is bounded). By the way we defined ν, the operator $\mathcal{C}_{\nu_g}$ is bounded as well with norm at most 1. Thus, we may apply Theorem 3.1 and conclude that $\mathcal{C}_{\sigma_g}$ is also bounded with a certain absolute bound of the norm. Therefore, using (3.11), we get

$$\gamma\Big(\bigcup_{D_j \subset B} L_j\Big) \geq \gamma\Big(\bigcup_{j \text{ is good}, \, D_j \subset B} L_j\Big) \geq c\|\sigma_g\| \geq c_1 \sum_{j \text{ is good}, \, D_j \subset B} \ell_j \geq c_2 \sum_{D_j \subset B} \ell_j .$$

In Lemma 3.5 we have proved that

$$\gamma\Big(\bigcup_{D_j \subset B} L_j\Big) \asymp \gamma\Big(\bigcup_{D_j \subset B} L_j \cap B\Big).$$

Moreover, for every j such that $D_j \subset B$, we have

$$\ell_j = \mathcal{H}^1(L_j) \asymp \mathcal{H}^1(L_j \cap B) \asymp \gamma(L_j \cap B).$$

Thus, we obtain (3.9), and Lemma 3.8 is proved. $\square$

The main Theorem of [NV] says:

Theorem 3.9. *Let $L \subset \mathbb{R}^2$, be a compact set of positive and finite Hausdorff measure $\mathcal{H}^1$, and let $\sigma = \mathcal{H}^1|L$. Then $\mathcal{C}_\sigma$ is bounded if and only if there exists a finite constant C_0 such that $\sigma(B \cap L) \leq C_0 \gamma(B \cap L)$ for any disc B.*

Starting with the main assumption of Theorem 1.3 (the inequality $\mu(B) \leq C_0\gamma(B \cap E)$ for any disc B) we proved in Lemmas 3.7, 3.8 that the uniform in B almost-additivity of γ holds for the union of all sets $\{L_j \cap B\}$. Namely, we proved that the following holds for any B with uniform positive c_2:

$$\gamma(B \cap L) \geq c_1 \sum_j \gamma(B \cap L_j) \geq c_2 \sum_j \sigma(B \cap L_j) = c_2 \sigma(B \cap L) \, , \tag{3.12}$$

where $\sigma := \mathcal{H}^1|L$. Hence the measure σ satisfies Theorem 3.9. So the boundedness of Cauchy integral on the union of crosses is obtained. The measures $\sigma|L_j$ and μ_j are supported on $\lambda' D_j$, the discs $\frac{\lambda}{\lambda'} \cdot (\lambda' D_j)$ are disjoint, and $\sigma(L_j) \asymp \mu_j$ (see (3.1)). We may apply Theorem 3.1 again to establish the boundedness of $\mathcal{C}_\mu$ in $L^2(\mu)$.

4. Proof of Theorem 1.3

We would like to explain why under the assumptions of Theorem 1.3, the conditions of Theorem 1.2 are satisfied. Indeed, since $\|\mathcal{C}_{\mu_j}\|_{\mu_j} \leq 1$, the relation (1.2) implies that $\mu(B \cap E_j) = \mu_j(B \cap E_j) \leq C\gamma(B \cap E_j)$ for every disc B. Therefore,

$$\mu(B) = \sum \mu(B \cap E_j) \leq C \sum \gamma(B \cap E_j) \leq C_0 \gamma(B \cap E),$$

where the latter is the condition of Theorem 1.2.

5. “Sharpness” of Theorem 1.2

We saw in Section 3 that the condition

$$\mu(B) \le C_0 \gamma(B \cap E) \quad \text{for every disc } B \tag{5.1}$$

is necessary for the boundedness of the Cauchy operator $\mathcal{C}_\mu$ with any Borel measure μ. It is not difficult to see that this condition alone is not enough for the boundedness of $\mathcal{C}_\mu$. Indeed, let $\mu_n^{1/4}$ be the probability measure uniformly distributed on the set $E_n^{1/4}$ defined in Introduction. Let $\mu^{1/4}$ be the weak limit of some weakly convergent subsequence $\{\mu_{n_k}^{1/4}\}$, $E^{1/4} = \bigcap E_n^{1/4}$, E is the initial unit square, and $\mu := \mu^{1/4} + \mathcal{H}^2|E$. Then μ satisfies (5.1), but $\mathcal{C}_\mu$ is unbounded – see for example [MT, MTV]. We are going to demonstrate more: in general the condition (5.1) is not sufficient for the boundedness even if μ consists of countably many pieces, and each of them gives a bounded Cauchy operator.

