THE s-RIESZ TRANSFORM OF AN s-DIMENSIONAL MEASURE IN $\mathbb{R}^2$ IS UNBOUNDED FOR $1 < s < 2$

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Abstract. In this paper, we prove that for $s \in (1,2)$ there exists no totally lower irregular finite positive Borel measure μ in $\mathbb{R}^2$ with $\mathcal{H}^s(\operatorname{supp}\mu) < +\infty$ such that $\|R\mu\|_{L^\infty(m_2)} < +\infty$, where $R\mu = \mu * \frac{x}{|x|^{s+1}}$ and m_2 is the Lebesgue measure in $\mathbb{R}^2$. Combined with known results of Prat and Vihtilä, this shows that for any non-integer $s \in (0,2)$ and any finite positive Borel measure in $\mathbb{R}^2$ with $\mathcal{H}^s(\operatorname{supp}\mu) < +\infty$, we have $\|R\mu\|_{L^\infty(m_2)} = \infty$.

1. Introduction

Let μ be a finite strictly positive Borel measure on the plane $\mathbb{R}^2$. We will say that μ is s-dimensional if $\mathcal{H}^s(\operatorname{supp}\mu) < +\infty$ where $\mathcal{H}^s$ is the s-dimensional Hausdorff measure. Another way to state it is that there exists some positive $H < +\infty$ such that for every $r > 0$, one can find a (countable) sequence of disks $D_i = D(c_i, r_i)$ with centers c_i and radii r_i such that $r_i < r$ for all i, $\sum_i r_i^s \leqslant H$, and $\mu(\mathbb{R}^2 \setminus \cup_i D_i) = 0$.

An s-dimensional measure μ is called totally lower irregular if

$$\liminf_{r\to 0+} r^{-s}\mu(D(x,r)) = 0 \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^2\,.$$

If ν is a finite (signed) measure on $\mathbb{R}^2$, its (s-dimensional) Riesz transform $R\nu$ is defined by

$$(R\nu)(x) = \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^{s+1}}\,d\nu(y)\,.$$

If $0 < s < 2$, the integral in this definition converges absolutely almost everywhere with respect to the 2-dimensional Lebesgue measure m_2 on $\mathbb{R}^2$. If, in addition to being finite, ν has bounded density with respect to m_2, the integral converges everywhere and is a continuous function on the plane that tends to 0 at infinity.

We will say that $R\nu$ is bounded if $\|R\nu\|_{L^\infty(m_2)} < +\infty$.

Our goal is to complete the proof of the following theorem.

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Theorem. *Let $s \in (0,2)\setminus\{1\}$ and let μ be a strictly positive finite Borel measure in $\mathbb{R}^2$ such that $\mathcal{H}^s(\operatorname{supp}\mu) < +\infty$. Then $\|R\mu\|_{L^\infty(m_2)} = \infty$.*

It is easy to see that for $s = 1, 2$ this statement is incorrect. Indeed, for any nonnegative $\varphi \in C_0^\infty(\mathbb{R}^2)$, the measures $\mu = \varphi\mathcal{H}^1|_L$, where L is some line in $\mathbb{R}^2$, and $\mu = \varphi m_2$ give counterexamples for $s = 1$ and $s = 2$ respectively.

For non-integer $s \in (0, 2)$, the theorem has been known in the following cases.

For $0 < s < 1$, it has been proved by Prat [7] using Melnikov's curvature techniques introduced in [4]. Unfortunately this tool is "cruelly missing" (by the expression of Guy David) for $s > 1$, because the natural analog of the squared Menger curvature can be negative.

For Riesz transforms in $\mathbb{R}^d$ corresponding to non-integer $s \in (0, d)$, the unboundedness of $R\mu$ was established by Vihtilä [11] under the additional assumption that the lower s-density of μ is positive for μ-almost all $x \in \operatorname{supp}\mu$. The main tool in [11] is the concept of the tangent measure. This method also fails in the general case because without any assumptions on lower density, the tangent measure may lose the property of being s-dimensional. On the other hand, [11] gives more than is formally claimed there. The same argument (but if one adds some non-homogeneous Harmonic Analysis consideration like in [12] for example) yields the desired assertion for any finite measure μ such that $\mu\{x : \liminf_{r\to 0+} r^{-s}\mu(D(x,r)) > 0\} > 0$. Thus, to finish the proof of the theorem, it is enough to consider the case of s-dimensional totally irregular measures, which is exactly what we will do in the current paper. This requires introducing several new techniques, which, we hope, may be of independent interest.

Note that our theorem, as well as the results of Prat and Vihtilä, apply to arbitrary s-dimensional measures. When μ is supported on a Cantor set of certain type in $\mathbb{R}^d$, the unboundedness of its Riesz transform follows immediately from explicit bounds for Calderón-Zygmund capacities of Cantor sets in [3], [9],[1], and other similar papers.

It is also worth mentioning that de Villa and Tolsa [8] proved that the Riesz transform of an s-dimensional measure in $\mathbb{R}^d$ cannot have principal values for non-integer s.

2. Definitions and notation

The operator R returns a vector-valued function and is often written as (R_1, R_2) where $R_j\nu$ is the j-th coordinate of $R\nu$ $(j = 1, 2)$. We shall denote by R^* the formal adjoint of R that acts on vector-valued finite measures η by the rule $R^*\eta = -\sum_j R_j\eta_j$ where η_j are the "coordinate measures" of η. The identity

$$\int_{\mathbb{R}^2} \langle R\nu, d\eta\rangle = \int_{\mathbb{R}^2} R^*\eta\, d\nu$$

holds every time when at least one of the finite measures involved has bounded density with respect to m_2 (here $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^2$).

By C with or without an index we shall denote a (large) positive constant that may depend only on s. This constant may change from line to line if it has no index. The indexed constants are fixed throughout the paper and the convention is that C_j can be chosen as soon as all C_i with $i < j$ are known.

For reader's convenience, we will list a few symbols that will occur rather frequently.

s a number in $(1, 2)$;
$D(x, r)$ the disk of radius r centered at x;
$\mathfrak{A} = \{2, 4, 8, 16, \dots\}$ the set of positive integer powers of 2;
ν, η generic measures (possibly signed or even vector-valued);
$N, \varepsilon, M, \delta$ positive parameters to be chosen in this order. N and M are large, ε and δ are small;
μ the totally lower irregular s-dimensional measure with bounded Riesz transform (i.e., the measure whose non-existence we want to prove);
m one half of the total mass of μ;
H twice the Hausdorff measure of the support of μ;
μ' the part of μ obtained by dropping everything supported outside the lowest level of the Cantor construction;
$\widetilde{\mu}$ the mollified μ' with smooth density consisting of small caps supported on $\widetilde{\Omega}_j$;
$\widetilde{\Omega}_j$ the disk of radius $\varepsilon\rho_j$ contained in Ω_j;
Ω_j the $\varepsilon\rho_j$-neighborhood of $\widetilde{B}_j$;
$\widetilde{B}_j = (1-3\varepsilon)B_j \setminus \cup_{i<j} B_i$;
B_j the disks in the bottom cover;
T_j the disks in the top cover;
$\widetilde{T}_j = T_j \setminus \cup_{i<j} T_i$;
ψ the vector-valued function associated with the top cover;
Ψ the majorant of $|\psi|$;
R the Riesz transform;
R^* the adjoint Riesz transform;
$R^\sharp$ the maximal Riesz transform;
$(R^*)^\sharp$ the maximal adjoint Riesz transform;

U the $(s-1)$-dimensional Newton potential;
V a smooth convex version of $\min(|x|^2, |x|)$.

We shall also assume that the reader is familiar with the basic theory of singular integral operators in non-homogeneous spaces.

3. Elementary properties of the Riesz transform

We shall use the following standard facts without any special references.

Translation invariance and scaling. If $f \in L^1(m_2)$, then

$$[R(f(\tfrac{\cdot-c}{r})m_2)](x) = r^{2-s}[R(fm_2)](\tfrac{x-c}{r}), \quad x \in \mathbb{R}^2.$$

Action on the Fourier side.

$$\widehat{R_j(fm_2)}(\xi) = i\sigma \frac{\xi_j}{|\xi|^{3-s}} \widehat{f}(\xi),$$

where $\sigma \neq 0$ is some real constant.

More precisely, if $f, g, \widehat{f}, \widehat{g} \in L^1(m_2) \cap L^\infty(m_2)$, then

$$\int_{\mathbb{R}^2} [R_j(fm_2)]\bar{g}\, dm_2 = i\sigma \int_{\mathbb{R}^2} \frac{\xi_j}{|\xi|^{3-s}} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\, dm_2(\xi).$$

The L^∞ bound. If $\operatorname{supp} f$ is contained in a disk of radius r, then

$$\|R(fm_2)\|_{L^\infty(m_2)} \leqslant C_1 \|f\|_{L^\infty(m_2)} r^{2-s}.$$

Relation to the Newton potential. Let $U\nu(x) = -\frac{1}{s-1}\int_{\mathbb{R}^2} \frac{d\nu(y)}{|x-y|^{s-1}}$. Then

$$R_j \nu = \frac{\partial}{\partial x_j} U\nu.$$

If $\nu = fm_2$ with smooth compactly supported f, we can pass the derivative to f and write

$$R_j \nu = U(\tfrac{\partial f}{\partial x_j} m_2).$$

4. The representation of the standard cap

Let $\varphi_\circ$ be any positive Schwartz function that is at least 1 on the unit disk centered at the origin. Define the vector field $\psi_\circ$ by

$$\widehat{\psi}_\circ(\xi) = i\sigma^{-1}\xi|\xi|^{1-s}\widehat{\varphi}_\circ(\xi).$$

We claim that

$$|\psi_\circ(x)| \leqslant \frac{C_2}{(1+|x|)^{4-s}} \qquad \text{and} \qquad R^*(\psi_\circ m_2) = \varphi_\circ.$$

The second claim follows from the first at once if we check the action of both sides on nice test-functions and pass to the Fourier side (which is justified because $\psi_\circ, \widehat{\psi_\circ} \in L^1(m_2) \cap L^\infty(m_2)$). The first claim is a standard exercise in elementary Fourier analysis left to the reader.

5. The growth bound and its implications

Let μ be a finite positive measure satisfying $\|R\mu\|_{L^\infty(m_2)} \leqslant 1$. Take a disk $D = D(c, r)$ and write

$$\mu(D) \leqslant \int_{\mathbb{R}^2} \varphi_\circ(\tfrac{\cdot - c}{r})\, d\mu = \int_{\mathbb{R}^2} R^*[r^{s-2}\psi_\circ(\tfrac{\cdot - c}{r}) m_2]\, d\mu$$
$$= r^s \int_{\mathbb{R}^2} \langle R\mu, r^{-2}\psi_\circ(\tfrac{\cdot - c}{r})\rangle\, dm_2 \leqslant r^s \|R\mu\|_{L^\infty(m_2)} \|\psi_\circ\|_{L^1(m_2)} \leqslant C_3 r^s\,.$$

This a priori growth bound combined with the assumption

$$\|R\mu\|_{L^\infty(m_2)} \leqslant 1$$

allows one to apply to the measure μ the whole non-homogeneous singular integral machinery (see, e.g.. [6], [5]) and to conclude that the maximal singular operators $f \mapsto R^\sharp(f\mu)$ and $g \mapsto (R^*)^\sharp(g\mu)$ are bounded in $L^2(\mu)$ with norms not exceeding C_4. Here $R^\sharp$ is the maximal Riesz transform defined by

$$(R^\sharp \nu)(x) = \sup_{D : x \in D} \left| \int_{\mathbb{R}^2 \setminus 2D} \frac{x - y}{|x - y|^{s+1}}\, d\nu(y) \right|,$$

where the supremum is taken over all disks D containing x, and $2D$ stands for the disk with the same center as D but of twice larger radius. The operator $(R^*)^\sharp$ is defined in a similar way.

Note that, unlike the initial assumption $\|R\mu\|_{L^\infty(m_2)} \leqslant 1$, the growth bound and the operator norm condition are preserved if we drop any part of the measure μ. In what follows, we will rely on these two conditions only and never use the L^∞ bound itself.

From now on, μ will be a fixed finite measure of total mass $2m$, supported on a set of (s-dimensional) Hausdorff measure $H/2$, and satisfying the growth bound and the operator norm condition above. Note that the growth bound implies that we automatically have $m \leqslant C_3 H$.

6. The good old Cantor set argument

The main motivation for our construction is the following well-known argument for Frostman measures on sparse Cantor squares. Assume that we have a sparse Cantor square K of dimension s on the plane in which the squares of each generation are separated by distances much larger than their diameters.

For $x \in K$, let $K^{(n)}(x)$ be the square of the n-th generation containing x. Let $\nu = \mathcal{H}^s|_K$. Define

$$R^{(n)}\nu(x) = \int_{K^{(n)}(x)\setminus K^{(n+1)}(x)} \frac{x-y}{|x-y|^{s+1}}\, d\nu(y)\,.$$

Then $R^\sharp\nu$ dominates every partial sum $\sum_{n=0}^{N-1} R^{(n)}\nu$.

The key observation is that the norms $\|R^{(n)}\nu\|_{L^2(\nu)}$ are uniformly bounded from below in $L^2(\nu)$ because for every x, the differences $x-y$ point pretty much in the same direction when $y \in K^{(n)}(x) \setminus K^{(n+1)}(x)$ and the kernel blow-up near the diagonal is perfectly balanced with the decay of the measure. On the other hand, the oscillation $\operatorname{osc}_{K_j^{(n+1)}} R^{(n)}\nu$ is very small for every Cantor square $K_j^{(n+1)}$ of the $(n+1)$-st generation and we also have the cancellation property

$$\int_{K_j^{(n+1)}} [R^{(n+1)}\nu]\, d\nu =$$

$$\iint_{x,y\in K_j^{(n+1)},\, K^{(n+2)}(x)\neq K^{(n+2)}(y)} \frac{x-y}{|x-y|^{s+1}}\, d\nu(x)\, d\nu(y) = 0\,.$$

Together they imply that the functions $R^{(n)}\nu$ are almost orthogonal in $L^2(\nu)$, so

$$\int_{\mathbb{R}^2} \Big|\sum_{n=0}^{N-1} R^{(n)}\nu\Big|^2 d\nu \approx \sum_{n=0}^{N-1} \int_{\mathbb{R}^2} |R^{(n)}\nu|^2\, d\nu \approx N\,,$$

and we can conclude that $R^\sharp$ is unbounded in $L^2(\nu)$.

We will use this simple argument as a guideline. The difficulty is that an arbitrary s-dimensional set has no a priori Cantor type structure and an attempt to introduce it using the standard dyadic scales encounters severe difficulties with both almost orthogonality and the lower bounds for $R^{(n)}\mu$. We will use a slightly different partition that gives the almost orthogonality for free in the case of totally lower irregular measures. Still, we will have to fight hard for the lower bounds.

7. The top cover and the associated Ψ-function

Fix $N \in \mathbb{N}$, $\varepsilon > 0$, $M > 1$, $\delta > 0$ to be chosen in this order. The reader should think of N, M as of very large parameters and of ε, δ as of very small ones. Choose some $r^* > 0$. We start with choosing a finite sequence of disks $T_j = D(c_j, r_j)$ such that $r_j \leqslant r^*$, $\sum_j r_j^s \leqslant H$, and $\mu(\mathbb{R}^2 \setminus \cup T_j) < \varepsilon m$. Without loss of generality, we may also assume that the union of the boundaries of T_j has zero μ-measure. Put $\widetilde{T}_j = T_j \setminus \cup_{i<j} T_i$.

Define

$$\psi = \sum_j \mu(\widetilde{T}_j) r_j^{-2} \psi_\circ\left(\tfrac{\cdot - c_j}{r_j}\right)$$

and

$$\Psi_A = \sum_j \frac{\mu(\widetilde{T}_j)}{\pi A^2 r_j^2} \chi_{AT_j}\,, \qquad \Psi = \sum_{A \in \mathfrak{A}} A^{s-2} \Psi_A\,,$$

where $\mathfrak{A} = \{2^k : k \in \mathbb{N}\}$ and χ_E is the characteristic function of the set E.