Proposition 5.1. *There exists a family of measures $\{\mu_j\}_{j=0}^\infty$ with the following properties: (a) $\|\mathcal{C}_{\mu_j}\|_{\mu_j} \le 1$; (b) $\|\mu_j\| \asymp \gamma(E_j)$, where $E_j = \operatorname{supp}\mu_j$; (c) $2E_j \cap 2E_k = \emptyset$, $j \ne k$, $j,k \ge 1$; (d) the measure $\mu = \sum_{j=0}^\infty \mu_j$ satisfies (5.1); (e) $\|\mathcal{C}_\mu\|_\mu = \infty$.*

Proof. We use the idea of David-Semmes (see [VE, Example 8.7] for more detailed exposition). Let $N_0 = 0$, and let $\{N_k\}_{k=0}^\infty$ be a sequence of natural numbers such that $N_{k+1} - N_k \to \infty$ as $k \to \infty$. Start the construction with the unit square E_0 and make $N_1 - N_0$ steps of the construction of the corner 1/4-Cantor set $E^{1/4}$. We get $4^{N_1 - N_0}$ squares with side length 4^{-N_1}. Choose one (any) of them, denote it by Q_1, and continue the construction only with this square. Other $4^{N_1-N_0} - 1$ squares are the sets E_j which have already been defined. For the chosen square Q_1 we make next $N_2 - N_1$ steps of the construction of $E^{1/4}$, obtaining $4^{N_2 - N_1}$ squares with side length 4^{-N_2}. Again, continue the construction only for one of them, say, for a square Q_2, and so on.

Let μ_j, $j = 0, 1, \dots$, be the 2-dimensional measure uniformly distributed on E_j such that $\|\mu_j\| = c\ell_j$, where ℓ_j is the side length of E_j, and the absolute constant c is chosen in such a way that $\|\mathcal{C}_{\mu_j}\|_{\mu_j} = 1$. Then properties (a), (b), (c) are obvious. To demonstrate (d) we notice that $E := \bigcup_{j \ge 0} E_j$ is equal to E_0, and thus $\gamma(B \cap E) \asymp \operatorname{diam}(B \cap E) =: d_0$. On the other hand, for any $j \ge 0$ and $d_j := \operatorname{diam}(B \cap E_j)$, we have $\mu(B \cap E_j) \le c\ell_j^{-1} d_j^2 < C d_j$ (the density of μ_j is c/ℓ_j). Hence, $\mu(B \cap E) < C \sum_{j=0}^\infty d_j \asymp d_0$, and (d) is established.

Finally, to prove (e), we apply the operator $\mathcal{C}_\mu$ to the characteristic functions χ_{Q_k}, $k = 0, 1, \dots$ We have

$$\|\mathcal{C}_\mu(\chi_{Q_k})\|_{L^2(\mu)} = \|\mathcal{C}_{\mu|Q_k}(\mathbf{1})\|_{L^2(\mu)} \ge \|\mathcal{C}_{\mu|Q_k}(\mathbf{1})\|_{L^2(\mu|Q_k)}.$$

But

$$\|\mathcal{C}_{\mu|Q_k}(\mathbf{1})\|^2_{L^2(\mu|Q_k)} \ge c(N_{k+1} - N_k) 4^{-N_k}$$

with an absolute constant c – see [MT]. Hence, $\|\mathcal{C}_\mu\|_\mu \ge c(N_{k+1} - N_k) \to \infty$, and (e) is proved. □

Remark that the measures $\{\mu_j\}_{j=1}^\infty$ satisfy all assumptions of Theorem 1.2 except (5.1). Therefore, we have to add μ_0 and change the structure of μ.

6. Proof of Theorem 1.1

The main result of this section can be related to Theorem 2.1. It is known that a compact chord-arc curve is a bi-lipschitz image of a straight segment, see [Po], Chapter 7. On the other hand analytic capacity can be only finitely distorted by bi-lipschitz maps. This is a difficult result by X. Tolsa, [To3]. So if we allow the separation constant $\lambda > 1$ to depend on the Lipschitz constant of our chord-arc curve (so, the separation of the discs to be large if the constant of the curve is large), then we can obtain Theorem 1.1 directly from Theorem 2.1.

However, we do not want the separation constant to depend on chord-arc constant. Then we need another proof, which follows.