Note that the pointwise bound for $\psi_\circ$ implies that $|\psi| \leqslant C_5 \Psi$. Also observe that

$$R^*(\psi m_2) = \sum_j \frac{\mu(\widetilde{T}_j)}{r_j^s} \varphi_\circ\left(\tfrac{\cdot - c_j}{r_j}\right) \geqslant \sum_j \frac{\mu(\widetilde{T}_j)}{r_j^s} \chi_{\widetilde{T}_j}\,.$$

Let ν be any finite positive measure supported on $\cup_j T_j$ and satisfying $\nu(\widetilde{T}_j) \leqslant 2\mu(\widetilde{T}_j)$, $\nu(\mathbb{R}^2) \geqslant m$. Write

$$\begin{aligned} C_5 \int_{\mathbb{R}^2} |R\nu| \Psi\, dm_2 &\geqslant \int_{\mathbb{R}^2} \langle R\nu, \psi \rangle\, dm_2 = \int_{\mathbb{R}^2} R^*(\psi m_2)\, d\nu \\ &\geqslant \sum_j \frac{\mu(\widetilde{T}_j)}{r_j^s} \nu(\widetilde{T}_j) \geqslant \frac{1}{2} \sum_j \frac{\nu(\widetilde{T}_j)^2}{r_j^s} \\ &\geqslant \frac{1}{2} \Big(\sum_j \nu(\widetilde{T}_j) \Big)^2 \Big(\sum_j r_j^s \Big)^{-1} \geqslant \frac{m^2}{2H}\,. \end{aligned}$$

On the other hand, we, clearly, have

$$\int_{\mathbb{R}^2} \Psi_A\, dm_2 = \sum_j \mu(\widetilde{T}_j) \leqslant 2m \qquad \text{whence} \qquad \int_{\mathbb{R}^2} \Psi\, dm_2 \leqslant C_6 m\,.$$

8. The function V

Consider any C^∞ function v on $[0, +\infty)$ such that $v(0) = v'(0) = 0$ and v'' is a non-increasing function that is identically 2 on $[0, 1]$ and identically 0 on $[2, +\infty)$. The function $v(t)$ is increasing, convex, equals t^2 on $[0, 1]$, satisfies the inequalities $\min(t, t^2) \leqslant v(t) \leqslant t^2$ and $v' \leqslant 4$ for all $t \geqslant 0$. Also we have $v'(t) = \int_0^t v''(\tau)\, d\tau \geqslant t v''(t)$, that is, $(tv')' \leqslant (2v)'$. Hence, $tv' \leqslant 2v$. Integrating the inequality $\frac{v'(t)}{v(t)} \leqslant \frac{2}{t}$ from t to at, $a \geqslant 1$, we get $v(at) \leqslant a^2 v(t)$. Moreover, we have $v'(t)^2 \leqslant \frac{4v^2(t)}{t^2} \leqslant 4v(t)$.

Define $V(x) = v(|x|)$. In what follows, we will need a good lower bound for the integral $\int_{\mathbb{R}^2} V(R\nu)\Psi\, dm_2$ under the same assumptions on ν as in the previous section. Put $I = \int_{\mathbb{R}^2} \Psi\, dm_2$ and apply Jensen's inequality to get

$$\int_{\mathbb{R}^2} V(R\nu)\Psi\, dm_2 \geqslant Iv\left(I^{-1}\int_{\mathbb{R}^2} |R\nu|\Psi\, dm_2\right) \geqslant Iv\left(I^{-1}\frac{m^2}{2C_5H}\right)$$
$$\geqslant \min\left(\frac{m^2}{2C_5H}, \frac{m^4}{4C_5^2H^2I}\right) \geqslant \min\left(\frac{m^2}{2C_5H}, \frac{m^3}{4C_5^2C_6H^2}\right) \geqslant C_7^{-1}\frac{m^3}{H^2}$$

(we used that $v(t) \geqslant \min(t,t^2)$, $\int_{\mathbb{R}^2} \Psi\, dm_2 \leqslant C_6 m$, and $m \leqslant C_3 H$ here).

9. The Marcinkiewicz g-function

For $A \geqslant 2$, define

$$g_A = A^{-s}\sum_j \frac{\mu(\widetilde{T}_j)}{r_j^s}\chi_{AT_j}\,.$$

We claim that $\int_{\mathbb{R}^2} g_A^2\, d\mu \leqslant C_8 m$. Indeed, let f be any (positive) function with $\|f\|_{L^2(\mu)} = 1$. Then

$$\int_{\mathbb{R}^2} g_A f\, d\mu = \sum_j \mu(\widetilde{T}_j)\frac{1}{(Ar_j)^s}\int_{AT_j} f\, d\mu \leqslant 3^s C_3 \sum_j \mu(\widetilde{T}_j)\frac{1}{\mu(3AT_j)}\int_{AT_j} f\, d\mu$$

because $\mu(3AT_j) \leqslant C_3(3Ar_j)^s$. But the normalized integral factor is dominated by the non-homogeneous Hardy-Littlewood maximal function

$$\mathcal{M}f(x) = \sup_{D:\, x\in D}\frac{1}{\mu(3D)}\int_D f\, d\mu$$

on $\widetilde{T}_j$. Thus, the last sum does not exceed $\int_{\mathbb{R}^2} \mathcal{M}f\, d\mu \leqslant C\sqrt{m}\|\mathcal{M}f\|_{L^2(\mu)} \leqslant C\sqrt{m}$ because the operator norm of $\mathcal{M}$ in $L^2(\mu)$ is bounded by some absolute constant. The desired inequality follows by duality now.

10. The $L^2(\mu)$ bound for the Riesz transform of the Ψ-function

This section is devoted to the proof of the inequality

$$\int_{\mathbb{R}^2} |R(\Psi m_2)|^2\, d\mu \leqslant C_9 m\,.$$

It will suffice to get a uniform bound of the same kind for each Ψ_A ($A \geqslant 2$) separately. We shall compare $R(\Psi_A m_2)$ to $\sum_j \chi_{\mathbb{R}^2\setminus 2AT_j} R(\chi_{\widetilde{T}_j}\mu)$.

Note that the $L^2(\mu)$ norm of the latter is bounded by $C\sqrt{m}$, which can be shown by exactly the same duality argument as in the previous section only with $(R^*)^\sharp$ instead of $\mathcal{M}$.

We start with estimating each difference

$$R\big(\tfrac{\mu(\widetilde{T}_j)}{\pi A^2 r_j^2}\chi_{AT_j} m_2\big) - \chi_{\mathbb{R}^2\setminus 2AT_j} R(\chi_{\widetilde{T}_j}\mu)$$

pointwise. If $x \in 2AT_j$, then only the first term matters and we can use the trivial L^∞ bound

$$\big|R(\tfrac{\mu(\widetilde{T}_j)}{\pi A^2 r_j^2}\chi_{AT_j} m_2)\big| \leqslant C\frac{\mu(\widetilde{T}_j)}{(Ar_j)^s}\,.$$

If $x \notin 2AT_j$, we can use the smoothness of the kernel $\frac{x-y}{|x-y|^{s+1}}$ and the cancellation property of the measure $\frac{\mu(\widetilde{T}_j)}{\pi A^2 r_j^2}\chi_{AT_j} m_2 - \chi_{\widetilde{T}_j}\mu$ to get the bound $C\frac{\mu(\widetilde{T}_j)}{|x-c_j|^s}\frac{Ar_j}{|x-c_j|}$.

Combining these two bounds, we see that the difference under consideration is bounded by

$$C\mu(\widetilde{T}_j)\sum_{A'\in\mathfrak{A},A'\geqslant A}\frac{A}{A'}\frac{1}{(A'r_j)^s}\chi_{A'T_j}\,,$$

which implies that $R(\Psi_A m_2)$ differs from $\sum_j \chi_{\mathbb{R}^2\setminus 2AT_j} R(\chi_{\widetilde{T}_j}\mu)$ by at most $C\sum_{A'\in\mathfrak{A},A'\geqslant A}\frac{A}{A'}g_{A'}$. But the $L^2(\mu)$-norms of the Marcinkiewicz functions $g_{A'}$ are uniformly bounded by $\sqrt{C_8 m}$.

11. The bottom cover

Choose $\rho^* > 0$ so small that the μ-measure of the ρ^*-neighborhood of the union of the boundaries of the top cover disks T_j is less than εm and that $|R(\Psi m_2)|^2(x') - |R(\Psi m_2)|^2(x'') \leqslant 1$ whenever $|x'-x''| \leqslant 3\rho^*$ (note that $|R(\Psi m_2)|^2$ is a continuous function tending to 0 at infinity).

Take any point $x \in \cup_j T_j$ whose distance to the boundary of any T_j is greater than ρ^* and choose some disk $D(x,t_0)$ with $0 < t_0 < \rho^*$ satisfying

$$\mu(D(x,Mt_0)) \leqslant \delta t_0^s\,.$$

According to our assumptions, the points x for which such disk does not exist form a set of μ-measure 0. Now put $t_j = (1-3\varepsilon)^j t_0$ ($j \geqslant 1$). Let $k \geqslant 0$ be the least index such that

$$\mu(D(x,t_k)\setminus D(x,t_{k+1})) \leqslant 6\varepsilon\mu(D(x,t_k))\,.$$

It may happen, of course, that this inequality never holds but in that case

$$\frac{\mu(D(x,t_{j+1}))}{t_{j+1}^2} \leqslant \frac{1-6\varepsilon}{(1-3\varepsilon)^2}\frac{\mu(D(x,t_j))}{t_j^2}$$

for all $j \geqslant 0$. Since $\frac{1-6\varepsilon}{(1-3\varepsilon)^2} < 1$, this implies that at every such point x, the measure μ has zero density with respect to m_2 whence such bad points form a set of μ-measure 0.

Put $\rho(x) = t_k$. We claim that

$$\mu(D(x, M\rho(x))) \leqslant (1-3\varepsilon)^{-s} M^s \delta \rho(x)^s \leqslant 2M^s\delta\rho(x)^s,$$

provided that $\varepsilon < 0.01$, say.

If $M\rho(x) > t_0$, this follows from the choice of t_0 immediately. Otherwise, choose the largest j such that $t_j \geqslant M\rho(x)$. Note that the sequence $\frac{\mu(D(x,t_i))}{t_i^s}$ $(0 \leqslant i \leqslant j)$ is decreasing and its zeroth term is at most δ. Thus

$$\mu(D(x, M\rho(x))) \leqslant \delta t_j^s \leqslant (1-3\varepsilon)^{-s} M^s \delta\rho(x)^s\,.$$

Now use the Besicovitch covering lemma to find a finite sequence of disks $B_j = B(x_j, \rho(x_j))$ that has covering number not exceeding C_9 and covers all points outside an exceptional set of measure at most $3\varepsilon m$ (which includes the points outside $\cup_j T_j$, the points too close to the boundaries, various bad points, and a small extra piece that ensures that the covering is finite rather than countable). We shall write ρ_j instead of $\rho(x_j)$ from now on and assume that the sequence ρ_j is non-increasing.

Let $\widetilde{B}_j = (1-3\varepsilon)B_j \setminus \cup_{i<j} B_i$. Note that the sets $\widetilde{B}_j$ cover all points in the union $\cup_j B_j$ except those that lie in the set $\cup_j (B_j \setminus (1-3\varepsilon)B_j)$ whose μ-measure does not exceed $6\varepsilon \sum_j \mu(B_j) \leqslant 12 C_9 \varepsilon m$.

Another nice property of $\widetilde{B}_j$ is that the distance from $\widetilde{B}_i$ to $\widetilde{B}_j$ is at least $3\varepsilon \max(\rho_i, \rho_j)$ (the ordering of ρ_j was done exactly for this purpose). The sets $\widetilde{B}_j$ are nice but they may be a bit too thin, so let us also introduce for each j the set Ω_j, which is the $\varepsilon\rho_j$-neighborhood of $\widetilde{B}_j$. Ignoring the indices for which $\widetilde{B}_j = \varnothing$, we can say that the sets Ω_j are still well-separated: the distance from each Ω_j to any other Ω_i is at least $\varepsilon\rho_j$, and each set Ω_j contains some disk $\widetilde{\Omega}_j$ of radius $\varepsilon\rho_j$.

The sets Ω_j will be used as the first generation Cantor cells.

12. The full N-level Cantor construction and the associated measure μ'

For the zeroth level, we put $Q_1^{(0)} = \mathbb{R}^2$, $\mu_1^{(0)} = \mu$, $H_1^{(0)} = H$, $m_1^{(0)} = m$.

For the first level, we put $Q_j^{(1)} = \Omega_j$, $\mu_j^{(1)} = \chi_{\widetilde{B}_j}\mu$, $H_j^{(1)} = 2\mathcal{H}^s(\operatorname{supp}\mu_j^{(1)})$, $m_j^{(1)} = \mu_j^{(1)}(\mathbb{R}^2)/2 = \mu(\widetilde{B}_j)/2$.

To get the second level, we repeat the entire construction for each measure $\mu_j^{(1)}$ instead of μ (using the corresponding parameters $m_j^{(1)}$ and $H_j^{(1)}$ instead of m and H but the same ε, M, δ). We shall get some new cells $Q_j^{(2)}$. We can easily ensure that each $Q_j^{(2)}$ is contained in a unique cell $Q_i^{(1)}$ of the previous generation if we choose the radius bound r^* (depending on j) for the top cover of $\mu_j^{(1)}$ small enough (note that $\operatorname{supp}\mu_j^{(1)}$ lies deep inside $Q_j^{(1)}$). It will be also convenient to assume that the radius bound ρ^* for the bottom cover of $\mu_j^{(1)}$ is chosen so that $M\rho^*$ is much less than all the distances from $Q_j^{(1)}$ to all other cells $Q_i^{(1)}$.

Continuing this procedure for N steps, we get a Cantor structure $Q_j^{(n)}$ on the plane ($n = 0, \dots, N$). We define the rarefied measure μ' by

$$\mu' = \sum_j \mu_j^{(N)} .$$

Note that μ' is just the restriction of μ to some subset of the plane.

The important points to keep in mind are the following:

Small measure loss. Since every time we go one level down we get only $C_{10}\varepsilon$-portion of the entire measure outside the next level Cantor cells, we have

$$\mu'(Q_j^{(n)}) \geqslant (1 - C_{10}\varepsilon)^{N-n} \cdot 2m_j^{(n)} \geqslant m_j^{(n)}$$

if we choose ε so small that $(1 - C_{10}\varepsilon)^N \geqslant \frac{1}{2}$.

Subordination. μ' is dominated by $\mu_j^{(n)}$ on $Q_j^{(n)}$.

The total counts. For every fixed $n = 0, \dots, N-1$, we have $\sum_j m_j^{(n)} \geqslant \frac{m}{2}$, $\sum_j H_j^{(n)} \leqslant H$.

13. Partial Riesz potentials $R^{(n)}\mu'$ and the key estimates

For every $x \in \operatorname{supp}\mu'$, denote by $Q^{(n)}(x)$ the unique set $Q_j^{(n)}$ containing x. Put

$$R^{(n)}\mu'(x) = \int_{Q^{(n)}(x)\setminus Q^{(n+1)}(x)} \frac{x-y}{|x-y|^{s+1}}\, d\mu'(y)\,, \qquad n = 0, \dots, N-1\,.$$

The key observation is that, once N is fixed, the other three construction parameters ε, M, δ can be chosen so that the following three claims hold:

Claim 1. On $\operatorname{supp}\mu'$, one has

$$\Big|\sum_{n=0}^{N-1} R^{(n)}\mu'\Big| \leqslant R^{\sharp}\mu' + 1\,.$$

Claim 2. For every $n = 0, \dots, N-2$, one has

$$\Big|\int_{\mathbb{R}^2} \Big\langle R^{(n)}\mu', \sum_{k=n+1}^{N-1} R^{(k)}\mu'\Big\rangle\, d\mu'\Big| \leqslant \frac{m^{5/2}}{4NC_{20}H^2}\sum_{k=n+1}^{N-1}\|R^{(k)}\mu'\|_{L^2(\mu')}\,.$$

Claim 3.

$$\int_{\mathbb{R}^2} |R^{(n)}\mu'|^2\, d\mu' \geqslant C_{20}^{-2}\frac{m^5}{H^4}$$

for all $n = 0, \dots, N-1$.

Once these claims are established, we can finish the argument as follows. On one hand, Claim 1 implies that

$$\int_{\mathbb{R}^2}\Big|\sum_{n=0}^{N-1} R^{(n)}\mu'\Big|^2\, d\mu' \leqslant \int_{\mathbb{R}^2} |R^{\sharp}\mu' + 1|^2\, d\mu \leqslant 2(C_4+1)^2 m\,.$$

On the other hand, expanding the square and combining Claims 2 and 3, we get the lower bound

$$\sum_{n=0}^{N-1}\|R^{(n)}\mu'\|_{L^2(\mu')}\left(\|R^{(n)}\mu'\|_{L^2(\mu')} - \frac{m^{5/2}}{2C_{20}H^2}\right)$$
$$\geqslant \frac{1}{2}\sum_{n=0}^{N-1}\|R^{(n)}\mu'\|^2_{L^2(\mu')} \geqslant \frac{N}{2C_{20}^2}\frac{m^5}{H^4}\,.$$

If $N > 4(C_4+1)^2C_{20}^2\left(\frac{H}{m}\right)^4$, we get a clear contradiction.