The Melnikov–Menger curvature of a positive Borel measure μ in $\mathbb{C}$ is defined (see [M], [To], [Le]) as

$$c^2(\mu) = \iiint \frac{1}{R^2(x,y,z)}\, d\mu(x)\, d\mu(y)\, d\mu(z),$$

where $R(x,y,z)$ is the radius of the circle passing through points $x, y, z \in \mathbb{C}$, with $R(x,y,z) = \infty$ if x, y, z lie on the same straight line (in particular, if two of these points coincide). This notion was introduced by Melnikov [M]. The following relation characterizes the analytic capacity in terms of the curvature of a measure [To1], [To2, p. 104], [Vo], [To] : for any compact set F in $\mathbb{C}$,

$$\gamma(F) \asymp \sup\{\mu(F) :\ \operatorname{supp}\mu \subset F,\ \mu \in \Sigma,\ c^2(\mu) \le \mu(F)\}, \tag{6.1}$$

where Σ is the class of measures of linear growth defined in (1.2).

Lemma 6.1 (Main Lemma). *Let $D_j = D(x_j, r_j)$ be discs with centers on a chord-arc curve Γ, such that $\lambda D_j \cap \lambda D_k = \emptyset$, $j \ne k$, for some $\lambda > 1$. Let μ_j be positive measures with the following properties: (1)* $\operatorname{supp}\mu_j \subset D_j$*; (2) $\mu_j(B_j) =: \|\mu_j\| \le r_j$. Then for $\mu = \sum \mu_j$ we have*

$$c^2(\mu) \le \sum_j c^2(\mu_j) + C\|\mu\|, \quad C = C(\lambda, A_0), \tag{6.2}$$

where A_0 is the constant of Γ.

At the beginning let us show that Theorem 1.1 is a direct consequence of Main Lemma and (6.1).

Proof of Theorem 1.1. Consider measures μ_j participating in (6.1) for $F = E_j$, $j = 1, \dots$ Then $\mu(D(x,r)) \le Cr$ for any disc D, where $C = C(A_0)$ and $\mu = \sum \mu_j$. To prove this assertion, we fix a disc $D = D(x,r)$ and divide all discs D_j onto two groups: $\mathcal{D}_1 := \{D_j : D_j \cap D \ne \emptyset,\ r_j \le r\}$, $\mathcal{D}_2 := \{D_j : D_j \cap D \ne \emptyset,\ r_j > r\}$. Since Γ is chord-arc, $\sum_{D_j \in \mathcal{D}_1} r_i \le Cr$, $C = C(A_0)$. It is easy to see that $\#\mathcal{D}_2 \le 6$. Hence,

$$\mu(D) \le \sum_{D_j \in \mathcal{D}_1} \mu(D_j) + \sum_{D_j \in \mathcal{D}_2} \mu(D_j \cap D) \le \sum_{D_j \in \mathcal{D}_1} r_j + 6\mu(D) < Cr.$$

Furthermore, Main Lemma implies the inequality $c^2(\mu) \le C\|\mu\|$, $C = C(\lambda, A_0)$. Thus, the measure $c\mu$ with an appropriate constant c depending on λ, A_0, participates in (6.1) for $F = E = \cup E_j$. So, $\gamma(E) \ge c\|\mu\|$, that implies Theorem 1.1. $\square$

Proof of Lemma 6.1. It is enough to consider the case of a finite set of discs B_j, $j = 1, \dots, N$. We assume that these discs are enumerated in the order of increase of the natural parameters of their centers.

Let Γ_j be arcs of Γ such that $\Gamma_j \subset D_j$ and $\mathcal{H}^1(\Gamma_j) = \mu(D_j)$. Let $\sigma_j := \mathcal{H}^1|\Gamma_j$ and $\sigma := \sum \sigma_j$, so that $\sigma(D_j) = \mu(D_j)$. Obviously,

$$c^2(\mu) = \Big(\sum_j \iiint_{D_j^3} + \iiint_{\mathbb{C}^3 \setminus \bigcup_j D_j^3}\Big) \frac{1}{R^2(x,y,z)}\, d\mu(x)\, d\mu(y)\, d\mu(z) =: I_1 + I_2.$$

Since $I_1 = \sum_j c^2(\mu_j)$, we have to estimate only I_2. Our proof is based on the comparison of I_2 and the corresponding integral with respect to σ:

$$\bar I_2 := \iiint_{\mathbb{C}^3 \setminus \bigcup_j D_j^3} \frac{1}{R^2(x,y,z)}\, d\sigma(x)\, d\sigma(y)\, d\sigma(z).$$