14. The proof of Claim 1

Let $x \in \operatorname{supp}\mu'$. Let $Q_j^{(N-1)}$ be the unique Cantor cell from the $(N-1)$-st level containing $Q^{(N)}(x)$. Let B be the disk in the bottom cover of $\mu_j^{(N-1)}$ that gave birth to $Q^{(N)}(x)$. Let ρ be its radius. Recall that the radius bound ρ^* in the construction of the bottom cover for the measure $\mu_j^{(N-1)}$ was chosen much less than the distance from $Q_j^{(N-1)}$ to any other $Q_i^{(N-1)}$, so the disk $2B$ does not intersect any other $Q_i^{(N-1)}$. The value of the sum to estimate at the point x can be written as $\int_{\mathbb{R}^2\setminus Q^{(N)}(x)} \frac{x-y}{|x-y|^{s+1}}\, d\mu'(y)$. It differs from the integral over $\mathbb{R}^2\setminus 2B$ (which is dominated by $(R^\sharp\mu')(x)$ by the definition of the latter) only by the integral

$$\int_{2B\setminus Q^{(N)}(x)} \frac{x-y}{|x-y|^{s+1}}\,d\mu'(y)\,.$$

Now, the integrand is uniformly bounded by $\frac{1}{(\varepsilon\rho)^s}$ and the measure is not greater than $\mu_j^{(N-1)}(2B) \leqslant \mu_j^{(N-1)}(MB) \leqslant 2M^s\delta\rho^s$, provided that $M \geqslant 2$. Thus, the integral is at most 1, provided that $\frac{2M^s\delta}{\varepsilon^s} < 1$.

15. The oscillation bound

Let ν be any finite (signed) measure. Assume that $\Omega \subset \mathbb{R}^2$ is contained in a disk $B = B(x,\rho)$ and is $\varepsilon\rho$-separated from the support of ν. Then

$$\operatorname{osc}_\Omega R\nu \leqslant \frac{2}{(\varepsilon\rho)^s}|\nu|(\tfrac{M}{3}B) + \frac{C_{11}}{M}\sup_{r>0}\frac{|\nu|(D(x,r))}{r^s}\,.$$

Indeed, take $x', x'' \in \Omega$ and notice that the difference

$$\frac{x'-y}{|x'-y|^{s+1}} - \frac{x''-y}{|x''-y|^{s+1}}$$

is bounded by $\frac{2}{(\varepsilon\rho)^s}$ for all $y \in \operatorname{supp}\nu$ and by

$$\frac{C\rho}{|x-y|^{s+1}}$$

for $y \notin \frac{M}{3}B$ if $M \geqslant 6$, say. Integrating the first bound over $\frac{M}{3}B$ and the second one over its complement with respect to $|\nu|$, we get the desired estimate.

We will also need the dual form of this estimate, which says that if ν is a finite positive measure and η is a signed measure supported on Ω with perfect cancellation ($\eta(\Omega) = 0$), then

$$\int_{\mathbb{R}^2} |R\eta|\,d\nu \leqslant \left[\frac{2}{(\varepsilon\rho)^s}\nu(\tfrac{M}{3}B) + \frac{C_{11}}{M}\sup_{r>0}\frac{\nu(D(x,r))}{r^s}\right]|\eta|(\Omega)\,.$$

Similar bounds (with the same proofs, but, possibly, slightly larger constants) hold for R^* instead of R.

16. Proof of Claim 2

Apply the obtained oscillation bound to $\Omega = Q_j^{(n+1)} \subset Q_i^{(n)}$ and the measure $\nu = \chi_{Q_i^{(n)}}\mu'$ which is dominated by $\mu_i^{(n)}$. Let B be the disk in the bottom cover of $\mu_i^{(n)}$ that gave birth to the Cantor cell $Q_j^{(n+1)}$. Then the first term in the oscillation bound does not exceed $\frac{4M^s\delta}{\varepsilon^s}$ and the second term is bounded by $\frac{C_3C_{11}}{M}$. Thus, for every Cantor cell $Q_j^{(n+1)}$ of the $(n+1)$-st generation,

$$\operatorname{osc}_{Q_j^{(n+1)}} R^{(n)}\mu' \leqslant C_{12}\bigg(\frac{M^s\delta}{\varepsilon^s}+\frac{1}{M}\bigg).$$

On the other hand, the sum $\sum_{k=n+1}^{N-1} R^{(k)}\mu'$ has the cancellation property

$$\int_{Q_j^{(n+1)}}\Big[\sum_{k=n+1}^{N-1} R^{(k)}\mu'\Big]\,d\mu'$$
$$=\iint_{x,y\in Q_j^{(n+1)},Q^{(N)}(x)\neq Q^{(N)}(y)} \frac{x-y}{|x-y|^{s+1}}\,d\mu'(x)\,d\mu'(y)=0\,.$$

Thus

$$\Big|\int_{\mathbb{R}^2}\Big\langle R^{(n)}\mu', \sum_{k=n+1}^{N-1} R^{(k)}\mu'\Big\rangle\,d\mu'\Big| \leqslant C_{12}\left(\frac{M^s\delta}{\varepsilon^s}+\frac{1}{M}\right)\sum_{k=n+1}^{N-1}\|R^{(k)}\mu'\|_{L^1(\mu')}$$
$$\leqslant C_{12}\left(\frac{M^s\delta}{\varepsilon^s}+\frac{1}{M}\right)\sqrt{2m}\sum_{k=n+1}^{N-1}\|R^{(k)}\mu'\|_{L^2(\mu')}$$

by Cauchy-Schwarz, and Claim 2 will follow if M and δ satisfy

$$C_{12}\left(\frac{M^s\delta}{\varepsilon^s}+\frac{1}{M}\right)\sqrt{2}\leqslant \frac{m^2}{4NC_{20}H^2}\,.$$

17. The maximum principle

Suppose that η is a vector-valued measure with compactly supported C^∞ density with respect to m_2. Then

$$\max_{\mathbb{R}^2} R^*\eta = \max_{\operatorname{supp}\eta} R^*\eta$$

provided that the left hand side is positive.

Indeed, the function $u=R^*\eta$ can be written as the $(s-1)$-dimensional Newton potential $U\nu$ where ν is some scalar signed measure with compactly supported C^∞ density with respect to m_2 satisfying $\operatorname{supp}\nu\subset\operatorname{supp}\eta$ (see Section 3). We need the well-known fact that the density p of ν can be recovered from the potential $u=U\nu$ by the formula

$$p(x)=\sigma\int_{\mathbb{R}^2}\frac{u(x+y)-u(x)}{|y|^{5-s}}\,dm_2(y),$$

where σ is some non-zero real number and the integral is understood in the principal value sense.

To demonstrate it, define the class S_γ ($\gamma > 0$) of smooth functions in $\mathbb{R}^d$ by

$$S_\gamma = \{\varphi \in C^\infty(\mathbb{R}^d) : \varphi(x) = O(|x|^{-\gamma}) \text{ as } |x| \to \infty\}.$$

For $0 < \operatorname{Re}\alpha < \gamma$ and $\varphi \in S_\gamma$, define

$$K_\alpha \varphi = A(d,\alpha)\, \varphi * \frac{1}{|x|^{d-\alpha}} \quad \text{with} \quad A(d,\alpha) = \pi^{\alpha - \frac{d}{2}} \frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}.$$

Note that for every $x \in \mathbb{R}^d$, $K_\alpha\varphi(x)$ is analytic in α in the strip $0 < \operatorname{Re}\alpha < \gamma$. The argument on pages 45–46 in [2] shows that $K_\alpha\varphi(x)$ extends analytically to the wider strip $-2 < \operatorname{Re}\alpha < \gamma$ and is given for $\operatorname{Re}\alpha < 0$ by the formula

$$K_\alpha\varphi(x) = A(d,\alpha) \int_{\mathbb{R}^d} \frac{\varphi(x+y) - \varphi(x)}{|y|^{d-\alpha}}\, dy,$$

where the integral converges absolutely for $\operatorname{Re}\alpha > -1$ and should be understood as the principal value for $-2 < \operatorname{Re}\alpha \leqslant -1$. Note that Landkof writes p instead of d and $k_\alpha *$ instead of K_α. Also, if $\varphi \in C_0^\infty(\mathbb{R}^d)$, then, for $\operatorname{Re}\alpha \in (0,d)$, we have $K_\alpha\varphi \in S_{d-\operatorname{Re}\alpha}$.