Notice that

$$\bar I_2 < c^2(\sigma) \le C\|\sigma\|, \quad C = C(A_0). \tag{6.3}$$

The last inequality is a consequence of two well-known facts. (a) The boundedness of the Cauchy operator $\mathcal{C}_{\mathcal{H}^1|\Gamma}$ on chord-arc curves – see [MV, p. 330]. In particular,

$$\|\mathcal{C}_\sigma^\varepsilon \mathbf{1}\|^2_{L^2(\sigma)} \le \|\mathcal{C}_{\mathcal{H}^1|\Gamma} \chi_{\cup\Gamma_j}\|^2_{L^2(\mathcal{H}^1|\Gamma)} \le C\|\chi_{\cup\Gamma_j}\|^2_{L^2(\mathcal{H}^1|\Gamma)} = C\|\sigma\|, \quad \varepsilon > 0,$$

where C depends only on A_0. (b) The connection between the curvature of a measure and the norm of a Cauchy potential:

$$\|\mathcal{C}_\mu^\varepsilon \mathbf{1}\|^2_{L^2(\mu)} = \frac{1}{6} c_\varepsilon^2(\mu) + O(\|\mu\|)$$

for any measure $\mu \in \Sigma$ uniformly in ε – see for example [To2]. Here $c_\varepsilon^2(\mu)$ is the truncated version of $c^2(\mu)$ defined in the same way as $c_\varepsilon^2(\mu)$, but the triple integral is taken over the set $\{(x,y,z) \in \mathbb{C}^3 : |x-y|, |y-z|, |x-z| > \varepsilon\}$. This equality with $\mu = c\,\sigma \in \Sigma$, and the previous relations imply (6.3).

Obviously,

$$I_2 = \Big(\iiint_{\Omega_1} + \iiint_{\Omega_2}\Big) \frac{1}{R^2(x,y,z)}\, d\mu(x)\, d\mu(y)\, d\mu(z) =: I_{2,1} + I_{2,2}\,,$$

where

$$\Omega_1 := \{D_j \times D_k \times D_l : j = k \ne l \vee j \ne k = l \vee j = l \ne k\},$$
$$\Omega_2 := \{D_j \times D_k \times D_l : j \ne k,\ k \ne l,\ j \ne l\}.$$

To estimate the integral over Ω_1, it's sufficient to consider the subset

$$\Omega_1' := \{D_j \times D_k \times D_l : j \ne k = l\}.$$

For $x \in D_j = D(x_j, r_j),\ y, z \in D_k,\ j \ne k$, we have

$$2R(x,y,z) \ge |x-y| \ge c(r_j + r_{j+1} + \dots + r_k), \quad c = c(\lambda, A_0)$$

(here we assume that $j < k$; the case $k < j$ is analogous). Then

$$\iiint_{\Omega_1'} \frac{1}{R^2(x,y,z)}\, d\mu(x)\, d\mu(y)\, d\mu(z) \le C\Big[\sum_{j=1}^{N-1} \|\mu_j\| \sum_{k=j+1}^{N} \frac{\|\mu_k\|^2}{(r_j + r_{j+1} + \cdots + r_k)^2}$$

$$+ \sum_{j=1}^{N-1} \|\mu_{N+1-j}\| \sum_{k=j+1}^{N} \frac{\|\mu_{N+1-k}\|^2}{(r_{N+1-j} + r_{j+1} + \cdots + r_{N+1-k})^2} =: C[S_{N,1} + S_{N,2}].$$

Estimates for both terms on the right are the same. We estimate $S_{N,1}$ using the induction with respect to N.

1. $N = 2$. Then

$$S_{N,1} = \|\mu_1\| \cdot \frac{\|\mu_2\|^2}{(r_1 + r_2)^2} \le \|\mu_1\| \le \|\mu_1\| + \|\mu_2\|.$$

2. Suppose that the inequality

$$S_{N,1} = \sum_{j=1}^{N-1} \|\mu_j\| \sum_{k=j+1}^{N} \frac{\|\mu_k\|^2}{(r_j + \cdots + r_k)^2} \le \|\mu_1\| + \cdots + \|\mu_N\| \tag{6.4}$$

holds for some $N \ge 2$. For $N + 1$ discs we have

$$S_{N+1,1} = S_{N,1} + \sum_{j=1}^{N} \|\mu_j\| \frac{\|\mu_{N+1}\|^2}{(r_j + \cdots + r_{N+1})^2}$$

$$\le S_{N,1} + \|\mu_{N+1}\|^2 \sum_{j=1}^{N} \frac{r_j}{(r_j + \cdots + r_{N+1})^2}.$$

The last sum is dominated by the integral

$$\int_0^\infty \frac{dt}{(r_{N+1} + t)^2} = \frac{1}{r_{N+1}}.$$

Hence,

$$S_{N+1,1} \le S_{N,1} + \|\mu_{N+1}\|^2 / r_{N+1} \le \|\mu_1\| + \cdots + \|\mu_{N+1}\|.$$

Thus, we proved (6.4) and therefore estimated the triple integral over Ω_1.

By symmetry,

$$\iiint_{\Omega_2} \frac{1}{R^2(x,y,z)}\, d\mu(x)\, d\mu(y)\, d\mu(z) = 6 \iiint_{\Omega_2'} \frac{1}{R^2(x,y,z)}\, d\mu(x)\, d\mu(y)\, d\mu(z),$$

where $\Omega_2' := \{D_j \times D_k \times D_l : j < k < l\}$. Moreover, we may restrict ourself by the integration over

$$\Omega_{2,1}' := \{D_j \times D_k \times D_l : j < k < l,\ r_j + \cdots + r_k \ge \tfrac{1}{2}(r_j + \cdots + r_l)\}.$$

Indeed, if we prove the inequality

$$\iiint_{\Omega_{2,1}'} \frac{1}{R^2(x,y,z)}\, d\mu(x)\, d\mu(y)\, d\mu(z) \le C\|\mu\| \tag{6.5}$$

with $C = C(\lambda, A_0)$, then using the inverse parametrization of Γ, we get the same estimate for the triple integral over

$$\Omega_{2,2}' := \{D_j \times D_k \times D_l : j < k < l,\ r_k + \cdots + r_l \ge \tfrac{1}{2}(r_j + \cdots + r_l)\}$$

(here we use the same numeration of discs as before). Since $\iiint_{\Omega_2'} \le \iiint_{\Omega_{2,1}'} + \iiint_{\Omega_{2,2}'}$, (6.4) and (6.5) imply (6.2).

Fix indices j, k, l. For any triples $(x,y,z), (x',y',z') \in D_j \times D_k \times D_l$, the sine of the angle between the intervals (y,z) and (y',z') does not exceed

$$C\,\frac{r_k + r_l}{r_k + \cdots + r_l}, \quad C = C(\lambda, A_0).$$

For the angle between the intervals (x,z) and (x',z') we have $C\,\frac{r_j+r_l}{r_j+\cdots+r_l}$. Denote by α, α' the angles at z, z' of the triangles x,y,z and x',y',z', correspondingly. Since $\sin(\alpha+\beta+\gamma) \le \sin\alpha + \sin\beta + \sin\gamma$ as $\alpha,\beta,\gamma \in [0,\pi]$, we get the estimate

$$\sin\alpha < \sin\alpha' + C\,\frac{r_k + r_l}{r_k + \cdots + r_l} + C\,\frac{r_j + r_l}{r_j + \cdots + r_l}.$$

Hence,

$$\frac{1}{R(x,y,z)} = \frac{2\sin\alpha}{|x-y|} < \frac{C}{|x'-y'|}\left[2\sin\alpha' + \frac{r_k + r_l}{r_k + \cdots + r_l} + \frac{r_j + r_l}{r_j + \cdots + r_l}\right].$$

Therefore,

$$\begin{aligned}\iiint_{\Omega_{2,1}'} \frac{1}{R^2(x,y,z)}\, d\mu(x)\, d\mu(y)\, d\mu(z) \le C\Bigg[\iiint_{\Omega_{2,1}'} \frac{1}{R^2(x',y',z')}\, d\sigma(x')\, d\sigma(y')\, d\sigma(z')\\ + \sum_{j=1}^{N-2} \|\mu_j\| \sum_{l=j+2}^{N} \sum_{k=j+1}^{l-1} \frac{r_k^2\, \|\mu_k\|\, \|\mu_l\|}{(r_j + \cdots + r_k)^2 (r_k + \cdots + r_l)^2}\\ + \sum_{j=1}^{N-2} \|\mu_j\| \sum_{l=j+2}^{N} \sum_{k=j+1}^{l-1} \frac{r_l^2\, \|\mu_k\|\, \|\mu_l\|}{(r_j + \cdots + r_k)^2 (r_k + \cdots + r_l)^2}\\ + \sum_{j=1}^{N-2} \|\mu_j\| r_j^2 \sum_{l=j+2}^{N} \sum_{k=j+1}^{l-1} \frac{\|\mu_k\|\, \|\mu_l\|}{(r_j + \cdots + r_k)^2 (r_j + \cdots + r_l)^2}\Bigg] =: C[I + S^{(1)} + S^{(2)} + S^{(3)}].\end{aligned}$$