It is well-known that for $\varphi \in C_0^\infty(\mathbb{R}^d)$, we have

$$K_\alpha K_\beta\, \varphi = K_{\alpha+\beta}\, \varphi, \quad \alpha, \beta > 0, \quad \alpha + \beta < d.$$

Also $K_0\varphi = \varphi$ (see [2], p. 46). Now, consider the identity $K_\beta K_\alpha\, \varphi = K_{\beta+\alpha}\, \varphi$ for $0 < \alpha < \min(2,d)$ and $0 < \beta < d - \alpha$. Viewing both parts of this identity as analytic functions of β in the strip $0 < \operatorname{Re}\beta < d-\alpha$, we conclude that it holds in the entire strip and continues to hold in the wider strip $-2 < \operatorname{Re}\beta < d - \alpha$ for the analytic extensions. Plugging $\beta = -\alpha$, we obtain $K_{-\alpha}K_\alpha\, \varphi = \varphi$ for $0 < \alpha < \min(2,d)$, which is equivalent to our reproduction formula for $d = 2$, $\alpha = 3 - s$.

In particular, we can conclude that the integral on the right hand side vanishes for all $x \notin \operatorname{supp}\nu$. Now, since u is smooth and tends to 0 at infinity, the point of maximum is guaranteed to exist if the maximum is positive. But then at the point of maximum, the integral is certainly negative because the integrand is non-positive everywhere and negative for all sufficiently large $y \in \mathbb{R}^2$. Thus the point of maximum must belong to $\operatorname{supp}\nu \subset \operatorname{supp}\eta$, proving the claim.

We shall need a slightly more general fact below. If ν is a finite positive measure with compactly supported C^∞ density with respect to m_2, and g is any C^∞ vector-valued function, then

$$\max_{\mathbb{R}^2}[V(R\nu) + R^*(g\nu)] = \max_{\operatorname{supp}\nu}[V(R\nu) + R^*(g\nu)],$$

provided that the left hand side is positive.

Indeed, we can write $v(t) = \max_{\tau \geqslant 0}[\tau t - v^*(\tau)]$ where v^* is the Legendre transform of v (all we really need to know is that $v^* \geqslant 0$). Thus,

$$V(x) = \max_{\tau \geqslant 0,\, |e|=1} [\tau \langle e, x \rangle - v^*(\tau)]$$

and

$$V(R\nu) + R^*(g\nu) = \max_{\tau \geqslant 0,\, |e|=1} [R^*((g - \tau e)\nu) - v^*(\tau)] .$$

Again, if the maximum is positive, it is attained at some point x and equals to the value of $R^*((g-\tau e)\nu) - v^*(\tau)$ at x for some τ, e. But then the maximum of $R^*((g-\tau e)\nu)$ is also positive and is attained at some point $y \in \operatorname{supp} \nu$. The chain of inequalities

$$\begin{aligned}[V(R\nu) + R^*(g\nu)](y) &\geqslant [R^*((g-\tau e)\nu)](y) - v^*(\tau) \\ &\geqslant [R^*((g-\tau e)\nu)](x) - v^*(\tau) = [V(R\nu) + R^*(g\nu)](x)\end{aligned}$$

finishes the argument.

It will be convenient to restate the last result in the following form. If $\Lambda > 0$ and $V(R\nu) + R^*(g\nu) \leqslant \Lambda$ on $\operatorname{supp} \nu$, then $V(R\nu) + R^*(g\nu) \leqslant \Lambda$ on the entire plane.

Note that this part fails dramatically for $s < 1$ because the density reproduction formula then becomes more complicated and involves the Laplacian $\Delta u(x)$, which is (or, at least, seems) totally out of control.

18. The mollified measure $\widetilde{\mu}$

We now return to the zeroth level of the Cantor structure and to the notation of Sections 7–11. For each disk $\widetilde{\Omega}_j$, choose some positive C^∞ cap φ_j such that $\operatorname{supp} \varphi_j \subset \widetilde{\Omega}_j$, $\|\varphi_j\|_{L^\infty(m_2)} \leqslant \frac{\mu'(\Omega_j)}{(\varepsilon \rho_j)^2}$, and $\int_{\mathbb{R}^2} \varphi_j \, dm_2 = \mu'(\Omega_j)$. Put $\widetilde{\mu}_j = \varphi_j m_2$ and $\widetilde{\mu} = \sum_j \widetilde{\mu}_j$.

Our first task will be to get a decent growth bound for $\widetilde{\mu}$. Take any disk $D = D(x, r)$. Write

$$\widetilde{\mu}(D) = \sum_{j: \rho_j < r} \widetilde{\mu}(D \cap \Omega_j) + \widetilde{\mu}(D \cap (\cup_{j:\rho_j \geqslant r} \Omega_j)) .$$

Recall that Ω_j are disjoint (and even well-separated). Also note that every Ω_j with $\rho_j < r$ that intersects D is contained in $3D$, whence the first sum does not exceed

$$\sum_{j: \Omega_j \subset 3D} \widetilde{\mu}(\Omega_j) = \sum_{j: \Omega_j \subset 3D} \mu'(\Omega_j) \leqslant \mu'(3D) .$$

On the other hand, on each Ω_j with $\rho_j \geqslant r$, the density of the measure $\widetilde{\mu}$ with respect to m_2 is bounded by

$$\frac{\mu'(\Omega_j)}{(\varepsilon\rho_j)^2} \leqslant \frac{\mu(B_j)}{(\varepsilon\rho_j)^2} \leqslant \frac{\mu(MB_j)}{(\varepsilon\rho_j)^2} \leqslant \frac{2M^s\delta}{\varepsilon^2}\rho_j^{s-2} \leqslant \frac{2M^s\delta}{\varepsilon^2} r^{s-2}\,,$$

so the second term is at most $\frac{2\pi M^s\delta}{\varepsilon^2}r^s$. This yields the final growth bound

$$\widetilde{\mu}(D) \leqslant \mu'(3D) + \frac{2\pi M^s\delta}{\varepsilon^2}r^s\,,$$

which can be used in two ways. First, choosing δ so that $\frac{2\pi M^s\delta}{\varepsilon^2} < 1$, we conclude that $\widetilde{\mu}(D) \leqslant (3^sC_3+1)r^s = C_{13}r^s$ for all disks D. Second, taking $D = \frac{M}{3}B_j$, we conclude that

$$\widetilde{\mu}(\tfrac{M}{3}B_j) \leqslant 2M^s\delta\rho_j^s + \frac{2\pi M^s\delta}{\varepsilon^2}\left(\frac{M}{3}\rho_j\right)^s \leqslant \frac{9M^{2s}\delta}{\varepsilon^2}\rho_j^s\,.$$

We shall use these bounds in combination with the results of Section 15 in the next section. Now let us point out one more nice property of $\widetilde{\mu}$, which (in addition to having an infinitely smooth density) is its great advantage over the unmollified measure μ': for every j,

$$\|R(f\widetilde{\mu}_j)\|_{L^\infty(m_2)} \leqslant C(\varepsilon\rho_j)^{2-s}\frac{\mu'(\Omega_j)}{(\varepsilon\rho_j)^2}\|f\|_{L^\infty(m_2)} \leqslant C_{14}\frac{M^s\delta}{\varepsilon^s}\|f\|_{L^\infty(m_2)}\,.$$

The same bound holds for R^* as well.

19. The operator $\widetilde{R}$ and the mollified lower bound problem

For a (signed) measure ν supported on $\cup_j\Omega_j$ and a point $x \in \Omega_j$, define $(\widetilde{R}\nu)(x) = (R(\chi_{\mathbb{R}^2\setminus\Omega(x)}\nu))(x)$ where $\Omega(x)$ is the unique Ω_j containing x. Note that $\widetilde{R}\mu' = R^{(0)}\mu'$, of course. The reason we introduce this new notation now is that we want to view $\widetilde{R}$ as an operator while $R^{(0)}\mu'$ was rather a complex notation for a single function.

We want to compare $\int_{\mathbb{R}^2} V(\widetilde{R}\mu')\,d\mu'$ with $\int_{\mathbb{R}^2} V(R\widetilde{\mu})\,d\widetilde{\mu}$ now. One remark about the notation may be in order. It would be slightly more accurate to say that the integrals are taken over $\cup_j\Omega_j$ because $\widetilde{R}\nu$ is defined only there. Nevertheless, since we will integrate the expressions involving $\widetilde{R}$ exclusively with respect to measures supported on $\cup_j\Omega_j$, we can view the integrals over $\mathbb{R}^2$ just as integrals of functions defined almost everywhere rather than everywhere.

The comparison will be done in three steps.