By (6.3), $I \le c^2(\sigma) \le C\|\sigma\|$. We estimate each of sums separately. Write $S^{(1)}$ as

$$S^{(1)} = \sum_{j=1}^{N-2} \|\mu_j\| \sum_{k=j+1}^{N-1} \sum_{l=k+1}^{N} \frac{r_k^2\, \|\mu_k\|\, \|\mu_l\|}{(r_j + \cdots + r_k)^2 (r_k + \cdots + r_l)^2}.$$

Since the inner sum with respect to l does not exceed

$$\frac{r_k^2\, \|\mu_k\|}{(r_j + \cdots + r_k)^2} \int_{r_k}^{\infty} \frac{dx}{x^2} = \frac{r_k\, \|\mu_k\|}{(r_j + \cdots + r_k)^2},$$

we get the estimate

$$\begin{aligned} S^{(1)} &\le \sum_{j=1}^{N-2} \|\mu_j\| \sum_{k=j+1}^{N-1} \frac{r_k\, \|\mu_k\|}{(r_j + \cdots + r_k)^2} \le \sum_{k=2}^{N-1} \sum_{j=1}^{k-1} \frac{r_k\, \|\mu_k\|\, r_j}{(r_j + \cdots + r_k)^2} \\ &\le \sum_{k=2}^{N-1} \|\mu_k\| \sum_{j=1}^{k-1} r_k \int_{r_k}^{\infty} \frac{dx}{x^2} = \sum_{k=2}^{N-1} \|\mu_k\| < \|\mu\|. \end{aligned} \tag{6.6}$$

Now we will use the possibility to consider only those k for which $r_j + \cdots + r_k \geq \frac{1}{2}(r_j + \cdots + r_l)$ (the set of such k can be empty). Suppose that the last inequality holds for $p \leq k \leq l-1$. Then we estimate $S^{(2)}$:

$$\sum_{j=1}^{N-2} \|\mu_j\| \sum_{l=j+2}^{N} \sum_{k=p}^{l-1} \frac{r_l^2 \|\mu_k\| \|\mu_l\|}{(r_j + \cdots + r_k)^2 (r_k + \cdots + r_l)^2}$$
$$\leq 4 \sum_{j=1}^{N-2} \|\mu_j\| \sum_{l=j+2}^{N} \sum_{k=p}^{l-1} \frac{r_l^2 \|\mu_k\| \|\mu_l\|}{(r_j + \cdots + r_l)^2 (r_k + \cdots + r_l)^2}$$
$$\leq 4 \sum_{j=1}^{N-2} \|\mu_j\| \sum_{l=j+2}^{N} \frac{r_l \|\mu_l\|}{(r_j + \cdots + r_l)^2}$$

(we estimate the sum with respect to k in the same way as above). We may deal with the last double sum as in (6.6), or notice that this sum does not exceed

$$\sum_{j=1}^{N-2} \|\mu_j\| \left[1 + \sum_{l=j+1}^{N-1} \frac{r_l \|\mu_l\|}{(r_j + \cdots + r_l)^2}\right].$$

Now change the order of summation and and use (6.6) directly. Finally,

$$S^{(3)} \leq \sum_{j=1}^{N-2} \|\mu_j\| \sum_{l=j+2}^{N} \frac{r_j^2 \|\mu_l\|}{(r_j + \cdots + r_l)^2 r_j} < \sum_{j=1}^{N-2} \|\mu_j\| < \|\mu\|.$$

Lemma 6.1 is proved. □

7. Question on super-additivity

We make more accurate the question posed in Section 1. In Theorems 1.1, 2.1 discs were λ-separated, where $\lambda > 1$. But what if they are just disjoint? Namely, let D_j be circles with centers on a chord-arc curve (or even on the real line $\mathbb{R}$), such that $D_j \cap D_k = \emptyset$, $j \neq k$. Let $E_j \subset D_j$ be arbitrary compact sets. Is it true that there exists a universal constant $c > 0$, such that

$$\gamma\big(\bigcup E_j\big) \geq c \sum_j \gamma(E_j)\,?$$

We cannot either prove or construct a counter-example to this claim.

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