Step 1. Since V is Lipschitz with the Lipschitz constant 4, we have

$$\int_{\mathbb{R}^2} |V(\widetilde{R}\mu') - V(\widetilde{R}\widetilde{\mu})|\, d\mu' \leqslant 4 \int_{\mathbb{R}^2} |\widetilde{R}\mu' - \widetilde{R}\widetilde{\mu}|\, d\mu'\,.$$

Let $\eta_j = \chi_{\Omega_j}\mu' - \widetilde{\mu}_j$. Note that $\widetilde{R}\eta_j = 0$ on Ω_j. Applying the dual form of the oscillation bound from Section 15 with $\eta = \eta_j$, $\nu = \chi_{\mathbb{R}^2\setminus\Omega_j}\mu'$, we get

$$\int_{\mathbb{R}^2} |\widetilde{R}\eta_j|\, d\mu' \leqslant 2\left(\frac{2M^s\delta}{\varepsilon^s} + \frac{C_{11}C_3}{M}\right)\mu'(\Omega_j)\,.$$

Adding these estimates up, we conclude that

$$\int_{\mathbb{R}^2} |\widetilde{R}\mu' - \widetilde{R}\widetilde{\mu}|\, d\mu' \leqslant C\left(\frac{M^s\delta}{\varepsilon^s} + \frac{1}{M}\right) m$$

and the same estimate holds for $\int_{\mathbb{R}^2} |V(\widetilde{R}\mu') - V(\widetilde{R}\widetilde{\mu})|\, d\mu'$.

Step 2. The oscillation bound, combined with the growth bounds for $\widetilde{\mu}$ from the previous section, implies that

$$\operatorname{osc}_{\Omega_j} V(\widetilde{R}\widetilde{\mu}) \leqslant 4 \operatorname{osc}_{\Omega_j} \widetilde{R}\widetilde{\mu} \leqslant C\left(\frac{M^{2s}\delta}{\varepsilon^{2+s}} + \frac{1}{M}\right),$$

so, since $\widetilde{\mu}(\Omega_j) = \mu'(\Omega_j)$ for all j, we have

$$\left|\int_{\mathbb{R}^2} V(\widetilde{R}\widetilde{\mu})\, d\mu' - \int_{\mathbb{R}^2} V(\widetilde{R}\widetilde{\mu})\, d\widetilde{\mu}\right| \leqslant C\left(\frac{M^{2s}\delta}{\varepsilon^{2+s}} + \frac{1}{M}\right) m\,.$$

Step 3. Finally, recalling that $\|R\widetilde{\mu}_j\|_{L^\infty(m_2)} \leqslant C_{14}\frac{M^s\delta}{\varepsilon^s}$ (see Section 18), we observe that

$$\int_{\mathbb{R}^2} |V(\widetilde{R}\widetilde{\mu}) - V(R\widetilde{\mu})|\, d\widetilde{\mu} \leqslant 4C_{14}\frac{M^s\delta}{\varepsilon^s} m\,.$$

Bringing all the above inequalities together, we obtain

$$\int_{\mathbb{R}^2} V(\widetilde{R}\mu')\, d\mu' \geqslant \int_{\mathbb{R}^2} V(R\widetilde{\mu})\, d\widetilde{\mu} - C_{15}\left(\frac{M^{2s}\delta}{\varepsilon^{2+s}} + \frac{1}{M}\right) m\,.$$

20. The family of measures $\widetilde{\mu}^\alpha$ and the extremal problem

The direct estimate of $\int_{\mathbb{R}^2} V(R\widetilde{\mu})\, d\widetilde{\mu}$ is still a hard task because, despite we know that $V(R\widetilde{\mu})$ has noticeable values on the plane, our maximum principle, if we apply it to $V(R\widetilde{\mu})$ directly, allows us only to conclude that $V(R\widetilde{\mu})$ is not too small at some point on the support of $\widetilde{\mu}$, which seems next to useless for estimating any integral norm.

What saves the day is the idea of the equilibrium measure borrowed from the positive symmetric kernel capacity theory. Instead of proving the above energy type inequality for the original measure, we prove it for the "energy minimizer" $\widetilde{\mu}^a$ whose potential is, in some sense, almost constant on $\operatorname{supp}\widetilde{\mu}^a$, so an L^∞ lower bound translates into an integral lower bound automatically. The idea that the singular Riesz potential of extremal measure should be "almost constant" (like in the classical potential theory with positive kernel) was somewhat explored in Section 5.2 of [13]. For $d = 2, s = 1$ the Cauchy potential was replaced by Menger's curvature potential which is again strange but positive kernel, see Tolsa's [10].

To carry out the formal argument, consider all vectors $\alpha = \{\alpha_j\}$ with non-negative entries and define $\widetilde{\mu}^\alpha = \sum_j \alpha_j \widetilde{\mu}_j$. Fix $\lambda > 0$ and consider the functional

$$\Phi(\alpha) = \lambda m \max_j \alpha_j + \int_{\mathbb{R}^2} V(R\widetilde{\mu}^\alpha)\, d\widetilde{\mu}^\alpha\,.$$

Let a be the minimizer of $\Phi(\alpha)$ under the constraint $\widetilde{\mu}^\alpha(\mathbb{R}^2) = \widetilde{\mu}(\mathbb{R}^2)$ (recall that $\widetilde{\mu}(\mathbb{R}^2) \in [m, 2m]$). The minimizer exists because $\Phi(\alpha)$ is a continuous function of α tending to $+\infty$ as $\max_j \alpha_j \to +\infty$.

Let us assume that $\int_{\mathbb{R}^2} V(R\widetilde{\mu})\, d\widetilde{\mu} \leqslant \lambda m$. Then $\Phi(a) \leqslant 2\lambda m$ whence all $a_j \leqslant 2$, so the extremal measure $\widetilde{\mu}^a$ is dominated by $2\widetilde{\mu}$.

Now let us fix any j with $a_j > 0$, take a small $t > 0$, and try to replace $\widetilde{\mu}^a$ by $[1 - t\widetilde{\mu}(\mathbb{R}^2)^{-1}\widetilde{\mu}_j(\mathbb{R}^2)]^{-1}(\widetilde{\mu}^a - t\widetilde{\mu}_j)$, which is also an admissible measure.

If we just subtract $t\widetilde{\mu}_j$ without the renormalization, $\max_j a_j$ will not increase and the integral part will change in the first order by

$$\begin{aligned} -\,t\Big[\int_{\mathbb{R}^2} V(R\widetilde{\mu}^a)\, d\widetilde{\mu}_j + \int_{\mathbb{R}^2} \langle \nabla V(R\widetilde{\mu}^a), R\widetilde{\mu}_j\rangle\, d\widetilde{\mu}^a\Big] \\ = -t\int_{\mathbb{R}^2} [V(R\widetilde{\mu}^a) + R^*(\nabla V(R\widetilde{\mu}^a)\widetilde{\mu}^a)]\, d\widetilde{\mu}_j = -tI\,. \end{aligned}$$

Since the renormalization can raise the value of any part of $\Phi(a)$ at most $[1 - t\widetilde{\mu}(\mathbb{R}^2)^{-1}\widetilde{\mu}_j(\mathbb{R}^2)]^{-3}$ times, we should have

$$\big[1 - t\widetilde{\mu}(\mathbb{R}^2)^{-1}\widetilde{\mu}_j(\mathbb{R}^2)\big]^{-3}(\Phi(a) - tI) \geqslant \Phi(a) - o(t) \qquad \text{as } t \to 0+\,,$$

whence

$$I \leqslant 3\Phi(a)\widetilde{\mu}(\mathbb{R}^2)^{-1}\widetilde{\mu}_j(\mathbb{R}^2) \leqslant 6\lambda\widetilde{\mu}_j(\mathbb{R}^2)$$

because $\Phi(a) \leqslant 2\lambda m$ and $\widetilde{\mu}(\mathbb{R}^2) \geqslant m$.

Thus, $V(R\widetilde{\mu}^a) + R^*(\nabla V(R\widetilde{\mu}^a)\widetilde{\mu}^a)$ is at most 6λ on Ω_j on average (with respect to the measure $\widetilde{\mu}_j$). Now notice that $|\nabla V| \leqslant 4$ and $\widetilde{\mu}^a$ may have the growth bounds only twice worse than those for $\widetilde{\mu}$. The immediate conclusion is that

$$\operatorname{osc}_{\Omega_j}[V(R\widetilde{\mu}^a) + R^*(\nabla V(R\widetilde{\mu}^a)\widetilde{\mu}^a)] \leqslant C_{16}\left(\frac{M^{2s}\delta}{\varepsilon^{2+s}} + \frac{1}{M}\right)$$

(compare with Sections 16 and 19).

So

$$V(R\widetilde{\mu}^a) + R^*(\nabla V(R\widetilde{\mu}^a)\widetilde{\mu}^a) \leqslant 6\lambda + C_{16}\left(\frac{M^{2s}\delta}{\varepsilon^{2+s}} + \frac{1}{M}\right) = 6\lambda + \beta$$

on the entire Ω_j and, since j was chosen arbitrarily, on $\operatorname{supp}\widetilde{\mu}^a$. But then this estimate automatically extends to the entire plane by the maximum principle.

Integrating it against Ψdm_2, we get

$$(6\lambda + \beta)\int_{\mathbb{R}^2}\Psi\, dm_2 \geqslant \int_{\mathbb{R}^2} V(R\widetilde{\mu}^a)\Psi\, dm_2 + \int_{\mathbb{R}^2} R^*(\nabla V(R\widetilde{\mu}^a)\widetilde{\mu}^a)\Psi\, dm_2\,.$$

21. Proof of Claim 3

Now it is time to bring up everything we know about the top cover and the associated Ψ-function in one final effort. First, we have seen in Section 7 that

$$\int_{\mathbb{R}^2}\Psi\, dm_2 \leqslant C_6 m\,.$$

Second, the measure $\widetilde{\mu}^a$ satisfies the assumptions on the measure ν in Sections 7, 8. Thus

$$\int_{\mathbb{R}^2} V(R\widetilde{\mu}^a)\Psi\, dm_2 \geqslant C_7^{-1}\frac{m^3}{H^2}\,.$$

Third, the last remaining integral can be rewritten as

$$\int_{\mathbb{R}^2}\langle R(\Psi m_2), \nabla V(R\widetilde{\mu}^a)\rangle\, d\widetilde{\mu}^a$$

which, by Cauchy-Schwarz, does not exceed

$$\left(\int_{\mathbb{R}^2}|R(\Psi m_2)|^2\, d\widetilde{\mu}^a\right)^{1/2}\left(\int_{\mathbb{R}^2}|\nabla V(R\widetilde{\mu}^a)|^2\, d\widetilde{\mu}^a\right)^{1/2}$$

in absolute value.

Now, due to the second restriction on the radius bound ρ^* in Section 11 and the inequality $\widetilde{\mu}^a \leqslant 2\widetilde{\mu}$, the first integral is bounded by

$$2\int_{\mathbb{R}^2}|R(\Psi m_2)|^2\, d\mu' + 2m \leqslant 2(C_9 + 1)m$$

according to the result of Section 10. To estimate the second integral, we use the inequality $|\nabla V|^2 \leqslant 4V$ from Section 8 and obtain

$$\int_{\mathbb{R}^2} |\nabla V(R\widetilde{\mu}^a)|^2 \, d\widetilde{\mu}^a \leqslant 4 \int_{\mathbb{R}^2} V(R\widetilde{\mu}^a) \, d\widetilde{\mu}^a \leqslant 4\Phi(a) \leqslant 8\lambda m \, .$$

Putting all these estimates together, we see that either $\lambda \leqslant \beta$, or

$$7C_6\lambda \geqslant C_7^{-1} \left(\frac{m}{H}\right)^2 - 4\sqrt{C_9+1}\sqrt{\lambda} \, .$$

Taking $\lambda = C_{17}^{-1} \left(\frac{m}{H}\right)^4$ with sufficiently large C_{17}, and recalling that $m \leqslant C_3 H$ due to the growth bound, we see that the second possibility fails. So, either our initial assumption $\int_{\mathbb{R}^2} V(R\widetilde{\mu}) \, d\widetilde{\mu} \leqslant \lambda m$ was false, or $\lambda \leqslant \beta$. In both cases, we conclude that

$$\int_{\mathbb{R}^2} V(R\widetilde{\mu}) \, d\widetilde{\mu} \geqslant \lambda m - \beta m = C_{17}^{-1} \left(\frac{m}{H}\right)^4 m - C_{16} \left(\frac{M^{2s}\delta}{\varepsilon^{2+s}} + \frac{1}{M}\right) m \, .$$

Recalling the comparison inequality between $\int_{\mathbb{R}^2} V(\widetilde{R}\mu') \, d\mu'$ and $\int_{\mathbb{R}^2} V(R\widetilde{\mu}) \, d\widetilde{\mu}$ from Section 19, we finally obtain

$$\int_{\mathbb{R}^2} V(\widetilde{R}\mu') \, d\mu' \geqslant C_{17}^{-1} \left(\frac{m}{H}\right)^4 m - C_{18} \left(\frac{M^{2s}\delta}{\varepsilon^{2+s}} + \frac{1}{M}\right) m \, .$$

Returning to the notation of Section 12 and considering $\mu_j^{(n)}$ instead of μ, we get the inequalities

$$\int_{Q_j^{(n)}} V(R^{(n)}\mu') \, d\mu' \geqslant C_{17}^{-1} \left(\frac{m_j^{(n)}}{H_j^{(n)}}\right)^4 m_j^{(n)} - C_{18} \left(\frac{M^{2s}\delta}{\varepsilon^{2+s}} + \frac{1}{M}\right) m_j^{(n)} \, .$$

Summing these estimates over j and taking into account that $V(x) \leqslant |x|^2$, we arrive at the estimate

$$\int_{\mathbb{R}^2} |R^{(n)}\mu'|^2 \, d\mu' \geqslant C_{17}^{-1} \sum_j \left(\frac{m_j^{(n)}}{H_j^{(n)}}\right)^4 m_j^{(n)} - C_{18} \left(\frac{M^{2s}\delta}{\varepsilon^{2+s}} + \frac{1}{M}\right) \sum_j m_j^{(n)} \, .$$

According to Section 12 we have $2m \geqslant 2\sum_j m_j^{(n)} \geqslant m$, which allows us to estimate the second sum from above by m. To estimate the first sum from below, note that for any positive numbers a_j, b_j, we have

$$\sum \frac{a_j^5}{b_j^4} \geqslant \frac{(\sum a_j)^5}{(\sum b_j)^4} \, ,$$

which is just the Hölder inequality

$$\sum b_j^{4/5} \frac{a_j}{b_j^{4/5}} \leqslant \left[\sum (b_j^{4/5})^{5/4}\right]^{4/5} \left[\sum \left(\frac{a_j}{b_j^{4/5}}\right)^5\right]^{1/5}$$

in disguise. Applying it with $a_j = m_j^{(n)}$ and $b_j = H_j^{(n)}$, we obtain the bound

$$\int_{\mathbb{R}^2} |R^{(n)}\mu'|^2 \, d\mu' \geqslant C_{19}^{-1} \left(\frac{m}{H}\right)^4 m - C_{18}\left(\frac{M^{2s}\delta}{\varepsilon^{2+s}} + \frac{1}{M}\right) m$$

with $C_{19} = 32C_{17}$. Thus, we will get Claim 3 in Section 13 with $C_{20} = \sqrt{2C_{19}}$ if M and δ satisfy

$$C_{18}\left(\frac{M^{2s}\delta}{\varepsilon^{2+s}} + \frac{1}{M}\right) \leqslant \frac{1}{2C_{19}}\left(\frac{m}{H}\right)^4 .$$

It remains to note that, once N and ε are fixed, we can always choose first $M > 1$ and then $\delta > 0$ to satisfy this condition simultaneously with the conditions

$$\frac{2M^s\delta}{\varepsilon^s} < 1 \quad \text{and} \quad C_{12}\left(\frac{M^s\delta}{\varepsilon^s} + \frac{1}{M}\right)\sqrt{2} \leqslant \frac{m^2}{4NC_{20}H^2}$$

in Sections 14 and 16 correspondingly.

22. Concluding remarks

The same proof works in any dimension d for $s \in (d-1, d)$. To cover the other values of s, we need some form of the maximum principle (no matter how week; the equilibrium measure idea should allow one to turn any decent statement of the kind “small on the support, hence small everywhere” into the desired L^2 bound). Of course, more direct ways to get the lower bound may be even more interesting.

Notice also that we also proved the following theorem

Theorem. *Let $s \in (0,2)\setminus\{1\}$ and let μ be a strictly positive finite Borel measure in $\mathbb{R}^2$ such that $\mathcal{H}^s(\operatorname{supp}\mu) < +\infty$. Then $\sup_{\epsilon>0} |R_\epsilon\mu(x)| = \infty$ for μ a. e. x, and the operator norm $\|R_\mu : L^2(\mu) \to L^2(\mu)\| = \infty$.*

The same is again true for $s \in (d-1, d)$ in any $\mathbb{R}^d$. This is just reformulations of our main theorem. This is easy to see by using several non-homogeneous Harmonic Analysis reductions as in [5], see also [12].

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